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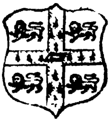
THE ELEMENTS

OF

ALGEBRA.



Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

THE ELEMENTS

OF

ALGEBRA

DESIGNED FOR THE USE OF STUDENTS IN THE UNIVERSITY.

BY

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DEAN OF ELY, AND MASTER OF ST JOHN'S COLLEGE,
CAMBRIDGE.

SIXTEENTH EDITION,

CAREFULLY REVISED AND MUCH ENLARGED

BY

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CAMBRIDGE.

LONDON:
LONGMAN, GREEN, LONGMAN, AND ROBERTS,
PATERNOSTER-ROW.
1861.

ADVERTISEMENT TO THE SIXTEENTH EDITION.

THIS Edition possesses several advantages over the preceding one:

1st. It has been printed with new type; and an unusual amount of labour has been expended in correcting the press. Also every Example, Problem, and Exercise, has been worked out by myself, and its Answer verified. So that the book now appears, it is hoped, as nearly free from errors, and as attractive, as it is possible for such a book to be.

2ndly. Numerous slight, but not unimportant, improvements have been made throughout the work, for the most part according to suggestions contained in the valuable MS. NOTES of the late Rev. G. B. Wildig, of Caius College. And I have re-produced a portion of the 2nd Part of Algebra, (as it was formerly called), which, though not furnishing a *complete* discussion of the *General Theory of Equations*, will yet be found sufficient for the majority of Students.

Moreover, some additional NOTES are given at the end of the book, of which one, compiled by permission from the late Professor Peacock's Algebra, and two others by the Rev. J. R. Lunn, and the Rev. H. G. Day, Fellows of St. John's College, will suggest views on *Symbolical*, or *Formal*, *Algebra*, in advance of those which are at present generally entertained.

3rdly. Although the Collection of Examples and Problems in this volume (presenting the marrow of the Cambridge Examinations for the last 40 years) contains some of extreme difficulty, not one is given of which a detailed *Solution* will not be found either n the "COMPANION" designed for *Students*, or in the "KEY" for *Schoolmasters*, lately published.

As before, I have printed Dr. Wood's matter in large type to distinguish it from the rest, having too great respect both for Dr. Wood's memory, and for my own character, to appropriate to myself that which is another's.

T. L.

Morton Rectory, near Alfreton,
Aug. 1, 1861.

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INTRODUCTION.

VULGAR FRACTIONS*.

ART. 1. A FRACTION is a quantity which represents a part or parts of an integer or whole.

2. A *simple fraction* consists of two members (or *terms*), the *numerator* and the *denominator;* the denominator shews into how many equal parts the whole, or unity, is divided; and the numerator the number of those parts taken. The numerator is usually placed over the denominator with a line between them. Thus $\frac{2}{3}$, (*two thirds*,) signifies that unity is divided into three equal parts, and that two of those parts are taken.

It must be observed, that we suppose every integer to be divisible into any number of equal parts at pleasure.

3. A *proper fraction* is one whose numerator is less than its denominator, as $\frac{7}{8}$.

4. An *improper fraction* is one whose numerator is equal to, or greater than, its denominator, as $\frac{6}{6}$, $\frac{7}{5}$.

5. A *compound fraction* is a fraction *of* a fraction, as $\frac{3}{4}$ of $\frac{5}{6}$,

---

* Such is the importance of a right knowledge of Arithmetical *Fractions*, and so common is it for the Student to fail herein, that the Author has wisely prefixed this chapter to his treatise on *Algebra*, as knowing that no sure progress can be made in the latter subject, while there is any unsoundness as to the former. ED.

where $\frac{5}{6}$ is the whole quantity of which $\frac{3}{4}$ is to be taken; also $\frac{2}{3}$ o of $\frac{9}{11}$ is a compound fraction; &c.

6. A quantity consisting of a whole number and a fraction called a *mixed number*, as $7\frac{3}{10}$, which signifies 7 integers togeth with $\frac{3}{10}$ of an integer.

7. Every integer may be considered as a fraction whose den minator is 1; thus 5, or 5 units, is $\frac{5}{1}$.

And $\frac{5}{1}$ may be read 5 *integers*, or 5 *wholes*.

8. A *continued fraction* is one whose denominator is continued by bein itself a *mixed number*, and the denominator of the fractional part agai continued as before, and so on: thus

$$\frac{1}{2+\frac{1}{3+\frac{1}{4}}},\quad 7+\frac{5}{3+\frac{4}{9+\&c.}}\ \text{are called } \textit{continued fractions}^*.$$

9. *To multiply a fraction by any whole number.*

Rule. Multiply the numerator by that number and retain the same denominator.

Thus $\frac{2}{15}$ multiplied by 7 is $\frac{14}{15}$. For the unit in each of the fractions $\frac{2}{15}$ and $\frac{14}{15}$ is divided into 15 equal parts, and 7 times as many of those parts are taken in the latter case as in the former.

10. *To divide a fraction by any whole number.*

Rule. Multiply the denominator by that number and retain the same numerator.

Thus $\frac{3}{5}$ divided by 4 is $\frac{3}{20}$. For, the unit being divided into four times as many equal parts in $\frac{3}{20}$ as it is in $\frac{3}{5}$, each of the parts in the latter case is four times as great as in the former, and the same number of parts is taken in both cases; therefore the former fraction is one fourth of the latter.

---

* To avoid repetition the reader is referred to the first section of the Algebra for the explanation of the signs +, −, ×, =, ÷, &c.

11. *A simple fraction may be considered as representing the quotient arising from the division of the numerator by the denominator.*

Thus the fraction $\frac{3}{4}$ represents the quotient of 3 divided by 4; for 3 is $\frac{3}{1}$ (Art. 7)[*], and this divided by 4 is the fraction $\frac{3}{4}$ (Art. 10). If the integer be supposed a pound, or twenty shillings, $\frac{3}{4}$ of £1, which is 15 shillings, is equal to $\frac{1}{4}$ of £3, which is also 15 shillings.

12. *If the numerator and denominator of a fraction be both multiplied by the same number, the value of the fraction is not altered.*

For, if the numerator be multiplied by any number, the fraction is multiplied by that number (Art. 9); and if the denominator be multiplied by the same number, the fraction is divided by it (Art. 10); and if a quantity be both multiplied and divided by the same number, its value is not altered.

$$\text{Thus } \frac{5}{14} = \frac{15}{42} = \frac{150}{420}, \text{ \&c.}$$

COR. Hence, if the numerator and denominator of a fraction be both *divided* by the same number, its value is not altered.

$$\text{Thus } \frac{150}{420} = \frac{15}{42} = \frac{5}{14}.$$

REDUCTION.

The operation by which a quantity is changed from one denomination to another, or by which a fraction has its *terms* diminished, *without altering its value*, is called *Reduction.*

---

[*] This is the usual way of referring, either for illustration or proof, to some other clause, or *Article. Ed.*

13. *To reduce a whole number to a fraction with a give denominator.*

Rule. Multiply the proposed number by the given denominator, and the product will be the numerator of the fractio required.

Ex. Reduce 5 to a fraction whose denominator is 6.

This is $\frac{5\times6}{6}$, or $\frac{30}{6}$; because 5 may be considered as a fraction (Art. 7), the numerator and denominator of which are multiplied b 6, therefore its value is not altered (Art. 12).

14. *To reduce a mixed number to an improper fraction.*

Rule. Multiply the integral by the denominator of the fractional part, to this product add the numerator of the fractiona part, and make its denominator the denominator of the sum.

Ex. 1. Reduce $7\frac{4}{5}$ to an improper fraction.

The quantity $7\frac{4}{5}$ is $7+\frac{4}{5}$, which is equal to $\frac{35}{5}+\frac{4}{5}$, or $\frac{39}{5}$; for 7 (by last Art.), is equal to $\frac{35}{5}$, and if to this $\frac{4}{5}$ be added, the whole is $\frac{39}{5}$.

Ex. 2. Also $23\frac{9}{11}=\frac{11\times23+9}{11}=\frac{253+9}{11}=\frac{262}{11}$.

15. *To reduce an improper fraction to a mixed number.*

Rule. Divide the numerator by the denominator for the integral part, and make the remainder the numerator of the fractional part, and the divisor its denominator.

Ex. Reduce $\frac{39}{5}$ to a mixed number.

The fraction $\frac{39}{5}=7\frac{4}{5}$; because the unit being divided into 5 equal parts, 39 such parts are to be taken, that is, 7 units and 4 such parts.

16. *To reduce a compound fraction to a simple one.*

Rule. Multiply all the numerators together for a new numerator, and all the denominators for a new denominator.

Ex. 1. $\frac{2}{3}$ of $\frac{4}{5}=\frac{8}{15}$; for one third of $\frac{4}{5}$ is $\frac{4}{15}$ (Art. 10); therefore two thirds of the same quantity, which must be twice as great, is $\frac{8}{15}$ (Art. 9).

Ex. 2. $\frac{3}{4}$ of $5=\frac{3}{4}$ of $\frac{5}{1}=\frac{15}{4}$.

Ex. 3. $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{9}{11}=\frac{72}{165}$.

Mixed numbers must be reduced to improper fractions before the rule can be applied. Thus

Ex. 4. $\frac{5}{8}$ of $\frac{2}{9}$ of $3\frac{1}{12}=\frac{10}{72}$ of $3\frac{1}{12}=\frac{10}{72}$ of $\frac{37}{12}$ (Art. 14) $=\frac{370}{864}$.

17. *To reduce a continued fraction to a simple one.*

Apply the rule (Art. 14) for reducing a mixed number to an improper fraction, commencing at the lowest extremity of the continued fraction, and proceeding gradually upwards until the whole is reduced to a simple fraction. But as this operation requires the use of a rule not yet proved, the example is deferred to Art. 38.

18. *To reduce a fraction to lower terms.*

RULE. Whenever the numerator and denominator of a fraction have a *common measure*, that is, *a number which divides each of them without remainder*, greater than unity, the fraction may be reduced to lower terms by dividing both the numerator and denominator by this common measure.

Ex. $\frac{105}{120}$ is reduced to $\frac{21}{24}$ by dividing both the numerator and denominator by 5; and $\frac{21}{24}$ is again reduced to $\frac{7}{8}$ by dividing its numerator and denominator by 3. That the value of the fraction is not altered appears from Art. 12, COR.

In the same manner $\frac{168}{210}=\frac{84}{105}=\frac{28}{35}=\frac{4}{5}$.

19. *The "Greatest Common Measure" of two numbers is found by dividing the greater by the less, and the preceding divisor by the remainder, continually, till nothing is left: the last* Divisor *is the Greatest Common Measure required.*

Def. The Greatest Common Measure of two or more numbers is the *greatest* number which will divide each of them without remainder.

Ex. To find the Greatest Common Measure of 189 and 224.

$$
\begin{array}{rrrr}
189)\underline{224}(1 & & & \\
\underline{189} & & & \\
35) & \underline{189}(5 & & \\
 & \underline{175} & & \\
 & 14) & 35(2 & \\
 & & \underline{28} & \\
 & & 7) & 14(2 \\
 & & & \underline{14} \\
 & & & 0
\end{array}
$$

By proceeding according to the rule, it appears that 7 is the last divisor, or the Greatest Common Measure sought.

The proof of this rule will be given hereafter. See Art. 103.

If the Greatest Common Measure of *three* numbers is to be found, find the G. C. M. of two of them, and then the G. C. M. of this and the third number, which will be the G. C. M. required.

Also, observe, it will abridge the operation to begin with those two which are the nearest to each other in value.

20. *A fraction is reduced to its* lowest *terms by dividing its numerator and denominator by their* greatest *Common Measure.*

Ex. To reduce $\frac{385}{396}$ to its lowest terms.

By the Rule given in the last Art. the *Greatest Common Measure* of the numerator and denominator is found to be 11; and therefore $\frac{35}{36}$ is the fraction in its lowest terms.

Cor. If unity be the Greatest Common Measure of the numerator and denominator, the fraction is already in its lowest terms.

21. *To reduce any number of fractions to a common denominator.*

Rule. Having reduced, if there be any, compound fractions to simple ones, and mixed numbers to improper fractions, multiply each numerator by all the denominators except its own for the new numerator, and all the denominators together for a common denominator.

Ex. 1. Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ to a common denominator.

$$\frac{1\times3\times4}{2\times3\times4},\ \frac{2\times2\times4}{2\times3\times4},\ \text{and}\ \frac{2\times3\times3}{2\times3\times4},\ \text{or}\ \frac{12}{24},\ \frac{16}{24},\ \text{and}\ \frac{18}{24},$$

are the fractions required. These fractions are respectively equal to the former, the numerator and denominator in each case having been multiplied by the same numbers, namely, the product of the denominators of the rest (Art. 12),

$$\frac{1\times3\times4}{2\times3\times4}=\frac{1}{2};\quad \frac{2\times2\times4}{2\times3\times4}=\frac{2}{3};\quad \text{and}\ \frac{2\times3\times3}{2\times3\times4}=\frac{3}{4}.$$

Ex. 2. Reduce $\frac{2}{5}$ of $\frac{3}{4}$ and $4\frac{1}{3}$ to a common denominator.

These are $\frac{6}{20}$ and $\frac{13}{3}$, or $\frac{3}{10}$ and $\frac{13}{3}$; therefore $\frac{9}{30}$ and $\frac{130}{30}$ are the fractions required.

22. *If the denominator of one of two fractions contain the denominator of the other a certain number of times exactly, multiply the numerator and denominator of the latter by that number, and it will be reduced to the same denominator with the former.*

Ex. Reduce $\frac{5}{12}$ and $\frac{2}{3}$ to a common denominator.

Since 12 contains 3 four times exactly, multiply both the numerator and denominator of $\frac{2}{3}$ by 4, and it becomes $\frac{8}{12}$, a fraction having the same denominator with $\frac{5}{12}$.

In the reduction of fractions to a common denominator the following rule is frequently required, in order that the reduced fractions may be *in their lowest terms*:—

23. *To find the "Least Common Multiple" of any numbers.*

Def. The "Least Common Multiple" of any numbers is the *least* number which is divisible by each of them without remainder.

Rule. To find it, write down in one line the numbers of which the least common multiple is required, separating them by some mark, as a comma. Divide *all those which have a common measure* by that common measure*, and bring down the other numbers placed in a line with the quotients, separated as before; and repeat this process as long as any common measure exists between two or more of them. The Least Common Multiple required will be the continued product of the divisors and of the final quotients.

---

* That is, the divisor must be *prime* to those numbers which it does not *measure*. We are sure we comply with the direction, if we divide by *prime numbers* only.

Ex. Required the Least Common Multiple of 8, 12, and 18.

$$\begin{array}{r|lll} 2 & 8, & 12, & 18 \\ \hline 2 & 4, & 6, & 9 \\ \hline 3 & 2, & 3, & 9 \\ \hline & 2, & 1, & 3 \end{array}$$

Least Com. Mult. is $2 \times 2 \times 3 \times 2 \times 1 \times 3$, or 72.

The proof of this Rule will be given hereafter. See Art. 115.

N.B. In finding the Least Common Multiple of any numbers care must be taken to follow the Rule strictly, viz. to "divide *all* those which have a common measure by that common measure". Thus, as above, the Least Common Multiple of 8, 12, and 18 is correctly found to be 72; but the operation might be carelessly attempted as follows:

$$\begin{array}{r|lll} 6 & 8, & 12, & 18 \\ \hline 2 & 8, & 2, & 3 \\ \hline & 4, & 1, & 3 \end{array}$$

first taking for a divisor 6, the common measure of 12 and 18, instead of 2, the common measure of *all*;

from which we should conclude that the Least Common Multiple required is $6 \times 2 \times 4 \times 1 \times 3$, or 144; which is twice as great as the true Least Common Multiple.

Also, it is obvious, that in any proposed case those numbers may be entirely omitted in the operation which are contained in any of the others. Thus, to find the Least Common Multiple of 6, 7, 12, and 14, we observe that 6 is contained in 12, and 7 in 14; therefore it remains only to find the Least Common Multiple of 12 and 14, which is 84.

24. *To reduce fractions to a common denominator, in their* lowest terms, *find the "Least Common Multiple" of all the denominators, and make that the common denominator by multiplying both the numerator and denominator of each fraction by the quotient of Least Common Multiple divided by the denominator.*

Ex. Reduce to a common denominator $\frac{1}{3}$, $\frac{2}{7}$, $\frac{3}{14}$, $\frac{12}{21}$, and $\frac{3}{4}$.

The Least Common Multiple of the denominators, by Art. 23, is found to be 84; and therefore the required fractions are

$$\frac{28 \times 1}{28 \times 3}, \quad \frac{12 \times 2}{12 \times 7}, \quad \frac{6 \times 3}{6 \times 14}, \quad \frac{4 \times 12}{4 \times 21}, \quad \frac{21 \times 3}{21 \times 4},$$

$$\text{or } \frac{28}{84}, \quad \frac{24}{84}, \quad \frac{18}{84}, \quad \frac{48}{84}, \quad \frac{63}{84}.$$

25. Cor. By reducing fractions to a common denominator their *values* may be compared.

Thus $\frac{4}{7}$ and $\frac{7}{12}$, when reduced to a common denominator, are $\frac{48}{84}$ and $\frac{49}{84}$; that is, the fractions have the same relative *values* that 48 and 49 have.

26. By reducing fractions to a common *numerator* also their values may be compared.

Thus $\frac{3}{13}$ and $\frac{4}{17}$, when reduced to a common numerator, are $\frac{12}{52}$ and $\frac{12}{51}$; and since the former of these fractions signifies that the unit is divided into 52 equal parts of which 12 are taken, and the latter signifies that the unit is divided into 51 equal parts of which 12 are taken, it is obvious that the latter fraction is the greater of the two, or that which has the smaller denominator.

27. *To find the value of a fraction of a proposed denomination in terms of a lower denomination.*

RULE. Multiply the fraction by the number of integers of the lower denomination contained in one integer of the higher, and the product is the value required. The value of any fractional part of the lower denomination may be obtained in the same manner, till we come to the lowest.

Ex. 1. What is the value of $\frac{5}{7}$ of £1?

First, $\frac{5}{7}$ of £1 is $\frac{5}{7}$ of 20 shillings,

$$\text{or } \frac{5}{7} \text{ of } \frac{20}{1} \text{ shillings} = \frac{100}{7} = 14\tfrac{2}{7} \text{ shillings.}$$

$$\text{Next, } \frac{2}{7} \text{ of a shilling} = \frac{2}{7} \text{ of } \frac{12}{1} \text{ pence,}$$

$$= \frac{24}{7} \text{ pence} = 3\tfrac{3}{7} \text{ pence.}$$

$$\text{Lastly, } \frac{3}{7} \text{ of a penny} = \frac{3}{7} \text{ of 4 farthings} = \frac{3}{7} \text{ of } \frac{4}{1},$$

$$= \frac{12}{7} \text{ farthings} = 1\tfrac{5}{7} \text{ farthings:}$$

$$\text{hence, } \frac{5}{7} \text{ of 1£ is } \overset{s.}{14} \,.\, \overset{d.}{3} \,.\, \overset{q.}{1\tfrac{5}{7}}.$$

The operation is usually performed in the following manner:

£5
20
7)100
14--2*s.*
12
7)24
3--3*d.*
4
7)12
1--5*q.*

Ans. $\overset{s.}{14} . \overset{d.}{3} . \overset{q}{1\frac{5}{7}}$.

Ex. 2. What is the value of $\frac{5}{9}$ of a crown?

5*C*
5
9)25
2--7*s.*
12
9)84
9--3*d.*
4
9)12
1--3*q.*

Ans. $\overset{s.}{2} . \overset{d.}{9} . \overset{q.}{1\frac{3}{9}}$.

28. *To reduce a quantity to a fraction of any denomination.*

Rule. Make the given quantity the numerator, and the number of integers of its denomination in one of the proposed denomination the denominator, and the fraction required is determined.

Ex. What fraction of a pound is $\overset{s.}{12} . \overset{d.}{7} . \overset{q.}{3}$?

$$\overset{s.}{12} . \overset{d.}{7} . \overset{q.}{3} = \overset{q.}{607}; \text{ and one pound} = \overset{q.}{960};$$

therefore $\frac{607}{960}$ is the fraction sought; because the integer being divided into 960 equal parts, $\overset{s.}{12} . \overset{d.}{7} . \overset{q.}{3}$ contains 607 such parts.

29. In the last example we were obliged to reduce the whole to farthings; and in general, if the higher denomination do not

contain the lower an exact number of times, reduce them to a common denomination, and proceed as before.

Ex. What fraction of a guinea is half a crown?

Here sixpence is the greatest common denomination, of which a guinea contains 42, and half a crown 5, therefore $\frac{5}{42}$ is the fraction required.

Any common denomination would answer the purpose; but, if the *greatest* be taken, the resulting fraction is *in the lowest terms.*

30. *To reduce a fraction to any denomination.*

Rule. Find what fraction of the proposed denomination an integer of the denomination of the given fraction is, and the fraction required will be found by Art. 16.

Ex. 1. What fraction of a pound is $\frac{2}{3}$ of a shilling?

1 shilling is $\frac{1}{20}$ of a pound, therefore $\frac{2}{3}$ of 1 shilling is $\frac{2}{3}$ of $\frac{1}{20}$ of a pound, $= \frac{2}{60} = \frac{1}{30}$ of a pound.

Ex. 2. What fraction of a yard is $\frac{5}{7}$ of an inch?

1 inch is $\frac{1}{36}$ of a yard, therefore $\frac{5}{7}$ of an inch is $\frac{5}{7}$ of $\frac{1}{36}$ of a yard, $= \frac{5}{252}$ of a yard.

Ex. 3. What fraction of a guinea is $\frac{4}{9}$ of a pound?

1 pound is $\frac{20}{21}$ of a guinea (Art. 29); hence $\frac{4}{9}$ of a pound is $\frac{4}{9}$ of $\frac{20}{21}$ of a guinea $= \frac{80}{189}$ of a guinea.

ADDITION OF FRACTIONS.

31. *To find the sum of two or more fractions.*

Rule I. *If fractions have a common denominator, their sum is found by taking the sum of the numerators, and subjoining the common denominator.*

Thus $\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$. For, if an integer be divided into five equal parts, one of those parts, together with two parts of the same kind, must make three such parts.

32. RULE II. *If the fractions have not a common denominator, reduce them to a common denominator, and proceed as before.*

Ex. Required the sum of $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$.

These reduced to a common denominator are $\frac{40}{60}$, $\frac{45}{60}$, and $\frac{48}{60}$, whose sum is $\frac{133}{60}$, or $2\frac{13}{60}$.

33. When *mixed numbers* are to be added, to the sum of the fractional parts, found as before, add the sum of the integers.

Ex. Add together $5\frac{3}{4}$, $6\frac{1}{3}$, and $\frac{2}{5}$ of $\frac{1}{7}$.

$$\frac{3}{4}+\frac{1}{3}+\frac{2}{35}=\frac{315}{420}+\frac{140}{420}+\frac{24}{420}=\frac{479}{420}=1\tfrac{59}{420},$$

therefore the whole sum required is $5+6+1\frac{59}{420}$, or $12\frac{59}{420}$.

SUBTRACTION.

34. *To find the difference of two fractions.*

RULE I. *The difference of two fractions which have a common denominator is found by taking the difference of their numerators, and subjoining the common denominator.*

Thus $\frac{4}{5}-\frac{3}{5}=\frac{1}{5}$. For, if the unit be supposed to be divided into five equal parts, and three of those parts be taken from four, the remainder must be one of the parts, or $\frac{1}{5}$.

35. RULE II. *If the fractions have not a common denominator, let them be reduced to a common denominator, and then take the difference as before.*

Ex. 1. From $\frac{9}{11}$ take $\frac{4}{5}$.

$$\frac{9}{11}-\frac{4}{5}=\frac{45}{55}-\frac{44}{55}=\frac{1}{55}.$$

Ex. 2. From $\frac{11}{12}$ of $\frac{3}{5}$ take $\frac{1}{3}$ of $\frac{7}{8}$.

$$\frac{11}{12}\text{ of }\frac{3}{5}=\frac{33}{60},\text{ and }\frac{1}{3}\text{ of }\frac{7}{8}=\frac{7}{24};$$

$$\frac{33}{60}-\frac{7}{24}=\frac{66}{120}-\frac{35}{120}\text{ (Art. 24)}=\frac{31}{120}.$$

When *mixed numbers* are to be subtracted, the integers may be subtracted separately, and then the fractional parts. Thus,

$$3\tfrac{1}{2}-2\tfrac{1}{4}=3-2+\tfrac{1}{2}-\tfrac{1}{4}=1+\tfrac{1}{4}=1\tfrac{1}{4}.$$

And if the fractional part of the mixed number to be subtracted is *greater* than that of the other, deduct 1 from the greater number and add it *in a fractional form* to the smaller fractional part; then proceed as before.

Thus, $6\tfrac{1}{9}-3\tfrac{7}{9}=5\tfrac{10}{9}-3\tfrac{7}{9}=5-3+\tfrac{10}{9}-\tfrac{7}{9}=2+\tfrac{3}{9}=2\tfrac{1}{3}$.

MULTIPLICATION.

36. **Def.** *To multiply one fraction by another is to take such part or parts of the former as the latter expresses.*

Rule. *This is done by multiplying the numerators of the two fractions together for a new numerator, and the denominators for a new denominator.*

Thus $\frac{3}{4}\times\frac{5}{7}=\frac{15}{28}$; for $\frac{3}{4}$ multiplied by $\frac{5}{7}$ is, according to the definition of multiplication, $\frac{5}{7}$ of $\frac{3}{4}$, or $\frac{15}{28}$, (Art. 16). (See '*Companion*[*],' p. 2).

If there be more than two fractions to be multiplied together, a similar rule applies:—multiply all the numerators together for a new numerator, and all the denominators for a new denominator.

Thus, $\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{6}{24}=\frac{1}{4}$; for $\frac{1}{2}$ of $\frac{2}{3}=\frac{1}{3}$, and $\frac{1}{3}$ of $\frac{3}{4}=\frac{1}{4}$.

Compound fractions must be reduced to simple ones, and *mixed numbers* to improper fractions, and they may then be multiplied as before.

---

[*] Companion to *Wood's Algebra*, by Lund, 3rd *Edition*.

Ex. 1. Multiply $\frac{2}{5}$ of $\frac{9}{13}$ by $7\frac{1}{8}$.

$$\frac{2}{5} \text{ of } \frac{9}{13} = \frac{18}{65}; \text{ and } 7\tfrac{1}{8} = \frac{57}{8};$$

therefore their product is

$$\frac{18}{65}\times\frac{57}{8} = \frac{1026}{520} = 1\tfrac{506}{520} = 1\tfrac{253}{260}.$$

Ex. 2. Multiply $\frac{2}{217}$ by 7.

$$\frac{2}{217}\times 7 = \frac{2\times 7}{217} = \frac{2}{31}.$$

Hence it appears, that a fraction may be multiplied by a whole number by dividing the denominator by that number, when this division can take place.

Much trouble is frequently saved by observing what multipliers are *common* to the new numerator and denominator, and striking them out (Art. 12, Cor.) *before the multiplication is effected.*

Thus the product of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$, is, by the rule, $\frac{1\times 2\times 3\times 4}{2\times 3\times 4\times 5}$, which we see at once to be $\frac{1}{5}$, by striking out the quantity $2\times 3\times 4$ common to the numerator and denominator.

Again, if in the proposed fractions, to be multiplied together, there be *any* numerator and *any* denominator which have a *common measure*, divide them both by that common measure, and use the resulting quotients for the purpose of forming the required product.

Thus $\frac{18}{65}\times\frac{57}{8} = \frac{9}{65}\times\frac{57}{4} = \frac{513}{260}$; since 18 and 8 have the common measure 2.

Also $\frac{3}{4}\times\frac{7}{27}\times\frac{16}{21} = \frac{1}{4}\times\frac{7}{9}\times\frac{16}{21}$, (dividing 3 and 27 by 3,)

$= \frac{1}{1}\times\frac{7}{9}\times\frac{4}{21}$, (.......... 4 and 16 by 4,)

$= \frac{1}{1}\times\frac{1}{9}\times\frac{4}{3}$, (.......... 7 and 21 by 7,)

$= \frac{4}{27}$.

DIVISION.

37. *To divide one fraction by another.*

Rule. *To divide one fraction by another, or to determine how often one is contained in the other, invert the numerator and denominator of the* divisor, *and proceed as in multiplication.*

Ex. 1. $\frac{3}{4}$ divided by $\frac{5}{7}$ is $\frac{3}{4}\times\frac{7}{5}=\frac{21}{20}=1\frac{1}{20}$.

For, from the nature of division, the divisor multiplied by the quotient must produce the dividend: therefore $\frac{5}{7}\times$quotient$=\frac{3}{4}$; let these equal quantities be multiplied by the same quantity $\frac{7}{5}$, and the products must be equal; that is, $\frac{7}{5}\times\frac{5}{7}\times$quotient$=\frac{3}{4}\times\frac{7}{5}$, or $\frac{35}{35}\times$ quotient$=\frac{21}{20}$; but $\frac{35}{35}=1$; therefore the quotient$=\frac{21}{20}$, as was found by the rule. And the same method of proof is applicable to all cases.

Compound fractions must be reduced to simple ones, and mixed numbers to improper fractions, before the rule can be applied.

Ex. 2. Divide $\frac{5}{9}$ of $\frac{4}{7}$ by $3\frac{1}{3}$.

$$\frac{5}{9}\text{ of }\frac{4}{7}=\frac{20}{63},\text{ and }3\tfrac{1}{3}=\frac{10}{3};$$

therefore the quotient required is $\frac{20}{63}\times\frac{3}{10}=\frac{2}{21}$.

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38. It will often happen in practice that fractions present themselves which require the application, not of one single rule *only*, as of Addition, or Subtraction, or Multiplication, &c., but of several rules in one operation. Thus,

Ex. 1. Required to find the single fraction which is equivalent to

$$\frac{5}{7}\times\left\{100-\frac{2}{3}\text{ of }100+\frac{7\frac{1}{3}}{2\frac{1}{4}}\right\}.$$

$$\text{First, }100-\frac{2}{3}\text{ of }100=\frac{1}{3}\text{ of }100=\frac{100}{3},$$

$$\text{and } \frac{7\frac{1}{3}}{2\frac{1}{4}} = \frac{\frac{22}{3}}{\frac{9}{4}} = \frac{22}{3} \div \frac{9}{4} = \frac{88}{27};$$

therefore the whole quantity is equivalent to

$$\frac{5}{7} \times \left\{ \frac{100}{3} + \frac{88}{27} \right\},$$

$$= \frac{5}{7} \times \frac{900+88}{27},$$

$$= \frac{5}{7} \times \frac{988}{27} = \frac{4940}{189},$$

$$= 26\tfrac{26}{189}.$$

Ex. 2. Reduce $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{4}}}$ to a simple fraction.

Here $3+\frac{1}{4} = \frac{13}{4}$; and $\dfrac{1}{3+\frac{1}{4}} = \dfrac{1}{\frac{13}{4}} = \dfrac{4}{13}$. Therefore

$2+\dfrac{1}{3+\frac{1}{4}} = 2+\dfrac{4}{13} = \dfrac{30}{13}$, and the fraction required is $\dfrac{1}{\frac{30}{13}}$, or $\dfrac{13}{30}$.

DECIMAL FRACTIONS.

39. In order to lessen the trouble which in many cases attends the use of Vulgar Fractions, *Decimal Fractions* have been introduced, which differ from the former in this respect, that their denominators are always 10 or some power of 10, as 100, 1000, 10000, &c. and instead of *writing* the denominator under the numerator, it is *expressed* by pointing off from the right of the numerator as many figures as there are cyphers in the denominator;

thus ·2 signifies $\frac{2}{10}$,

·23 ...... $\frac{23}{100}$,

·127 ...... $\frac{127}{1000}$,

·0013 ...... $\frac{13}{10000}$,

43·7 ...... $43\frac{7}{10}$ or $\frac{437}{10}$.

40. Cor. 1. The value of each figure in a decimal decreases from the left to the right in a tenfold proportion, that is, each figure is ten times as great as if it were removed one place to the right, as in whole numbers; thus

$$\cdot 2 = \frac{2}{10}; \text{ but } \cdot 02 = \frac{2}{100}, \text{ and } \cdot 002 = \frac{2}{1000}, \text{ \&c.};$$

and the decimal ·127 is one tenth, two hundredths, and seven thousandths, of an unit.

41. Cor. 2. Adding cyphers to the right of a decimal does not alter its value; thus

$$\begin{aligned}\cdot 2 &= \frac{2}{10} = \frac{20}{100} = \cdot 20,\\ &= \frac{200}{1000} = \cdot 200,\\ &= \frac{2000}{10000} = \cdot 2000,\\ &= \text{\&c.}\end{aligned}$$

the numerator and denominator having been multiplied by the same number. (Art. 12.)

42. Cor. 3. Decimals may be reduced to a common denominator by adding cyphers to the right, where it is necessary, till the number of decimal places is the same in all.

Ex. ·5, ·01, and ·311, reduced to a common denominator, are ·500, ·010 and ·311;

$$\text{that is } \frac{500}{1000}, \frac{10}{1000}, \text{ and } \frac{311}{1000}.$$

43. Cor. 4. Hence in all complicated numerical reductions decimal fractions possess great advantages over vulgar fractions. For, 1st, the denominators of the former being always 10 or some power of 10, their reduction to a common denominator is easily effected, and consequently all operations requiring that previous reduction are facilitated: 2nd, the numerators and denominators of decimal fractions being usually written in one line, and the value of each figure decreasing in a tenfold proportion, from left to right, as in whole numbers, the common rules of *Arithmetic* are immediately applicable to such fractions, care only being taken, by means of rules for that purpose, to mark off correctly the decimal result.

As decimals are only fractions of a particular description, their operations must depend upon the principles already laid down.

ADDITION OF DECIMALS.

44. **Rule.** *To find the sum of any number of decimals place the figures in such a manner that those of the same denomination may stand under each other; add them together as in whole numbers, and place the decimal point in the sum under the other points.*

Ex. Add together 7·9, 51·43, and ·0118.

These, when reduced to a common denominator, are 7·9000, 51·4300, and ·0118; and proceeding according to the rule,

$$\begin{array}{rl} 7{\cdot}9000 & \\ 51{\cdot}4300 & \\ {\cdot}0118 & \\ \hline 59{\cdot}3418 & = \text{the sum required. (Art. 31.)} \end{array}$$

In the operation the cyphers may be omitted, if the several decimal points stand exactly under each other thus,

$$\begin{array}{l} 7{\cdot}9 \\ 51{\cdot}43 \\ {\cdot}0118 \\ \hline 59{\cdot}3418 \end{array}$$

SUBTRACTION.

45. **Rule.** *To find the difference of two decimals place the figures of the same denomination under each other; then subtract as in whole numbers, and place the decimal point under the other points.*

Ex. From 61·3 take 42·012.

These, reduced to a common denominator, are 61·300 and 42·012; therefore their difference is 19·288 (Art. 34). In the operation the cyphers may be omitted thus,

$$\begin{array}{l} 61{\cdot}3 \\ 42{\cdot}012 \\ \hline 19{\cdot}288 \end{array}$$

MULTIPLICATION.

46. **Rule.** *To multiply one decimal by another multiply the figures as in whole numbers, and point off as many decimal places*

in the product as there are in the multiplier and multiplicand together.

Ex. $51{\cdot}3 \times 4{\cdot}6 = 235{\cdot}98$.

For $\frac{513}{10} \times \frac{46}{10} = \frac{23598}{100}$ = (according to the decimal notation) $235{\cdot}98$. And a similar proof may be given in all other cases.

47. When there are fewer figures in the product than there are decimals in the multiplier and multiplicand together, cyphers must be annexed to the *left* of the product, that the decimal places may be properly represented.

Ex. ${\cdot}25 \times {\cdot}3 = {\cdot}075$; for $\frac{25}{100} \times \frac{3}{10} = \frac{75}{1000}$ = (according to the decimal notation) ${\cdot}075$.

DIVISION.

48. Rule. *Division in decimals is performed as in whole numbers, observing to point off as many decimals in the quotient as the number of decimal places in the dividend exceeds the number in the divisor.*

Ex. Divide $77{\cdot}922$ by $3{\cdot}7$.

$$\frac{77{\cdot}922}{3{\cdot}7} = 21{\cdot}06:$$

here there are three decimals in the dividend and one in the divisor; therefore, there are two in the quotient.

The truth of this rule is apparent from the nature of multiplication; for the product of the divisor and quotient is the dividend; there are, therefore, as many places of decimals in the dividend, as there are in the divisor and quotient together (Art. 46); consequently there are as many in the quotient as the number in the dividend exceeds the number in the divisor*.

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* The proof here given does not directly meet the most frequent case, viz. when, the division being performed as in whole numbers, a *remainder* is left. The general proof is given in Art. 131. But, in fact, division in decimals is best performed without rule; thus, to divide ${\cdot}336$ by 42, since ${\cdot}336 = \frac{336}{1000}$, therefore ${\cdot}336 \div 42 = \frac{1}{1000} \times \frac{336}{42} = \frac{8}{1000} = {\cdot}008$.

Again, to divide $25{\cdot}4$ by $12{\cdot}5$; $\frac{254}{10} \div \frac{125}{10} = \frac{254}{125} = 2 + \frac{4}{125} = 2 + \frac{32}{1000} = 2{\cdot}032$. Also $36 \div {\cdot}012 = 36 \div \frac{12}{1000} = 1000 \times \frac{36}{12} = 3000$. Ed.

49. If figures be wanting in the quotient to make up the proper number of decimal places, cyphers must be added to the *left.*

Ex. Divide ·336 by 42.

$$\frac{336}{42}=8;$$

and as the quotient of ·336 divided by 42 must contain three decimal places, that quotient is ·008.

For $\frac{336}{1000}$ divided by 42 is $\frac{336}{42000}$, or $\frac{8}{1000}$, that is, (according to the decimal notation) ·008.

50. When the dividend does not contain as many decimals as the divisor, cyphers must be added to the *right* of the decimals in the dividend, till that is the case. (Art. 41.)

Ex. Divide 36 by ·012.

$$36=36\cdot000,$$

and 36·000 divided by ·012 is 3000, according to the rule.

REDUCTION.

51. *To reduce a vulgar fraction to a decimal.*

Rule. Add cyphers at pleasure, as decimals, in the numerator, and divide by the denominator according to the rule for the division of decimals. The truth of this rule is evident from Art. 11.

Ex. 1. $\frac{3}{4}=\frac{3\cdot00}{4}=\cdot75.$

Ex. 2. $\frac{7}{8}=\frac{7\cdot000}{8}=\cdot875.$

Ex. 3. $\frac{4}{625}=\frac{4\cdot0000}{625}=\cdot0064.$

Ex. 4. $\frac{1}{3}=\frac{1\cdot000\ \&c.}{3}=\cdot333\ \&c.$

Ex. 5. $\frac{4}{33}=\frac{4\cdot0000\ \&c.}{33}=\cdot1212\ \&c.$

52. In some cases, as in the last two examples, the vulgar fraction cannot exactly be made up of tenths, hundredths, &c., but

the decimal will go on without ever coming to an end*, *the same figure or figures recurring in the same order*†; but though we cannot represent the exact value of the vulgar fraction, yet, by increasing the number of decimal places, we may approach to it *as near as we please.* Thus $\frac{1}{9}=\cdot1111$ &c. Now ·1, or $\frac{1}{10}$, is less than the true value by $\frac{1}{90}$; ·11, or $\frac{11}{100}$, is too little by $\frac{1}{900}$; and so on.

Again $\frac{41}{333}=\cdot123123$ &c., the figures 123 being repeated without end; $\frac{4}{27}=\cdot148148$ &c.; $\frac{5}{36}=\cdot138888$ &c.; and so on.

Decimals of this kind are called *recurring* or *circulating* decimals.

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* It is evident that no vulgar fraction can be *exactly* expressed by a decimal, unless it either has, or can be reduced to another which has, 10 or some power of 10, for its denominator, (Art. 39). Thus, reverting to the Exs. of the last Art.

$$\frac{3}{4}=\frac{3\times5^2}{2^2\times5^2}=\frac{75}{10^2}=0\cdot75,$$

$$\frac{7}{8}=\frac{7\times5^3}{2^3\times5^3}=\frac{875}{10^3}=0\cdot875,$$

$$\frac{4}{625}=\frac{4\times2^4}{5^4\times2^4}=\frac{64}{10^4}=0\cdot0064;$$

each of which vulgar fractions is expressed decimally with perfect exactness. But the following Exs. viz. $\frac{1}{3}$, and $\frac{4}{33}$, since they *cannot* be expressed by equivalent fractions with a denominator of 10 or some power of 10, are not capable of being expressed by terminating decimals.

Also since 10, and its powers, are divisible by 2 and 5 only, and their powers, it follows that *no vulgar fraction can be expressed by a terminating decimal unless, when it is in its lowest terms, its denominator is divisible by one or both of the numbers,* 2 *and* 5, *or their powers, and by no other number.* ED.

† This is easily shewn by a particular instance; and it may thence be seen to be true in all cases. Thus, suppose it is required to find the decimal equivalent to $\frac{4}{27}$. The required decimal is found by dividing 4·00000 &c. by 27; and if the quotient does not terminate, after each division there will be a remainder less than 27. Therefore, under the most unfavourable circumstances, at least after the quotient has reached to 26 figures, one of the remainders 1, 2, 3, &c....26 *must* recur; and consequently after that the figures in the quotient will recur. In the case proposed the remainder for one figure in the quotient is 13; for two figures, 22; for three figures, 4; which is a recurrence of the original figure: consequently the decimal is 0·148148, &c. the figures 148 being repeated *in infinitum.* Similarly also in other cases. ED.

Hence, although some vulgar fractions cannot be *accurately* represented by decimals, this affords no objection to the use of decimals, because for such fractions equivalent decimals can be found approximating to the true value *as nearly as we please**.

53. The method of reducing a *terminating* decimal to a vulgar fraction is pointed out in Art. 39. The following method will serve for converting *recurring* decimals into their equivalent vulgar fractions.—

It appears, by actual division, that

$$\frac{1}{9}=0\cdot11111 \text{ .... } \textit{in inf.} \ (1),$$

$$\text{Hence } \frac{1}{99}=0\cdot010101 \text{ .... [dividing (1) by 11]},$$

$$\frac{1}{999}=0\cdot001001 \text{ .... [dividing (1) by 111]},$$

$$\frac{1}{9999}=0\cdot00010001 \text{ .... [dividing (1) by 1111]},$$

and so on; where the recurring part of the decimal is always 1, preceded by as many ciphers as make the number of recurring digits equal to the number of 9's recurring in the denominator of the fraction.

If then, for instance, the vulgar fraction equivalent to 0·1212 &c. be required, we have

$$0\cdot1212 \text{ \&c.}=0\cdot0101 \text{ \&c.}\times12=\frac{1}{99}\times12=\frac{12}{99}=\frac{4}{33}.$$

$$\text{Again } 0\cdot123123 \text{ \&c.}=0\cdot001001 \text{ \&c.}\times123=\frac{1}{999}\times123=\frac{123}{999}=\frac{41}{333}.$$

Hence, when the recurring period begins immediately after the decimal point, the Rule is :—*Make the recurring period the Numerator with as many 9's for Denominator as there are figures in the Numerator.*

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* "The addition, subtraction, multiplication, and division, of decimal fractions, are much easier than those of common fractions; and though we cannot reduce all common fractions to decimals, yet we can find decimal fractions so near to each of them, that the error arising from using the decimal instead of the common fraction will not be perceptible. For example, if we suppose an inch to be divided into ten million of equal parts, one of those parts by itself will not be visible to the eye. Therefore, in finding a length, an error of a ten-millionth part of an inch is of no consequence, even where the finest measurement is necessary."

"In applying Arithmetic to practice, nothing can be measured so accurately as to be represented in numbers without any error whatever, whether it be length, weight, or any other species of magnitude. It is therefore unnecessary to use any other than decimal fractions; since, by means of them, any quantity may be represented with as much correctness as by any other method." *De Morgan's Arithmetic*, pp. 68-9.

If the recurring period does *not* begin with the first figure after the decimal point, multiply and divide by such a power of 10 as will move the decimal point to the required position; then proceed as before. Thus,

Ex. 1. $0{\cdot}04545\ \&c. = \dfrac{0{\cdot}4545\ \&c.}{10} = \dfrac{1}{10} \times 0{\cdot}0101\ \&c. \times 45,$

$$= \frac{1}{10} \times \frac{45}{99} = \frac{45}{990} = \frac{1}{22}.$$

Ex. 2. $0{\cdot}13888\ \&c. = \dfrac{13{\cdot}888\ \&c.}{100} = \dfrac{13}{100} + \dfrac{0{\cdot}888\ \&c.}{100}.$

Now $0{\cdot}888\ \&c. = 0{\cdot}111\ \&c. \times 8 = \dfrac{1}{9} \times 8 = \dfrac{8}{9};$

therefore $0{\cdot}13888\ \&c. = \dfrac{13}{100} + \dfrac{8}{900} = \dfrac{125}{900} = \dfrac{5}{36}.$

Similarly it may be shewn, that $0{\cdot}09009009\ \&c. = \dfrac{10}{111}.$

54. If it be sufficient for the purposes of any calculation to take a number of decimals less than the number given or obtained, the following rule is to be observed :—

Rule. When the first of the figures struck off is 5, or > 5, add 1 to the last remaining figure.

Thus, if 2·7182818 be the decimal under consideration, 2·72 is nearer to the true value than 2·71, for 2·7182818 − 2·71, is 0·0082818; and 2·72 − 2·7182818 is 0·0017182, which is considerably less than the former difference. Also 2·7183 is nearer to the true value than 2·7182, as may be shewn in a similar manner.

It may also be observed here, that in the *multiplication* of decimals some caution is requisite in taking the product as correct to a certain number of places of decimals, when either the multiplicand or multiplier is only approximately correct. Thus, if 3·12 express a certain length in inches, and is known to be correct within the thousandth part of an inch, the true length may be any thing between $3{\cdot}12 + \dfrac{1}{1000}$ and $3{\cdot}12 - \dfrac{1}{1000}$, that is, between 3·121 and 3·119; and if the proposed number is to be multiplied by 10, for example, the product is 31·2; whereas it may be any thing between 31·21 and 31·19; and therefore may not be correct even to one decimal place.

55. *To find the value of a decimal of one denomination in terms of a lower denomination.*

This may be done by the rule laid down in Art. 27.

Ex. Required the value of ·615625 £.

·615625 £
20
12·312500 *shillings*
12
3·7500 *pence*

3·00 *farthings.*

The value required is $\overset{s.}{12}\ .\ \overset{d.}{3}\ .\ \overset{q.}{3}$.

First, ·615625 £ = 12·3125 *shillings.*
Next, ·3125 *s.* = 3·75....*pence.*
Lastly, ·75 *d.* = 3·......*farthings.*

56. *To reduce a quantity to a decimal of a superior denomination.*

Rule. Divide the quantity by the number of integers of its denomination contained in one of the superior denomination, and the quotient is the decimal required.

Ex. 1. What decimal of a shilling is threepence?

12)3·00
·25 *Ans.*

For in the denomination shillings its numerical value must be $\frac{1}{12}$ of its value in the denomination pence.

Ex. 2. What decimal of a pound is $\overset{s.}{13}\ .\ \overset{d.}{4}\ .\ \overset{q.}{3}$?

4) 3·00
12) 4·75
20)13·3958333 &c.
·66979166 &c.

First, we find what decimal of a penny $\overset{q.}{3}$ is; this, by the rule, is ·75; then, what decimal of a shilling $\overset{d.}{4}\ .\ \overset{q.}{3}$ or 4·75*d.* is; this is found in the same manner to be ·3958333 &c.; lastly, we find, by the same rule, what decimal of a pound 13·3958333 &c. *sh.* is, which appears to be ·66979166 &c.

The conclusion will be the same if we reduce the quantity to a vulgar fraction (Art. 28), and this fraction to a decimal (Art. 51).

57. It will often happen in practice that a whole *series* of vulgar fractions, instead of a single one, is to be reduced to a decimal, and in such cases considerable trouble may frequently be saved by making each fraction, when reduced, subservient to the reduction of some one or more of the others. Thus,

Ex. 1. *Required to reduce to a single decimal having* 5 *decimal places the following series of fractions:—*

$$\frac{2}{1}+\frac{1}{1\times2}+\frac{1}{1\times2\times3}+\frac{1}{1\times2\times3\times4}+\&c.$$

$$\begin{array}{llcl}
\text{Here} & \frac{2}{1} & & =2{\cdot}000000 \\
& \frac{1}{1\times2} & =\frac{1{\cdot}000}{2} & = {\cdot}500000 \\
& \frac{1}{1\times2\times3^{*}} & =\frac{{\cdot}5000}{3} & = {\cdot}166667 \text{ (Art. 54).} \\
& \frac{1}{1\times2\times3\times4} & =\frac{{\cdot}166667}{4} & = {\cdot}041667 \\
& \frac{1}{1\times2\times3\times4\times5} & =\frac{{\cdot}041667}{5} & = {\cdot}008333 \\
& \frac{1}{1\times2\times3\times4\times5\times6} & =\frac{{\cdot}008333}{6} & = {\cdot}001389 \\
& \frac{1}{1\times2\times3\times4\times5\times6\times7} & =\frac{{\cdot}001389}{7} & = {\cdot}000198 \\
& \frac{1}{1\times2\times3\times4\times5\times6\times7\times8} & =\frac{{\cdot}000198}{8} & = {\cdot}000025 \\
& \frac{1}{1\times2\times3\times4\times5\times6\times7\times8\times9} & =\frac{{\cdot}000025}{9} & = {\cdot}000003 \\
\hline
& & \text{Sum} & =2{\cdot}718282
\end{array}$$

Hence the decimal required is 2·71828.

Each single fraction is calculated to *six* places of decimals, that the figure which occupies the 5th place in their sum may be correct; and no more *terms* of the series need be added, because the first five places of decimals are not affected by them.

Ex. 2. *Required to convert into a single decimal having five decimal places the following series:—*

$$2\times\left\{\frac{1}{7}+\frac{1}{3}\times\frac{1}{7^3}+\frac{1}{5}\times\frac{1}{7^5}+\frac{1}{7}\times\frac{1}{7^7}+\dots\right\}.$$

---

* Some trouble in writing may be saved in this and similar examples by making the symbol $\underline{|3}$ stand for 1x2x3, $\underline{|4}$ for 1x2x3x4, $\underline{|5}$ for 1x2x3x4x5; and so on.

Here $\frac{1}{7} = 0{\cdot}1428571$

$$\frac{1}{7^3} = 0{\cdot}0029154$$

$$\frac{1}{7^4} = 0{\cdot}0004164$$

$$\frac{1}{7^5} = 0{\cdot}0000594$$

$$\frac{1}{7^6} = 0{\cdot}0000084$$

$$\frac{1}{7^7} = 0{\cdot}0000012.$$

Since 5 decimal places only are required, it is not necessary to add any more terms of the series. Therefore we have

$$\begin{array}{rr}
\frac{1}{7} = & 0{\cdot}1428571 \\
\frac{1}{3}\times\frac{1}{7^3} = & 0{\cdot}0009718 \\
\frac{1}{5}\times\frac{1}{7^5} = & 0{\cdot}0000118 \\
\frac{1}{7}\times\frac{1}{7^7} = & 0{\cdot}0000001 \\ \hline
\text{Sum} = & 0{\cdot}1438408 \\
\text{Mult. by} & 2 \\ \hline
 & 0{\cdot}2876816
\end{array}$$

Therefore the required decimal is 0·28768.

The proofs of the rules for the management of vulgar and decimal fractions here given are necessarily confined to particular instances, though the same reasoning may be applied in every case; and by using general signs the proofs may be made general.

But this requires a knowledge of Algebra.

The Student is recommended to accustom himself to *reason out* the place of the *decimal point*. He will find the subject admirably treated in *De Morgan's Arithmetic*, Section VI.

EXAMPLES.

1. Reduce to mixed numbers $\frac{173}{18}$, $\frac{41}{40}$, and $\frac{2001}{200}$.

2. Reduce to improper fractions $9\frac{11}{18}$, $1\frac{1}{40}$, and $10\frac{1}{200}$.

3. What is $\frac{3}{4}$ of $\frac{5}{7}$ of $2\frac{1}{4}$? Ans. $1\frac{23}{112}$.

4. Reduce to lowest terms $\frac{5184}{6912}$, $\frac{7631}{26415}$, $\frac{236432}{2347432}$

(1) Ans. $\frac{3}{4}$. (2) Ans. $\frac{13}{45}$. (3) Ans. $\frac{14}{139}$.

5. Reduce to a common denominator $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$.

Ans. $\frac{2520}{5040}$, $\frac{1680}{5040}$, $\frac{1260}{5040}$, $\frac{1008}{5040}$, $\frac{840}{5040}$, $\frac{720}{5040}$.

6. Reduce to a common denominator 21, $\frac{27}{16}$, $\frac{5}{8}$ and $1\frac{1}{2}$.

Ans. $\frac{336}{16}$, $\frac{27}{16}$, $\frac{10}{16}$, $\frac{24}{16}$.

7. Find the Least Common Mult. of 1, 2, 3, 4, 5, 6, 7, 8, and 9. Ans. 2520.

8. Find the Least Com. Mult. of 21, 22, 23, and 24. Ans. 42504.

9. Find the Least Com. Mult. of 24, 7, 4, 21, and 14. Ans. 168.

10. Reduce to a common denominator $\frac{1}{7}$, $\frac{3}{8}$, $\frac{4}{9}$, $\frac{5}{24}$, $\frac{7}{72}$.

Ans. $\frac{72}{504}$, $\frac{189}{504}$, $\frac{224}{504}$, $\frac{105}{504}$, $\frac{49}{504}$.

11. What is the value of $\frac{3}{14}$ of half a guinea? Ans. 2*s*. 3*d*.

12. What is $\frac{2}{5}$ of a day $-\frac{3}{7}$ of an hour? Ans. 9*h*. $10\frac{2}{7}$*m*.

13. What fraction of half a crown is $\frac{3}{5}$ of 6*s*. 8*d*.? Ans. $1\frac{3}{5}$.

14. Add together $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$. Ans. $3\frac{11}{20}$.

15. Add together $387\frac{1}{2}$, $285\frac{1}{4}$, $394\frac{1}{3}$, and $\frac{2}{5}$ of 3704. Ans.

16. From $201\frac{2}{3}$ take $97\frac{8}{9}$. Ans. $103\frac{7}{9}$.

17. Divide $1\frac{1}{2}$ by $1\frac{1}{4}$; and $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$ by $\frac{1}{3}+\frac{1}{5}+\frac{1}{9}+\frac{1}{15}$.

(1) Ans. $1\frac{1}{5}$. (2) Ans. $1\frac{155}{256}$.

18. Convert 0·08, 0·00125, and 0·0078125 into their equivalent vulgar fractions. Ans. $\frac{2}{25}$, $\frac{1}{800}$, $\frac{1}{128}$.

19. Divide 3·1 by 0·0025; and 365 by 0·18349 to 6 places of decimals.

(1) Ans. 1240. (2) Ans. 1989·209221.

20. Reduce 2*s.* $11\frac{3}{4}$*d.* to the decimal of £1. Ans. £0·14895833 &c.

21. Find the value of 0·07 of £2 10*s.* and express the result as the decimal of £1. Ans. 3*s.* 6*d.* or £0·175.

22. Prove that 0·304565321 is more nearly represented by 0·30457 than by 0·30456.

THE

ELEMENTS OF ALGEBRA.

DEFINITIONS AND EXPLANATION OF SIGNS.

58. THE method of representing the relation of abstract quantities by letters and characters, which are made the signs of such quantities and their relations, is called ALGEBRA.

ALGEBRA may also be called the science of *generalization* as regards number and magnitude. Thus, for example, whilst *Arithmetic* teaches that the sum of the two numbers 6 and 4 multiplied by their difference is equal to the difference of their squares, *Algebra* teaches that the same is true for *any* two numbers whatever, whole or fractional.

59. Known or determined quantities are usually represented by the first letters of the alphabet, a, b, c, d, &c., and unknown or undetermined quantities, by the last, z, y, x, w, &c.

It must be observed, that this is simply a matter of agreement amongst Algebraical writers, for the sake of convenience, and not essential to the subject. Also, sometimes the letters of the *Greek* alphabet are used; sometimes A, B, C, D, &c. X, Y, Z; and sometimes others, according to circumstances or the will of the writer.

The following *signs* or *symbols* are made use of to express the relations which the quantities bear to each other:—

60. $+$ (which is read *Plus*) signifies that the quantity to which it is prefixed must be added. Thus $a+b$ signifies that the quantity represented by b is to be added to the quantity represented by a; if a represent 5, and b represent 7, then $a+b$ represents 12.

Also $a+b+c$ signifies that the *sum* of the quantities represented by a, b, and c, is to be taken.

If no sign be placed before a quantity, the sign $+$ is understood: thus a signifies $+a$. Such quantities are called *positive* quantities.

61. $-$ (which is read *Minus*) signifies that the quantity to which it is prefixed must be subtracted. Thus $a-b$ signifies

that b must be taken from a; if a be 7, and b be 5, $a-b$ expresses 7 diminished by 5, or 2.

Quantities to which the sign $-$ is prefixed are called *negative* quantities.

$\pm$, or $\mp$, (the former of which is read *plus or minus*, the latter *minus or plus*) signifies that the quantity to which it is prefixed may be either added or subtracted. Thus 6 ± 4 is either 10 or 2.

62. $\times$ (which is read *Into*) signifies that the quantities between which it stands are to be multiplied together. Thus $a\times b$ signifies that the quantity represented by a is to be multiplied by the quantity represented by b*.

This sign is frequently omitted; thus abc signifies $a\times b\times c$. Or a full point is used instead of it; thus $1\times2\times3$, and $1.2.3$, signify the same thing.

But the sign must never be *omitted*, for obvious reasons, when two or more *numerals* are to be multiplied together.

Any quantity which, as a multiplier, serves to *make up* a product, is called a *factor* of that product. Thus, of the product $3abc$, each of the quantities, 3, a, b, c is a *factor;* as also each of the quantities $3a$, $3b$, $3c$, $3ab$, $3ac$, $3bc$, ab, ac, bc, abc; the former being called *simple factors*, and the latter *compound factors.*

63. If in multiplication the same quantity be repeated (*as a factor*) any number of times, the product is usually expressed by placing, above the quantity, the number which represents how *often* it is repeated; thus a, $a\times a$, $a\times a\times a$, $a\times a\times a\times a$, &c. have respectively the same signification as a^1, a^2, a^3, a^4, &c. These quantities are called *powers;* thus a^1, or a, is called the first power of a; a^2 the second power, or square, of a; a^3 the third power, or cube, of a, &c. The numbers 1, 2, 3, &c. (thus affixed to a) are called the *indices* of a, or *exponents of the powers* of a.

Likewise a^1, a^2, a^3, &c. are said to be of one, two, three, &c. *dimensions* respectively; and, in general, any product is said to be of n *dimensions*, if the sum of the *indices* of its several *literal factors* is equal to n. Thus ab, that is a^1b^1, is of two dimensions; $3a^2b^3$ is of five dimensions; and so on.

64. $\div$ (which is read *Divided by*) signifies that the former of the quantities between which it is placed is to be divided by the latter. Thus $a\div b$ signifies that the quantity a is to be divided by b.

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* By quantities we understand such magnitudes as can be represented by numbers; we may therefore without impropriety speak of the multiplication, division, &c. of quantities by each other.

The division of one quantity by another is frequently represented by placing the dividend over the divisor with a line between them, in which case the expression is called a *Fraction*. Thus $\frac{a}{b}$ signifies a divided by b[*]; and a is the numerator, and b the denominator, of the fraction; also $\frac{a+b+c}{e+f+g}$ signifies that a, b, and c added together, are to be divided by e, f, and g added together.

65. A *power* in the denominator of a fraction is also expressed by placing it in the numerator, and prefixing the negative sign to its index; thus a^{-1}, a^{-2}, a^{-3}, a^{-n}, signify $\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^n}$ respectively; these are called the negative powers of a.

66. The sign $\sim$ between two quantities signifies that their *difference* is to be taken. Thus $a \sim x$, is $a-x$, or $x-a$, according as a or x is the greater; and $a \pm x$ signifies that the sum or difference of a and x is to be taken.

67. When several quantities are to be taken collectively, they are enclosed by *brackets*, as ( ), { }, [ ]. Thus $(a-b+c)\times(d-e)$ signifies that the quantity represented by $a-b+c$ is to be multiplied by the quantity represented by $d-e$.

Let a stand for 6; b, 5; c, 4; d, 3; and e, 1; then $a-b+c$ is $6-5+4$, or 5; and $d-e$ is $3-1$, or 2;

therefore $(a-b+c)\times(d-e)$ is 5×2, or 10.

Also $(ab-cd)\times(ab-cd)$, or $(ab-cd)^2$, signifies that the quantity represented by $ab-cd$ is to be multiplied by itself.

Sometimes a line, called a *vinculum*, is drawn over quantities, when taken collectively. Thus $\overline{a-b+c}\times\overline{d-e}$ means the same as

68. = (which is read *Equals*, or *is Equal to*) signifies that the quantities between which it is placed are equal to each other; thus

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[*] Since $\frac{a}{b}$ has already received a distinct signification by *Definition* in Art. 2, it seems scarcely allowable to *define* it again, as the Author has done here, without shewing that the two Definitions are coincident. It is true that $\frac{a}{b}$ *is equal to* $a \div b$, but it requires to be *proved*. It was shewn to be true in a particular case in Art. 11. The *general* proof will be given hereafter (See Art. 96). Ed.

$ax - by = cd + ad$ signifies that the quantity $ax - by$ *is equal to* the quantity $cd + ad$.

69. The sign $>$ between two quantities signifies that the former *is greater than* the latter, and the sign $<$ that the former *is less than* the latter.

The sign $\therefore$ signifies *therefore,* and $\because$ *since* or *because.*

70. The *square root* of any proposed quantity is that quantity whose square, or second power, gives the proposed quantity. The *cube root* is that quantity whose cube gives the proposed quantity; and so on.

The n^{th} root is that quantity whose n^{th} power gives the proposed quantity.

The signs $\sqrt{\ }$ or $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, &c. $\sqrt[n]{\ }$, are used to express the square, cube, biquadrate, &c. n^{th}, roots respectively of the quantities before which they are placed.

$$\sqrt[2]{a^2} = a,\quad \sqrt[3]{a^3} = a,\quad \sqrt[4]{a^4} = a,\quad \text{\&c.}\quad \sqrt[n]{a^n} = a.$$

These roots are also represented by the fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c. placed a little above the quantities, to the right. Thus $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{1}{n}}$, represent the square, cube, fourth, and n^{th}, root of a, respectively; $a^{\frac{5}{2}}$, $a^{\frac{7}{3}}$, $a^{\frac{3}{5}}$, represent the square root of the fifth power, the cube root of the seventh power, the fifth root of the cube, of a, respectively.

71. If any of these roots cannot be exactly determined, the quantities are said to be *irrational,* or are called *surds.*

The quantities are called *rational,* when the roots expressed can be exactly determined.

72. Certain points are made use of to denote *proportion;* thus $a : b :: c : d$ signifies that a bears the same proportion to b that c bears to d.

Sometimes this is written thus, $a : b = c : d$.

73. The *number* prefixed to any quantity, (*as a factor*), and which shews how often the quantity is to be taken, is called its *coefficient.* Thus, in $2a$, which signifies *twice* a, the coefficient of a is 2; and in the quantities $7ax$, $6by$, $3dz$, the numerals 7, 6 and 3 are called the coefficients of ax, by, and dz, respectively.

When no number is prefixed, the quantity is to be taken once, or the coefficient 1 is understood.

A *fraction* is not excluded from being a *coefficient.* Thus in $\frac{1}{2}x$ the *coefficient* of x is $\frac{1}{2}$.

These numbers are sometimes represented by letters, which are also called *coefficients.*

Thus in the quantities px^3, qx^2, rx, we call p, q, and r the *coefficients* of x^3, x^2, and x, respectively; since they may be read p times x^3, q times x^2, and r times x, respectively.

In fact *coefficient* simply means *co-factor;* so that in the product ab, for example, a is the coefficient of b, and b is the coefficient of a.

74. *Similar* or *like* algebraical quantities are such as differ only in their coefficients; $4a$, $6ab$, $9a^2$, $3a^2bc$, &c. are respectively *similar* to $15a$, $3ab$, $12a^2$, $15a^2bc$, &c.

Unlike quantities are different combinations of letters; thus ab, a^2b, abc, &c. are unlike.

But a distinction must be made in those cases where *letters* are taken to represent *coefficients;* for ax^3 and px^3 are *like* quantities when a and p are *coefficients* of x^3.

75. A quantity is said to be a *multiple* of another, when it contains it a certain number of times exactly; thus $16a$ is a multiple of $4a$, as it contains it exactly four times.

76. A quantity is called a *measure* of another, when the former is contained in the latter a certain number of times exactly; thus $4a$ is a measure of $16a$.

77. When two numbers have no common measure but unity, they are said to be *prime* to each other.

Thus 3 is *prime* to 7; 13 to 31; and so on.

A *prime number* is one which is prime to every other number; thus 3, 7, 11, 13, &c. are *prime numbers.*

78. A *simple* algebraical quantity is one which consists of a single term; as $4a$, or a^2bc, or $6xy$, &c.

A *compound* algebraical quantity consists of more terms than one—the number of *terms* meaning the number of quantities connected together by + or −; as $a+b$, or $2a-3x+4y$, &c.

A *binomial* is a quantity consisting of two terms, as $a+b$, or $2a-3bx$. A *trinomial* is a quantity consisting of three terms, as $2a+bd-3c$.

A *polynomial* or *multinomial* is a quantity consisting of many terms, as $a+bx+cx^2+dx^3+$ &c.

The following examples will serve to illustrate the method of representing quantities algebraically.

Let $a=8$, $b=7$, $c=6$, $d=5$, and $e=1$; then

Ex. 1. $3a - 2b + 4c - e = 24 - 14 + 24 - 1 = 33.$

Ex. 2. $ab + ce - bd = 56 + 6 - 35 = 27.$

Ex. 3. $\frac{a+b}{c-e} + \frac{3b-2c}{a-d} = \frac{8+7}{6-1} + \frac{21-12}{8-5} = \frac{15}{5} + \frac{9}{3} =$

Ex. 4. $d^2 \times (a - c) - 3ce^2 + d^3 = 25 \times 2 - 18 + 125,$

$= 50 - 18 + 125 = 157.$

Ex. 5. $\sqrt{a^2 - 3d} \times \sqrt[3]{b^3 - c^3 - 2e} = \sqrt{49} \times \sqrt[3]{125} = 7 \times 5 = 35.$

[The student is recommended at this stage to test the accuracy of his knowledge of the preceding Definitions by working out the *Exercises* A, placed at the end of the book.]

AXIOMS.

79. If equal quantities be added to equal quantities, the sums will be equal.

80. If equal quantities be taken from equal quantities, the remainders will be equal.

81. If equal quantities be multiplied by the same, or equal quantities, the products will be equal.

82. If equal quantities be divided by the same, or equal quantities, the quotients will be equal.

83. If the same quantity be added to and subtracted from another, the value of the latter will not be altered.

84. If a quantity be both multiplied and divided by another, its value will not be altered.

ADDITION OF ALGEBRAICAL QUANTITIES.

85. RULE. *The addition of algebraical quantities is performed by connecting those that are* unlike *with their proper signs, and collecting those that are* similar *into one sum.*

Add together the following *unlike* quantities;

Ex. 1.
$$\begin{array}{l} ax \\ -by \\ +e^2 \\ -ed \end{array}$$
Sum $= ax - by + e^2 - ed$

Ex. 2.
$$\begin{array}{l} a + 2b - c \\ \quad d - 5e + f \\ \hline \end{array}$$
Sum $= a + 2b - c + d - 5e + f$

It is immaterial in what order the quantities are set down, if we take care to prefix to each its proper sign.

Generally speaking, however, it is convenient to arrange algebraical quantities in the order in which the letters occur in the alphabet.

When any terms are *similar*, they may be incorporated, and the general expression for the sum shortened.

1^{st}. When *similar* quantities have the *same* sign, their sum is found by taking the sum of the coefficients with that sign, and annexing the common letters.

Ex. 3.
$$\begin{array}{rr} & 5a-3b \\ & 4a-7b \\ \hline \text{Sum}= & 9a-10b \\ \hline \end{array}$$

Ex. 4.
$$\begin{array}{rr} & 4a^2c-10bde \\ & 6a^2c-\ \ 9bde \\ & 11a^2c-\ \ 3bde \\ \hline \text{Sum}= & 21a^2c-22bde \\ \hline \end{array}$$

The reason is evident; $5a$ to be added (Ex. 3), together with $4a$ to be added, makes $9a$ to be added; and $3b$ to be subtracted, together with $7b$ to be subtracted, is $10b$ to be subtracted.

2^d. If *similar* quantities have *different* signs, their sum is found by taking the difference of the coefficients with the sign of the greater, and annexing the common letters as before.

Ex. 5.
$$\begin{array}{rr} & 7a+3b \\ & -5a-9b \\ \hline \text{Sum}= & 2a-6b \\ \hline \end{array}$$

Ex. 6.
$$\begin{array}{rr} & \frac{4}{5}x-\frac{1}{2}y \\ & -\frac{1}{5}x+\frac{1}{4}y \\ \hline \text{Sum}= & \frac{3}{5}x-\frac{1}{4}y \\ \hline \end{array}$$

In the first part of the operation (Ex. 5) we have 7 times a to add, and 5 times a to take away; therefore upon the whole we have $2a$ to add. In the latter part, we have 3 times b to add, and 9 times b to take away; therefore we have upon the whole 6 times b to take away; and thus the sum of all the quantities is $2a-6b$.

Ex. 7.
$$\begin{array}{rr} & a+b \\ & a-b \\ \hline \text{Sum}= & 2a \\ \hline \end{array}$$

Ex. 8.
$$\begin{array}{rr} & 1-x+x^2 \\ & 1+2x-2x^2 \\ \hline \text{Sum}= & 2+x-x^2 \\ \hline \end{array}$$

It must be borne in mind that when any quantity, as a, or x, or x^2, has no coefficient expressed, the coefficient 1 is understood.

3^d. If *several* similar quantities are to be added together, some with positive and some with negative signs, take the difference between the sum of the positive and the sum of the negative coefficients, prefix the sign of the greater sum, and annex the common letters.

$$\begin{array}{rl}
\text{Ex. 9.} & 3a^2 + 4bc - e^2 + 10 \\
 & -5a^2 + 6bc + 2e^2 - 15 \\
 & -4a^2 - 9bc - 10e^2 + 21 \\
\hline
\text{Sum} = & -6a^2 + bc - 9e^2 + 16 \\
\hline
\end{array}$$

The method of reasoning in this case is the same as in Ex. 5.

$$\begin{array}{rl}
\text{Ex. 10.} & 4ac - 15bd + ex \\
 & 11ac + 7b^2 - 19cx \\
 & -41a^2 + 6bd - 7dc \\
\hline
\text{Sum} = & 15ac - 41a^2 - 9bd + 7b^2 - 18cx - 7de \\
\hline
\end{array}$$

$$\begin{array}{rl}
\text{Ex. 11.} & px^3 - qx^2 - rx \\
 & ax^3 - bx^2 - x \\
\hline
\text{Sum} = & (p+a)x^3 - (q+b)x^2 - (r+1)x \\
\hline
\end{array}$$

In this example, letters are taken to represent coefficients, and the coefficients of like powers of x are enclosed within *brackets;* for it is evident, that p times x^3 together with a times x^3 is the same as $(p+a)$ times x^3; also q times x^2 to be subtracted together with b times x^2 to be subtracted is the same as $(q+b)$ times x^2 to be subtracted; and r times x to be subtracted together with x, or $1x$, to be subtracted, is $(r+1)$ times x to be subtracted.

The Rules above given for the addition of algebraical quantities differ in no respect from those employed in *Arithmetic.* For in adding together *like* quantities, as 3 hundreds, and 4 hundreds, we take the sum of the coefficients 3 and 4, so as to make 7 hundreds. But if we have to add together 3 hundreds, 5 tens, and 6 units, these, being *unlike* quantities, cannot be added in the same way, but only put together in one line, 3 hundreds + 5 tens + 6 units, which for shortness is written 356.

[*Exercises* B.]

SUBTRACTION.

86. **RULE.** ***Subtraction, or the taking away of one quantity from another, is performed by changing the sign of the quantity to be subtracted, and then adding it to the other by the rules laid down in Art. 85.***

Ex. 1. From $2bx$ take cy, and the difference is properly represented by $2bx - cy$; because the $-$ prefixed to cy shews that it is to be subtracted from the other; and $2bx - cy$ is the sum of $2bx$ and $-cy$, (Art. 85).

Ex. 2. Again, from $2bx$ take $-cy$, and the difference is $2bx + cy$; because $2bx = 2bx + cy - cy$, (Art. 83); take away $-cy$ from these equal quantities, and the differences will be equal (Art. 80); that is, the difference between $2bx$ and $-cy$ is $2bx + cy$, the quantity which arises from adding $+cy$ to $2bx$.

Ex. 3. From $a + b$
take $a - b$
Difference $= +2b$

Ex. 4. From $6a - 12b$
take $-5a - 10b$
Diff. $= 11a - 2b$

Ex. 5. From $5a^2 + 4ab - 6xy$
take $11a^2 + 6ab - 4xy$
Diff. $= -6a^2 - 2ab - 2xy$

Ex. 6. From $7a - 2b + 4c - 2$
take $6a - 6b + 4c - 1$
Diff. $= a + 4b - 1$

Ex. 7. From $4a - 3b + 6c - 11$
take $10x + a - 15 - 2y$
Diff. $= -10x + 3a - 3b + 4 + 6c + 2y$

Ex. 8. From $x + \frac{1}{2}y + 1$
take $\frac{1}{2}x + y + \frac{1}{2}$
Diff. $= \frac{1}{2}x - \frac{1}{2}y + \frac{1}{2}$

Ex. 9. From $ax^3 - bx^2 + x$
take $px^3 - qx^2 + rx$
Diff. $= (a - p)x^3 - (b - q)x^2 + (1 - r)x$

In this example the coefficients of like powers of x are *bracketed*, for reasons similar to those given in Ex. 11, Art. 85.

[*Exercises* C.]

ADDITION AND SUBTRACTION BY BRACKETS.

In actual practice it seldom happens that either Addition or Subtraction of Algebraical quantities is presented to us as in the Examples, Art. 85, and 86. All the quantities concerned are more commonly *in one line*, and are so retained through the whole operation, for the sake of convenience.

This arrangement renders necessary the frequent use of *Brackets*.

Thus Ex. 3. Art. 85 would stand $(5a - 3b) + (4a - 7b) = 9a - 10b$.
Ex. 3. Art. 86 .... $(a + b) - (a - b) = 2b$.

In the management of Brackets much care is needed, and the following rules are to be observed:—

87. RULE I. *If any number of quantities, enclosed within Brackets, be preceded by the sign* $+$, *the brackets may be struck out, as of no value or signification.*

RULE II. *If any number of quantities, enclosed within Brackets, be preceded by the sign* $-$, *the brackets may be struck out, if the signs of all the quantities within the brackets be changed, namely* $+$ *into* $-$, *and* $-$ *into* $+$.

Rule I. is obviously true; for in this case all that is meant is, that a number of quantities are to be added; and it can clearly make no difference whether they be added *collectively* or *separately*. Thus $a+(b+c)$ is equivalent to $a+b+c$; for the former signifies that the sum of b and c is to be added to a, which is evidently the sum of a, b, and c. Also $a+(b-c)$ is equivalent to $a+b-c$; for the former signifies that a quantity is to be added to a less than b by the quantity c; and the latter, that when b has been added to a, c must be subtracted, which is evidently the same thing.

Rule II. is proved thus:—

Let a, b, c represent any Algebraical quantities, simple or compound, of which $b+c$ is to be subtracted from a; this will be expressed by $a-(b+c)$.

Now if from a the portion b be taken, the result is $a-b$; but there is not enough subtracted from a by the quantity c, since $b+c$ was to be subtracted. Therefore c must also be subtracted, which leaves the result $a-b-c$; that is,

$$a-(b+c)=a-b-c.$$

Again, if $b-c$ is to be taken from a, this will be expressed by $a-(b-c)$.

Now if from a the quantity b be taken, the result is $a-b$, but there has been too much taken away by the quantity c, since $b-c$ only was to be subtracted; therefore c must be *added*, and the result becomes $a-b+c$; that is,

$$a-(b-c)=a-b+c.$$

The preceding rules apply also to quantities held together by a *vinculum*, since a *vinculum* serves the same purpose as *brackets*. Art. 67.

N.B. It is immaterial whether a vinculum or brackets be used in any case, that being dependent solely upon the will of the writer: but in some cases it is necessary, for distinction's sake, to use *both* at the same time, or else two kinds of brackets. Thus, to express a times the difference between b and $c-d$, we must write either $a.(b-\overline{c-d})$, or $a.\{b-(c-d)\}$. Consequently it requires to be especially noted, that in all cases when the Student meets with (or { or [, he must look, whatever may intervene, for the counterpart) or } or] respectively; and all that is included within the *complete bracket* must be treated, irrespective of other brackets or vincula, as the sign which precedes it directs. So that in striking out brackets by Rules I and II, each *pair* of symbols, as ( ), { }, [ ], must be struck out separately, and not all confusedly and at once.

A few examples will make this clearer.

Ex. 1. Perform the addition expressed by $(a+b)+(a-b)$.

$$\begin{aligned}(a+b)+(a-b)&=a+b+a-b, \text{ by Rule I,}\\ &=2a.\end{aligned}$$

Ex. 2. Perform the subtraction expressed by $(a+b)-(a-b)$.

$$\begin{aligned}(a+b)-(a-b)&=a+b-(a-b), \text{ by Rule I,}\\ &=a+b-a+b, \text{ by Rule II,}\\ &=2b.\end{aligned}$$

Ex. 3. Simplify $a-(x-a)-\{x-(a-x)\}$.

$$\begin{aligned}a-(x-a)-\{x-(a-x)\}&=a-x+a-x+(a-x),\\ &=a-x+a-x+a-x,\\ &=3a-3x.\end{aligned}$$

Ex. 4. Simplify $1-\{1-(1-\overline{1-x})\}$.

$$\begin{aligned}1-\{1-(1-\overline{1-x})\}&=1-1+(1-\overline{1-x}),\\ &=1-1+1-\overline{1-x},\\ &=1-1+1-1+x,\\ &=x.\end{aligned}$$

Ex. 5. Simplify $-[-\{-(-a)\}]$.

$$\begin{aligned}-[-\{-(-a)\}]&=+\{-(-a)\}, \text{ by Rule II,}\\ &=-(-a), \text{ by Rule I,}\\ &=a, \text{ by Rule II.}\end{aligned}$$

The converse of each of the Rules I and II evidently holds, viz. that any number of *terms*, following the sign +, may be inclosed within brackets; and also that any number of *terms*, following the sign −, may be inclosed within brackets, provided the sign of every term within the brackets be changed, + into −, and − into +.

[*Exercises* C*.]

MULTIPLICATION.

88. The multiplication of *simple* algebraical quantities must be represented according to the notation pointed out in Art. 62.

Thus $a\times b$, or ab, represents the product of a multiplied by b; abc the product of the three quantities a, b, and c; and so on.

It is also indifferent in what order they are placed, $a\times b$ and $b\times a$ being equal.

For $1\times a=a\times 1$, or 1 taken a times is the same with a taken once; also b taken a times, or $b\times a$, is b times as great as 1 taken a times; and a taken b times, or $a\times b$, is b times as great as a taken once; therefore (Art. 81) $b\times a=a\times b$. Also $abc=cab=bca=acb$; for, as in the former case, $1\times a\times b=a\times b\times 1$; and $c\times a\times b$ is c times as great as $1 \times a \times b$; also $a \times b \times c$ is c times as great as $a \times b \times 1$; therefore $a \times b \times c = c \times a \times b$ (Art. 81); and a similar proof may be applied to the other cases.

89. To determine the *sign* of the product, observe the following rule:—

If the multiplier and multiplicand have the same *sign, the product is positive; if they have* different *signs, the product is negative.*

1st. $+a \times +b = +ab$; because in this case a is to be taken positively b times; therefore the product ab must be positive.

2d. $-a \times +b = -ab$; because $-a$ is to be taken b times; that is, we must take $-ab$.

3d. $+a \times -b = -ab$; for a quantity is said to be multiplied by a negative number $-b$, if it be subtracted b times; and a subtracted b times is $-ab$. This also appears from Art. 92. Ex. 2.

4th. $-a \times -b = +ab$. Here $-a$ is to be subtracted b times, that is, $-ab$ is to be subtracted; but subtracting $-ab$ is the same as adding $+ab$ (Art. 86); therefore we have to add $+ab$.

The 2d and 4th cases may be thus proved; $a - a = 0$, multiply these equals by b, then ab together with $-a \times b$ must be equal to $b \times 0$, or nothing*; therefore $-a$ multiplied by b must give $-ab$, a quantity which when added to ab makes the sum nothing.

Again, $a - a = 0$; multiply these equals by $-b$, then $-ab$ together with $-a \times -b$ must be equal to 0; therefore $-a \times -b = +ab$.

Cor. Since $+a \times +b = +ab = -a \times -b$, the product of any two quantities is not affected by changing the signs of *both* multiplicand and multiplier.

90. If the quantities to be multiplied have *coefficients*, these must be multiplied together as in common arithmetic; the sign and the literal product being determined by the preceding rules.

Thus $3a \times 5b = 15ab$; because $3 \times a \times 5 \times b = 3 \times 5 \times a \times b = 15ab$ (Art. 88).

Again, $4x \times -11y = -44xy$; $-9b \times -5c = +45bc$;

and $-6d \times 4m = -24md$.

91. *The powers of the same quantity are multiplied together by adding the indices;* thus $a^2 \times a^3 = a^5$, for $aa \times aaa = aaaaa$. In the same manner $a^6 \times a^{10} = a^{16}$; and $-3a^2x^3 \times 5axy^2 = -15a^3x^4y^2$.

---

* It is a common mistake of beginners to say that an algebraical expression which appears under the form $a \times 0$ is equal to a, by supposing it to signify a not multiplied at all; whereas, since $a \times b$ signifies a taken b times, in the same manner $a \times 0$ signifies a taken 0 times, and is therefore equal to 0.—Ed.

To prove generally that $a^m \times a^n = a^{m+n}$, m *and* n *being any positive integers.*

By Def. Art. 63. $a^m = a \times a \times a \times a \times$ &c. continued to m factors;

also $a^n = a \times a \times a \times a \times$ &c. .... n ....;

$\therefore\ a^m \times a^n = a \times a \times a \ldots$ to m factors $\times a \times a \times a \ldots$ to n factors,

$= a \times a \times a \ldots$ to $m+n$ factors,

$= a^{m+n}$, by Def. Art. 63.

It will be *proved* hereafter that the same rule holds when m and n are either fractional or negative. (Arts. 132, 162.)

It follows, that $a^m.a^n.a^p = a^{m+n}a^p = a^{m+n+p}$. And, generally, $a^m.a^n.a^p.$&c. $= a^{m+n+p+\&c.}$

Also $a^m b^n \times a^r b^q = a^m.a^r.b^n.b^q = a^{m+r}b^{n+q}$.

Exs. $a^{m+1}.a^{m-1} = a^{2m}$; $ax^m.bx^n = abx^{m+n}$; $px^m.qx^{m-n}.rx^n = pqrx^{2m}$.

92. If the multiplier or multiplicand consist of several terms, each term of the latter must be multiplied by every term of the former, and the sum of all the products taken for the whole product of the two quantities.

$$\begin{array}{lll} \text{Ex. 1.} & \text{Mult.} & a+b \\ & \text{by} & c+d \\ \cline{3-3} & \text{Prod}^{t}. & = ac+bc+ad+bd \\ \cline{3-3} \end{array}$$

Here $a+b$ is to be taken $c+d$ times, that is, c times and d times, or $(a+b)c+(a+b)d$.

$$\begin{array}{lll} \text{Ex. 2.} & \text{Mult.} & a+b \\ & \text{by} & c-d \\ & \text{Prod}^{t}. & = ac+bc-ad-bd \\ \cline{3-3} \end{array}$$

Here $a+b$ is to be taken $c-d$ times, that is, c times wanting d times; or c times positively and d times negatively; that is, $(a+b)c-(a+b)d$, or $ac+bc-(ad+bd)$, or $ac+bc-ad-bd$, Art. 87.

$$\begin{array}{lll} \text{Ex. 3.} & \text{Mult.} & a+b \\ & \text{by} & a+b \\ \cline{3-3} \text{Prod}^{t}. & \text{by } a = & a^2+ab \\ \ldots\ldots & \text{by } +b = & \quad +ab+b^2 \\ \cline{3-3} \text{Whole prod}^{t}. & = & a^2+2ab+b^2 \end{array}$$

$$\begin{array}{lll} \text{Ex. 4.} & \text{Mult.} & a+b \\ & \text{by} & a-b \\ \cline{3-3} & & a^2+ab \\ & & \quad -ab-b^2 \\ \cline{3-3} & \text{Prod}^{t}. & = a^2 \quad -b^2 \end{array}$$

Ex. 5.

$$\begin{array}{ll}
\text{Mult.} & 3a^2-5bd \\
\text{by} & \underline{-5a^2+4bd} \\
 & -15a^4+25a^2bd \\
 & \underline{\qquad\qquad +12a^2bd-20b^2d^2} \\
\text{Prod}^{\text{t}}. = & \underline{-15a^4+37a^2bd-20b^2d^2}
\end{array}$$

Ex. 6.

$$\begin{array}{ll}
\text{Mult.} & x+a \\
\text{by} & \underline{x+b} \\
 & x^2+ax \\
 & \underline{\qquad +bx+ab} \\
\text{Prod}^{\text{t}}. = & \underline{x^2+(a+b)x+ab}
\end{array}$$

Ex. 7.

$$\begin{array}{ll}
\text{Mult.} & a^2+2ab+b^2 \\
\text{by} & \underline{a^2-2ab+b^2} \\
 & a^4+2a^3b+a^2b^2 \\
 & \quad -2a^3b-4a^2b^2-2ab^3 \\
 & \underline{\qquad\qquad +\ a^2b^2+2ab^3+b^4} \\
\text{Prod}^{\text{t}}. = & \underline{\quad a^4\ -2a^2b^2+\ b^4}
\end{array}$$

Ex. 8.

$$\begin{array}{ll}
\text{Mult.} & x^m+x \\
\text{by} & \underline{\qquad x^n} \\
\text{Prod}^{\text{t}}. = & \underline{x^{m+n}+x^{n+1}}
\end{array}$$

Ex. 9.

$$\begin{array}{ll}
\text{Mult.} & 1-x+x^2-x^3 \\
\text{by} & \underline{1+x} \\
 & 1-x+x^2-x^3 \\
 & \underline{\quad +x-x^2+x^3-x^4} \\
\text{Prod}^{\text{t}}. = & \underline{1\ -x^4}
\end{array}$$

Ex. 10.

$$\begin{array}{ll}
\text{Mult.} & \frac{1}{2}x-\frac{1}{3}y \\
\text{by} & \underline{2x-3y} \\
 & x^2-\frac{2}{3}xy \\
 & \underline{\qquad -\frac{3}{2}xy+y^2} \\
\text{Prod}^{\text{t}}. = & \underline{x^2-\frac{13}{6}xy+y^2}
\end{array}$$

Ex. 11.

$$\begin{array}{ll}
\text{Mult.} & x^2-px+q \\
\text{by} & \underline{x+a} \\
 & x^3-px^2+qx \\
 & \underline{\quad +ax^2-apx+aq} \\
\text{Prod}^{\text{t}}. = & \underline{x^3-(p-a)x^2+(q-ap)x+aq}
\end{array}$$

Here the coefficients of x^2 and x are bracketed, since $-(p-a)x^2$ $-px^2+ax^2$; and $(q-ap)x=qx-apx$.

Ex. 12.

$$\begin{array}{ll}
\text{Mult.} & ma^m+na^n \\
\text{by} & \underline{na^m+ma^n} \\
 & mna^{2m}+n^2a^{m+n} \\
 & \underline{\qquad +m^2a^{m+n}+mna^{2n}} \\
\text{Prod}^{\text{t}}. = & \underline{mna^{2m}+(m^2+n^2)a^{m+n}+mna^{2n}}
\end{array}$$

[*Exercises* D.]

It may be useful to exhibit the Rules for Multiplication as follows :—

Rules.
$$p(a+b)=pa+pb \ldots\ldots\ldots\ldots(1),$$
$$p(a-b)=pa-pb \ldots\ldots\ldots\ldots(2),$$
$$(a+b)(c+d)=ac+bc+ad+bd \ldots\ldots\ldots(3),$$
$$(a+b)(c-d)=ac+bc-ad-bd \ldots\ldots\ldots(4),$$
$$(a-b)(c-d)=ac-bc-ad+bd \ldots\ldots\ldots(5).$$

Assuming (1) and (2), which are too obvious to need a proof, to prove (3), let $a+b=m$, then,

$$\begin{aligned}(a+b)(c+d)&=m(c+d),\\ &=mc+md, \text{ by (1)},\\ &=(a+b)c+(a+b)d,\\ &=ac+bc+ad+bd, \text{ by (1)}.\end{aligned}$$

Similarly for (4).

To prove (5) let $a-b=m$, then

$$\begin{aligned}(a-b)(c-d)&=m(c-d),\\ &=mc-md, \text{ by (2)},\\ &=(a-b)c-(a-b)d,\\ &=ac-bc-(ad-bd),\\ &=ac-bc-ad+bd, \text{ by Art. 87}.\end{aligned}$$

N.B. The Rules for the management of *Brackets*, given in Art. 87, apply only to the *Addition* and *Subtraction* of quantities so enclosed. If a collection of quantities within brackets is to be *multiplied*, or *divided*, by any quantity or collection of quantities, the brackets must not be struck out until the multiplication or division is actually performed. Thus $(a+b)\times(c+d)$ signifies that $a+b$ is to be taken $c+d$ times, and is obviously not the same as either $a+b(c+d)$, or $(a+b)c+d$. Again, $(a+b)\div(c+d)$ is not equivalent to either $a+b\div(c+d)$, or $(a+b)\div c+d$; but it may be written $\frac{a+b}{c+d}$, the line which separates the numerator and denominator serving as a vinculum to *both*.

The learner would do well to practise multiplication of quantities by means of *brackets* as early as possible. Thus,

Ex. 1. $$\begin{aligned}(a-b)(c-d)&=(a-b)c-(a-b)d,\\ &=ac-bc-(ad-bd),\\ &=ac-bc-ad+bd.\end{aligned}$$

Ex. 2. $$\begin{aligned}(x+a)(x+b)&=(x+a)x+(x+a)b,\\ &=x^2+ax+bx+ab,\\ &=x^2+(a+b)x+ab.\end{aligned}$$

Ex. 3. $$\begin{aligned}(x+1)(x+2)(x+3)&=(x^2+\overline{2+1}\,.x+2)(x+3). \quad \text{(Ex. 2.)}\\ &=(x^2+3x+2)x+(x^2+3x+2)3,\\ &=x^3+3x^2+2x+3x^2+9x+6,\\ &=x^3+6x^2+11x+6.\end{aligned}$$

It is also useful *to commit to memory* at an early stage the three following results; as will appear from the subjoined Examples :—

$$(A+B).(A+B) \text{ or } (A+B)^2=A^2+B^2+2AB\text{.........(i)}$$
$$(A-B).(A-B) \text{ or } (A-B)^2=A^2+B^2-2AB\text{.........(ii)}$$
$$(A+B).(A-B)=A^2-B^2\text{...............(iii)}$$

whatever quantities A *and* B *may represent, simple or compound.*

Ex. 1. $(ax+by)(ax+by)=(ax)^2+(by)^2+2.ax.by$, by (i),
$=a^2x^2+b^2y^2+2abxy$.

Ex. 2. $(ax-by)(ax-by)=(ax)^2+(by)^2-2.ax.by$, by (ii),
$=a^2x^2+b^2y^2-2abxy$.

Ex. 3. $(ax+by)(ax-by)=(ax)^2-(by)^2$, by (iii),
$=a^2x^2-b^2y^2$.

Ex. 4. $(ax+b+cy+d)^2=(\overline{ax+b}+\overline{cy+d})^2$,
$=(ax+b)^2+(cy+d)^2+2(ax+b)(cy+d)$,
$=a^2x^2+b^2+2abx+c^2y^2+d^2+2cdy$
$+2acxy+2adx+2bcy+2bd$.

Ex. 5. $(a+b+c)(a+b-c)=(\overline{a+b}+c)(\overline{a+b}-c)$,
$=(a+b)^2-c^2$, by (iii).
$=a^2+b^2+2ab-c^2$.

Ex. 6. $(b+c-a)(c+a-b)=(c-\overline{a-b})(c+\overline{a-b})$,
$=c^2-(a-b)^2$, by (iii),
$=c^2-(a^2+b^2-2ab)$,
$=c^2-a^2-b^2+2ab$.

SCHOLIUM.

The method of determining the sign of a product from the consideration of abstract quantities has been found fault with by some algebraical writers, who contend that $-a$, without reference to other quantities, is imaginary, and consequently not the object of reason or demonstration. In answer to this objection we may observe, that whenever we make use of the notation $-a$, and say it signifies a quantity to be subtracted, we make a tacit reference to other quantities.

Thus, in numbers, $-a$ represents a number to be subtracted from those with which it is connected; and when we suppose $-a$ to be taken b times, we must understand that a is to be taken b times from some other numbers. In estimating lines, or distances, $-a$ represents a line, or distance, in a particular direction. The negative sign does not render quantities imaginary or impossible, but points out the relation of real quantities to others with which they are concerned.

DIVISION.

93. *To divide one quantity by another is to determine how often* the latter is contained in the former, or what quantity multiplied by the latter will produce the former.*

Thus to divide ab by a is to determine how often a must be taken to make up ab, that is, what quantity multiplied by a will give ab; which we know is b.

From this consideration are derived all the rules for the division of algebraical quantities.

94. If the divisor and dividend be affected with *like* signs, the sign of the quotient is +; but if their signs be *unlike,* the sign of the quotient is −. Thus,

If $-ab$ be divided by $-a$, the quotient is $+b$; because $-a \times +b$ gives $-ab$; and a similar proof may be given in the other cases.

95. *To divide one* simple *quantity by another.*

Rule. In the division of *simple* quantities, if the divisor be found as a factor in the dividend, the other part of the dividend, with the sign determined by the last rule, is the quotient.

This is obvious, since $Quotient \times Divisor = Dividend$ in all cases.

Thus $-7b \div b = -7$; $-ax \div -a = x$; $14ab \div 7b = 2a.7b \div 7b = 2a$.

Also $abc \div ab = c$; because ab multiplied by c gives abc.

If we first divide by a, and *then* by b, the result will be the same; for $abc \div a = bc$, and $bc \div b = c$, as before.

Cor. *If any* power *of a quantity be divided by any other* power *of the same quantity, the quotient is the same quantity with an index which is found by taking the index of the divisor from the index of the dividend.*

Thus $a^5 \div a^3 = a^2 \times a^3 \div a^3$ (Art. 91) $= a^2$, (Art. 84); and generally, if m and n be positive integers, and $m > n$, $a^m \div a^n = a^{m-n} \times a^n \div a^n = a^{m-n}$.

Similarly $6a^4b^6 \div 3a^2b^3 = 2a^2b^3 \times 3a^2b^3 \div 3a^2b^3 = 2a^2b^3$; and $(a^m b^n) \div (a^p b^q) = a^{m-p}b^{n-q}.a^p b^q \div a^p b^q = a^{m-p}b^{n-q}$.

---

* "*How often*" must be understood here in a very wide sense, because that which expresses it, called the "*Quotient,*" may be either a simple or compound quantity, integral, or fractional; in fact, *the* quantity, whatever it be, which multiplied by the divisor will produce the dividend.—Ed.

96. **Lemma.** *To shew that* $a \div b$ *is equal to the fraction* $\frac{a}{b}$.

According to the definition of a '*fraction*' the unit is divided into b equal parts, and a of them are taken, to make the quantity represented by $\frac{a}{b}$. Now, *each* of these parts is clearly the b^{th} part of the unit, and $\therefore \frac{a}{b}$ is equal to a times the b^{th} part of 1, $=x$, suppose. Multiplying these equals by b, (Art. 81), $b \times a$ times the b^{th} part of 1, or $a \times b$ times the b^{th} part of $1 = bx$; but b times the b^{th} part of 1 is clearly 1, $\therefore a = bx$. Now let $a \div b = y$, then by Definition (Art. 93), $a = by$, $\therefore bx = by$, or $x = y$, (Art. 82); that is $\frac{a}{b} = a \div b$.

97. In dividing one *simple* quantity by another, if only a *part* of the product which forms the divisor be contained in the dividend, the division must be represented by a *fraction* according to the direction in the last Art., and the *factors* which are common to the divisor and dividend expunged.

Thus $15a^3b^2c \div -3a^2bx = \dfrac{15a^3b^2c}{-3a^2bx} = \dfrac{-5abc}{x}$.

For, 1st, divide by $-3a^2b$, and the quotient is $-5abc$; this quantity is still to be divided by x (Art. 95), and as x is not contained in it, the division can only be *represented* in the usual way; that is, $\dfrac{-5abc}{x}$ is the quotient.

98. *To divide a quantity of two or more terms by a* simple *quantity.*

Rule. If the dividend consist of several terms, and the divisor be a *simple* quantity, every term of the dividend must be separately divided by it.

Thus to divide $a^3x^2 - 5abx^3 + 6ax^4$ by ax^2,

$$\text{Quotient} = \frac{a^3x^2}{ax^2} - \frac{5abx^3}{ax^2} + \frac{6ax^4}{ax^2} = a^2 - 5bx + 6x^2.$$

Also $(a+b+c) \div abc = \dfrac{a}{abc} + \dfrac{b}{abc} + \dfrac{c}{abc} = \dfrac{1}{bc} + \dfrac{1}{ac} + \dfrac{1}{ab}$.

99. *To divide one quantity by another when the* divisor *consists of two or more terms.*

Rule. When the divisor consists of several terms, *arrange both the divisor and dividend according to the powers of some one letter* contained in them,* (that is, beginning with the highest power and going regularly down to the lowest, or *vice versâ*,) then find how often the first term of the divisor is contained in the first term of the dividend, and write down this quantity for the first term in the quotient; multiply the whole divisor by it, subtract the product from the dividend and bring down to the remainder as many other terms of the dividend, as the case may require, and repeat the operation till all the terms are brought down.

---

* The operation will be shortest when that letter is chosen whose highest power in the dividend comes nearest to the highest power of the same letter in the divisor; and the same arrangement according to the powers of that letter must be kept up throughout the whole operation.—Ed.

Ex. 1. If $a^2-2ab+b^2$ be divided by $a-b$, the operation will be as follows:

$$\begin{array}{r|l|l}
a-b & a^2-2ab+b^2 & a-b \\
 & a^2-ab & \\ \cline{2-2}
 & \quad\;\; -ab+b^2 & \\
 & \quad\;\; -ab+b^2 & \\ \cline{2-2}
\end{array}$$

The reason of this and the foregoing rule is, that as the whole dividend is made up of all its parts, the divisor is contained in the whole as often* as it is contained in all the parts. In the preceding operation we inquire first, how often a is contained in a^2, which gives a for the first term of the quotient; then multiplying the whole divisor by it, we have a^2-ab to be subtracted from the dividend, and the remainder is $-ab+b^2$, with which we are to proceed as before.

The whole quantity $a^2-2ab+b^2$ is in reality divided into two parts by the process, each of which is divided by $a-b$; therefore the true quotient is obtained.

Ex. 2.
$$\begin{array}{r|l|l}
a+b & ac+ad+bc+bd & c+d \\
 & ac+bc & \\ \cline{2-2}
 & \qquad ad+bd & \\
 & \qquad ad+bd & \\ \cline{2-2}
\end{array}$$

Ex. 3.
$$\begin{array}{r|l|l}
1-x & 1 & 1+x+x^2+x^3+\&c.+\dfrac{\text{Remainder}}{1-x} \\
 & 1-x & \\ \cline{2-2}
 & \;\;+x & \\
 & \;\;+x-x^2 & \\ \cline{2-2}
 & \qquad +x^2 & \\
 & \qquad +x^2-x^3 & \\ \cline{2-2}
 & \qquad\qquad +x^3 & \\
 & \qquad\qquad +x^3-x^4 & \\ \cline{2-2}
 & \qquad\qquad\qquad +x^4\ \&c. & \\
\end{array}$$

---

* See Note p. 45.

Ex. 4.

$$\begin{array}{l}
y-1\,\big)\,y^3-1\,\big(\,y^2+y+1 \\
\qquad\;\; y^3-y^2 \\
\hline
\qquad\qquad +y^2-1 \\
\qquad\qquad +y^2-y \\
\hline
\qquad\qquad\qquad +y-1 \\
\qquad\qquad\qquad +y-1 \\
\hline
\end{array}$$

From this example it appears that y^3-1 is divisible by $y-1$ *without remainder*, the quotient being y^2+y+1. It may be shewn in the same manner that x^3-a^3 is divisible by $x-a$, the quotient being x^2+ax+a^2; and that x^3+a^3 is divisible by $x+a$, the quotient being x^2-ax+a^2. These results are worth remembering. Or thus,

$$y^3-1=(y-1)(y^2+y+1),$$
$$x^3-a^3=(x-a)(x^2+ax+a^2),$$
$$x^3+a^3=(x+a)(x^2-ax+a^2).$$

Ex. 5. To divide $4ab^3+51a^2b^2+10a^4-48a^3b-15b^4$ by $4ab-5a^2+3b^2$.

First arrange the terms of both dividend and divisor according to the powers of a, beginning with the highest.

$$\begin{array}{l}
-5a^2+4ab+3b^2\,\big)\,10a^4-48a^3b+51a^2b^2+4ab^3-15b^4\,\big(\,-2a^2+8ab-5b^2 \\
\qquad\qquad\qquad\quad 10a^4-\;8a^3b-6a^2b^2 \\
\hline
\qquad\qquad\qquad\qquad\quad -40a^3b+57a^2b^2+4ab^3 \\
\qquad\qquad\qquad\qquad\quad -40a^3b+32a^2b^2+24ab^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\quad 25a^2b^2-20ab^3-15b^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 25a^2b^2-20ab^3-15b^4 \\
\hline
\end{array}$$

Ex. 6.

$$\begin{array}{l}
x-y\,\big)\,x^m-y^m\,\big(\,x^{m-1}+x^{m-2}y+x^{m-3}y^2+\ldots+y^{m-1} \\
\qquad\quad x^m-x^{m-1}y \\
\hline
\qquad\qquad +x^{m-1}y-y^m \\
\qquad\qquad +x^{m-1}y-x^{m-2}y^2 \\
\hline
\qquad\qquad\qquad\quad +x^{m-2}y^2-y^m \\
\qquad\qquad\qquad\quad +x^{m-2}y^2-x^{m-3}y^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad +x^{m-3}y^3-y^m\ \text{\&c.}
\end{array}$$

This division will obviously terminate *without remainder* for any proposed integral and positive value of m, when the quotient has reached to m terms, the last term being y^{m-1}. Hence we have,

$$(x^3-y^3)\div(x-y)=x^2+xy+y^2,$$
$$(x^4-y^4)\div(x-y)=x^3+x^2y+xy^2+y^3,$$
$$(x^5-y^5)\div(x-y)=x^4+x^3y+x^2y^2+xy^3+y^4;\ \text{and so on.}$$

Or thus, generally, $x^m-y^m=(x-y)(x^{m-1}+x^{m-2}y+\ldots+y^{m-1})$.

Ex. 7.
$$\begin{array}{l}
x - a\,\big)\,x^3 - px^2 + qx - r\,\big(\,x^2 + (a-p)x + a^2 - pa + q \\
\qquad\quad x^3 - ax^2 \\
\hline
\qquad\quad (a-p)x^2 + qx \\
\qquad\quad (a-p)x^2 - (a^2 - pa)x \\
\hline
\qquad\qquad\qquad + (a^2 - pa + q)x - r \\
\qquad\qquad\qquad\ \ (a^2 - pa + q)x - (a^3 - pa^2 + qa) \\
\hline
\qquad\qquad\qquad \text{Remainder}\quad \underline{a^3 - pa^2 + qa - r}
\end{array}$$

[*Exercises* E.]

TRANSFORMATION OF FRACTIONS TO OTHERS OF EQUAL VALUE.

100. If the signs of all the terms both in the numerator and denominator of a fraction be changed, its value will not be altered.

For $\dfrac{-ab}{-a} = -ab \div -a$ (Art. 96) $= +b = +ab \div +a = \dfrac{+ab}{+a}$;

and $\dfrac{ab}{-a} = -b = \dfrac{-ab}{+a}$. Ex. $\dfrac{a-x}{b-x} = \dfrac{-a+x}{-b+x} = \dfrac{x-a}{x-b}$.

101. If the numerator and denominator of a fraction be both multiplied, or both divided, by the same quantity, its value is not altered.

For in $\dfrac{a}{b}$ the unit is divided into b equal parts, and a of them are taken,

in $\dfrac{a}{bc}$..............................bc...............c times as many as before,

and a of them taken; therefore in the former case each part is c times as great as in the latter. But if c times as many of the smaller as of the greater parts be taken, it is obvious that the result in either case will be the same, that is, that ac parts of the latter are equal to a parts of the former, or $\dfrac{ac}{bc} = \dfrac{a}{b}$.

Hence a fraction is reduced to its lowest terms by dividing both the numerator and denominator by the *greatest* quantity that *measures* them both.

This quantity is called the *Greatest Common Measure*, or *Highest Common Divisor*, of the numerator and denominator.

102. A fraction which has either its numerator or denominator a *simple* algebraical quantity is easily reduced to lowest terms; for the highest common divisor of the numerator and denominator is at once found by inspection, that is, by observing *at sight* what factors are common. Thus to reduce $\dfrac{3a^2bc}{5a^3b^2d}$ to its lowest terms, we *see* that a^2b is the highest

common divisor of the numerator and denominator; therefore the fraction required is $\frac{3c}{5abd}$.

Again, $\frac{2a+5ax}{7a}$ is at once reduced to $\frac{2+5x}{7}$; and $\frac{a^2}{a^2+ab}$ to $\frac{a}{a+b}$.

But in the case of fractions having for both numerator and denominator a *compound* algebraical quantity the following rule is often, though not always, needed.

103. *The Greatest Common Measure of two compound algebraical quantities is found by arranging them according to the powers of some letter*[*] *(as in division), and then dividing the greater by the less, and the preceding divisor always by the last remainder, till the remainder is nothing; the last divisor is the Greatest Common Measure required.*

Let a and b represent the two quantities, and let b be contained p times in a, with a remainder c; again, let c be contained q times in b, with a remainder d; and so on, till nothing remains; let d be the last divisor, and it will be the greatest common measure of a and b.

$$\begin{array}{l} b)\,a\,(p \\ \underline{pb} \\ \quad c)\,b\,(q \\ \quad \underline{qc} \\ \qquad d)\,c\,(r \\ \qquad\ \ \underline{rd} \\ \qquad\ \ \ 0 \end{array}$$

104. The truth of this rule depends upon these two principles;

1st. If one quantity measure another, it will also measure any multiple of that quantity. Let x measure y by the units in n, then it will measure my by the units in mn; for since $y = nx$, $my = mnx$ (Art. 81) $= mn.x$, that is, x is contained mn times in my, or measures my by the units in mn.

2d. If a quantity measure two others, it will measure their sum or difference. Let a be contained m times in x, and n times in y; then $ma = x$, and $na = y$; therefore $x \pm y = ma \pm na = (m \pm n)a$; that is, a is contained $m \pm n$ times in $x \pm y$, or it measures $x \pm y$ by the units in $m \pm n$.

105. Now it appears from the operation (Art. 103), that $a - pb = c$, and $b - qc = d$; every quantity therefore, which measures a and b, measures pb, and $a - pb$, or c; hence also it measures qc, and $b - qc$, or d; that is, every common measure of a and b measures d.

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[*] This letter is called the *letter*, or *symbol*, *of reference;* and of the two quantities that is said to be the *greater*, which contains the *highest power of this letter*. So also, the *Greatest* Common Measure is the *highest* common divisor, *highest*, that is, in its *dimensions* with respect to the *letter of reference*.—ED.

It appears also from the division that $a = pb + c$, $b = qc + d$, $c = rd$; therefore d measures c, and qc, and $qc + d$, or b; hence it measures pb, and $pb + c$, or a. Every common measure then of a and b measures d, and d measures a and b; therefore d is their greatest common measure*.

Ex. 1. To find the Greatest Common Measure of a^2+2a+1 and a^3+2a^2+2a+1; and to reduce $\frac{a^2+2a+1}{a^3+2a^2+2a+1}$ to its lowest terms.

$$\begin{array}{rll}
a^2+2a+1\,) & a^3+2a^2+2a+1 & (\,a \\
 & a^3+2a^2+a & \\
\hline
 & \qquad\quad a+1 & \\
 & a+1\,)\; a^2+2a+1 & (\,a+1 \\
 & \qquad\quad a^2+a & \\
\cline{2-2}
 & \qquad\qquad a+1 & \\
 & \qquad\qquad a+1 & \\
\cline{2-2}
\end{array}$$

$a+1$ is therefore the Greatest Common Measure of the two quantities; and if they be respectively divided by it, the fraction is reduced to $\frac{a+1}{a^2+a+1}$, and is in its lowest terms.

Ex. 2. To find the Greatest Common Measure of $a^4 - x^4$ and $a^3 - a^2x - ax^2 + x^3$; and to reduce $\frac{a^3 - a^2x - ax^2 + x^3}{a^4 - x^4}$ to its lowest terms.

$$\begin{array}{rll}
a^3 - a^2x - ax^2 + x^3\,) & a^4 - x^4 & (\,a + x \\
 & a^4 - a^3x - a^2x^2 + ax^3 & \\
\cline{2-2}
 & \qquad a^3x + a^2x^2 - ax^3 - x^4 & \\
 & \qquad a^3x - a^2x^2 - ax^3 + x^4 & \\
\cline{2-2}
 & \qquad\qquad 2a^2x^2 - 2x^4 = 2x^2(a^2 - x^2), &
\end{array}$$

leaving out the factor $2x^2$, the next divisor is $a^2 - x^2$.

$$\begin{array}{rll}
a^2 - x^2\,) & a^3 - a^2x - ax^2 + x^3 & (\,a - x \\
 & a^3 - ax^2 & \\
\cline{2-2}
 & \quad - a^2x + x^3 & \\
 & \quad - a^2x + x^3 & \\
\cline{2-2}
\end{array}$$

---

[*] This conclusion is more obvious when stated thus:—every common divisor of a and b is a divisor of d, but no quantity can be a divisor of d which is greater than d, therefore every common divisor of a and b is not greater than d; and since d is one of them, therefore d is the greatest.—ED.

Therefore $a^2 - x^2$ is the G.C.M. required; and the fraction is reduced to $\frac{a^2 + x^2}{a - x}$, and is in its lowest terms.

The quantity $2x^2$, found as a factor in every term of one of the divisors, $2a^2x^2 - 2x^4$, but not in every term of the dividend, $a^3 - a^2x - ax^2 + x^3$, must be left out; otherwise the quotient will be fractional, which is contrary to the supposition made in the proof of the rule: and by omitting this part, $2x^2$, no common measure of the divisor and dividend is left out, because, by the supposition, no part of $2x^2$ is found in all the terms of the dividend.

[*Exercises* F.]

106. The proof of the rule for finding the G.C.M. of two algebraical quantities (Art. 103—5) excludes every case in which any one of the quotients p, q, r, is fractional: for mb is not considered a "multiple" of b, unless m be either a *whole number*, or free from terms *in a fractional form*. But the fractional quotients may always be avoided by rejecting certain factors which can be detected by inspection.

(1) Let such factors be common to the proposed quantities; so that $a = ka'$; $b = kb'$, where k contains all such common factors. Then, as *all* factors that are common to a and b, excepting k, are also common to a' and b', and *all* that are common to a' and b' are also common to a and b, the G.C.M. of a and b will $= k \times$G.C.M. of a' and b'. Whence we see that such a common factor as k may be neglected, provided we multiply it into the G.C.M. that will afterwards be obtained.

(2) Let such factors be not common; and let $a' = ma''$, $b' = nb''$, where m and n contain *all* of them that can be detected by inspection; a'' and b'' consequently have no such factors; therefore m and n have no factors in common with b'' and a'', and by hypothesis they have none in common themselves; therefore all factors common to a' and b' are also common to a'' and b'', and conversely: therefore the G.C.M. of a' and b' = G.C.M. of a'' and b''. The factors m and n therefore may be entirely rejected.

From this also it is evident that such factors as the above can be *introduced* into one of the proposed quantities, provided that they do not contain any factor that already appears in the other.

If then we dispose at once of the factors that we can detect by inspection, according to the foregoing remarks, we can proceed to find the G.C.M. as follows:

$$\begin{array}{l}
b\,)\; a\; (p \\
\quad \underline{pb} \\
\quad\quad c = mc' \text{ suppose} \\
c'\,)\; b\; (q \\
\quad \underline{qc'} \\
\quad\quad d = nd' \text{ suppose} \\
d'\,)\; c'\; (r \\
\quad \underline{rd'} \\
\quad\quad 0
\end{array}$$

where m and n consist entirely of such factors as have been considered, or are not either of them a "*measure*" of the succeeding dividend.

Now every common measure of a and b measures $a-pb$ or c, and therefore is a common measure of b and c; also every common measure of b and c measures $c+pb$ or a, and therefore is a common measure of a and b.

Therefore the G.C.M. of a and b is also the G.C.M. of b and c.

But as m contains no factor that appears in b, the G.C.M. of b and c will be also the G.C.M. of b and c', by what has been said above.

And this, by the same reasoning as before, is also the G.C.M. of c' and d, and therefore also of c' and d'; and so on: d' being the last divisor, is evidently the G.C.M. of c' and d'; and therefore it is the G.C.M. of a and b.

All this reasoning will equally hold good, if we *introduce* such factors as m and n in the course of the operation, provided only that by such introduction they do not become common to the divisor and dividend; for otherwise the G.C.M. of the divisor and dividend, which has been proved to be the same as that of the proposed quantities, would be increased by such common factors, and the result would consequently be erroneous.

Care must be taken therefore, lest in rejecting a factor, we reject a part of the G.C.M.; as also in introducing a factor, lest we introduce one which will increase the common factors of the proposed quantities.

From what has been said, it will be seen that every common measure of a and b will measure c', and by the same reasoning, will measure d', and so on: i.e. every common measure of a and b will measure their G.C.M.

Ex. 1. Required the G.C.M. of $9x^3+53x^2-9x-18$ and $x^2+11x+30$.

$$\begin{array}{r}
x^2+11x+30)\,9x^3+53x^2-9x-18\,(9x-46 \\
9x^3+99x^2+270x \qquad \\
\hline
-46x^2-279x-18 \\
-46x^2-506x-1380 \\
\hline
227x+1362=227(x+6);
\end{array}$$

227 is evidently not a common divisor of the two proposed quantities, and may therefore be rejected.

$$\begin{array}{r}
x+6)\,x^2+11x+30\,(x+5 \\
x^2+6x \qquad\quad \\
\hline
5x+30 \qquad \\
5x+30 \qquad \\
\hline
\end{array}$$

$\therefore\ x+6$ is the G.C.M. required.

Ex. 2. Required the G.C.M. of $2x^3+x^2-8x+5$ and $7x^2-12x+5$.

To avoid a fractional quotient in the first step the former quantity must be multiplied by 7, which is not a factor of the other quantity, and will therefore not affect their G.C.M.

$$\begin{array}{r|l}
 & 2x^3+\ x^2-8x+5 \\
 & \qquad\qquad\qquad\ \ 7 \\
\hline
7x^2-12x+5 & 14x^3+\ 7x^2-56x+35\ (2x+4 \\
 & 14x^3-24x^2+10x \\
\hline
 & \qquad 31x^2-66x+35 \\
 & \qquad 28x^2-48x+20 \\
\hline
 & \qquad\ 3x^2-18x+15=3(x^2-6x+5);
\end{array}$$

3 is evidently not a common divisor of the proposed quantities, and may therefore be rejected.

$$\begin{array}{r|l}
x^2-6x+5 & 7x^2-12x+\ 5\ (7 \\
 & 7x^2-42x+35 \\
\hline
 & \qquad 30x-30=30(x-1);\ \text{reject the factor } 30;
\end{array}$$

$$\begin{array}{r|l}
x-1 & x^2-6x+5\ (x-5 \\
 & x^2-x \\
\hline
 & \quad -5x+5 \\
 & \quad -5x+5 \\
\hline
\end{array}$$

$\therefore\ x-1$ is the G.C.M. required.

Ex. 3. Required the G.C.M. of $36a^6-18a^5-27a^4+9a^3$ and $27a^5b^2-18a^4b^2-9a^3b^2$.

Here $36a^6-18a^5-27a^4+9a^3=9a^3(4a^3-2a^2-3a+1)$,

and $27a^5b^2-18a^4b^2-9a^3b^2=9a^3b^2(3a^2-2a-1)$;

$\therefore\ 9a^3$ is a factor of the G.C.M., and the factor b^2 in the latter quantity may be rejected.

Proceeding with the other factors,

$$\begin{array}{r|l}
 & 4a^3-2a^2-3a+1 \\
 & \qquad\qquad\qquad\ 3 \\
\hline
3a^2-2a-1 & 12a^3-6a^2-9a+3\ (4a \\
 & 12a^3-8a^2-4a \\
\hline
 & \qquad 2a^2-5a+3=2a(a-1)-3(a-1), \\
 & \qquad\qquad\qquad\qquad =(2a-3)(a-1);
\end{array}$$

$2a-3$ may be easily seen not to be a common divisor of the proposed quantities, and may therefore be rejected:

$$\begin{array}{r|l}
a-1 & 3a^2-2a-1\ (3a+1 \\
 & 3a^2-3a \\
\hline
 & \qquad a-1 \\
 & \qquad a-1 \\
\hline
\end{array}$$

$\therefore\ a-1$ is the G.C.M. of $4a^3-2a^2-3a+1$ and $3a^2-2a-1$.

Hence $9a^3(a-1)$ is the G.C.M. required.

107. To find the Greatest Common Measure of three quantities, a, b, c, find d the Greatest Common Measure of a and b; and the Greatest Common Measure of d and c is the Greatest Common Measure required.

Because every common measure of a, b, and c, measures d and c; and every measure of d and c measures a, b, and c (Art. 105); therefore, the Greatest Common Measure of d and c must be the Greatest Common Measure of a, b, and c.

108. In the same manner the Greatest Common Measure of four or more quantities may be found.

The Greatest Common Measure of four quantities, a, b, c, d, may be found by finding x the Greatest Common Measure of a and b, and y the Greatest Common Measure of c and d; then the Greatest Common Measure of x and y will be the common measure required.

109. It should be borne in mind that the Greatest Common Measure of two or more algebraical quantities found as above is not necessarily the *Arithmetical* Greatest Common Measure of the same quantities when *numerical values* are given to the letters contained in them; and the reason is this:—there may be *factors* in the several quantities which, in their *algebraical* state, are *prime* to one another, but become divisible by some common number, when certain values are given to the letters contained in them. For example, the factors $x-7$, and $x-4$, have no common measure greater than 1, as *algebraical* expressions; but if 10 be put for x, they become 3, and 6, which have a common measure, 3. Hence it is plain, that if we exclude certain factors from the Greatest Common Measure as having no common divisor, and *afterwards* change their form so as to make them to have a common divisor, the Greatest Common Measure obtained on the former supposition cannot possibly agree with the Greatest Common Measure obtained on the latter supposition. In fact, the phrases '*Greatest Common Measure*,' '*Lowest Terms*,' '*Least Common Multiple*,' applied to *algebraical* quantities, do not regard comparative *magnitude*.

110. In practice the Greatest Common Measure of two or more algebraical quantities is frequently found by a more expeditious method than the preceding, as follows. Taking Ex. 2, Art. 105,

Ex. Required the G. C. M. of a^4-x^4 and $a^3-a^2x-ax^2+x^3$.

$$\text{First, } a^4-x^4=(a^2+x^2)(a^2-x^2).$$

$$\text{Also } a^3-a^2x-ax^2+x^3=a^2(a-x)-x^2(a-x),$$
$$=(a^2-x^2)(a-x);$$

therefore a^2-x^2 is a common factor or divisor of the proposed quantities; and since the other factors a^2+x^2 and $a-x$ have no common measure greater than 1, therefore a^2-x^2 is the Greatest Common Measure required.

111. In the same manner fractions are usually reduced to their lowest terms without the application of the Rule for finding the G. C. M. of the numerator and denominator.

Ex. 1. Reduce to lowest terms $\dfrac{x^2+11x+30}{9x^3+53x^2-9x-18}$.

$$\frac{x^2+11x+30}{9x^3+53x^2-9x-18}=\frac{x(x+6)+5(x+6)}{9x^2(x+6)-(x^2+9x+18)},$$

$$=\frac{(x+5)(x+6)}{9x^2(x+6)-(x+3)(x+6)},$$

$$=\frac{(x+5)(x+6)}{(9x^2-x-3)(x+6)},$$

(divid^g. num^r. and denom^r. by $x+6)=\dfrac{x+5}{9x^2-x-3}$.

Ex. 2. Reduce $\dfrac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}$ to its lowest terms.

$$x^2+(a+c)x+ac=x^2+ax+cx+ac,$$

$$=x(x+a)+c(x+a),$$

$$=(x+c)(x+a);$$

also $x^2+(b+c)x+bc=(x+c)(x+b)$;

$\therefore$ the fraction becomes $\dfrac{(x+c)(x+a)}{(x+c)(x+b)}$,

and in lowest terms is $\dfrac{x+a}{x+b}$.

Ex. 3. Reduce $\dfrac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2}$ to its lowest terms.

$$\frac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2}=\frac{3a^2(a-b)+b^2(a-b)}{4a(a-b)-b(a-b)},$$

$$=\frac{(3a^2+b^2)(a-b)}{(4a-b)(a-b)},$$

$$=\frac{3a^2+b^2}{4a-b}.$$

Ex. 4. Reduce $\dfrac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$ to its lowest terms.

$$\frac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}=\frac{(a+b)(a+b+c)(a+b-c)}{(2bc+a^2-b^2-c^2)(2bc-a^2+b^2+c^2)},$$

$$=\frac{(a+b)(a+b+c)(a+b-c)}{\{a^2-(b-c)^2\}\{(b+c)^2-a^2\}},$$

$$=\frac{(a+b)(a+b+c)(a+b-c)}{(a+b-c)(a+c-b)(a+b+c)(b+c-a)},$$

$$= \frac{a+b}{(a+c-b)(b+c-a)},$$
$$= \frac{a+b}{c^2-(a-b)^2}.$$

[*Exercises* G.]

112. ***Fractions may be changed to others of equal value, with a common denominator, by multiplying each numerator by every denominator except its own, for the new numerator, and all the denominators together for the common denominator.***

Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ be the proposed fractions; then $\frac{adf}{bdf}, \frac{cbf}{bdf}, \frac{ebd}{bdf}$, are fractions of the same value respectively with the former, having the common denominator bdf. For $\frac{adf}{bdf}=\frac{a}{b}$; $\frac{cbf}{bdf}=\frac{c}{d}$; and $\frac{ebd}{bdf}=\frac{e}{f}$ (Art. 101); the numerator and denominator of each fraction having been multiplied by the same quantity, *viz.* the product of the denominators of all the other fractions.

113. When the denominators of the proposed fractions are not *prime* to each other, find their Greatest Common Measure; multiply both the numerator and denominator of each fraction by the denominators of all the rest, divided respectively by their Greatest Common Measure; and the fractions will be reduced to a common denominator *in lower terms* than they would have been by proceeding according to the former rule.

Thus $\frac{a}{mx}, \frac{b}{my}, \frac{c}{mz}$, reduced to a common denominator, are $\frac{ayz}{mxyz}, \frac{bxz}{mxyz}, \frac{cxy}{mxyz}$.

114. To obtain them in the *lowest* terms each must be reduced to another of equal value, with the denominator which is the *Least Common Multiple*, or *Lowest Common Multiple*, of all the denominators.

It becomes necessary therefore to investigate a rule for finding the Least Common Multiple of two or more quantities. And first,

115. *To find the Least Common Multiple of any number of* simple *quantities.*

To do this, observe the combinations of letters which form the several quantities, and resolve each quantity, by inspection, into its simple *factors.* The object then will be, to construct such a quantity as shall contain all the different factors found in the proposed quantities, but no factor repeated which is not also similarly repeated in some one of them; for thus we shall obviously form a quantity, and the *least* quantity, which is divisible by each of the proposed quantities without remainder, that is, the Least Common Multiple of them all. To this end, detach from each quantity all the factors, which are common to two or more of them, until the quantities are left prime to each other. The continued product of these common factors and prime results will be the Least Common Multiple required.

Thus, let the L. C. M. of $2a$, $6ab$, and $8ab$ be required. We see that the three quantities have a common factor $2a$, which being detached leaves the quantities 1, $3b$, and $4b$: of these again, the two latter have a common factor b, which being detached leaves the quantities 1, 3, and 4; and these are prime to each other. Therefore, the L. C. M. required is

$$2a\times b\times 1\times 3\times 4, \text{ or } 24ab.$$

OBS. Since the detaching of the Common Factors is the same thing as dividing the quantities by their Greatest Common Measures, it is clear that this method coincides with the arithmetical rule given in Art. 23.

It may also be observed that the preceding method is applicable to *compound* quantities, as well as *simple*, provided that each of the quantities can be readily resolved into its component factors. Thus, if the L. C. M. of $ab+ad$, and $ab-ad$ be required, we see that the quantities have a common factor a, and when stripped of this become $b+d$, and $b-d$, which are prime to each other. Therefore the L. C. M. required is

$$a(b+d)(b-d) \text{ or } ab^2-ad^2.$$

The following method is generally applicable to all quantities Simple or Compound.

116. *To find the Least Common Multiple of two quantities, or the least quantity which is divisible by each of them without remainder.*

Let a and b be the two quantities, x their greatest common measure, m their least common multiple, and let m contain a, p times, and b, q times, that is, let $m=pa=qb$; then dividing the two latter equal quantities by pb (Art. 82), $\frac{a}{b}=\frac{q}{p}$; and since m is the least possible, p and q are the least possible; therefore $\frac{q}{p}$ is the fraction $\frac{a}{b}$ in its lowest terms*, and consequently $q=\frac{a}{x}$; hence $m=qb=\frac{a}{x}\times b$.

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* For, if not, let some other fraction $\frac{q'}{p'}$ be the fraction $\frac{a}{b}$ in its lowest terms; then since $\frac{q'}{p'}=\frac{a}{b}$, multiplying these equal quantities by $p'b$ (Art. 81), $q'b=p'a$, or there are common multiples of a and b less than pa and qb, which is impossible, since pa and qb are the *least*.—ED.

The rule here proved may be thus enunciated:—

Find the G. C. M. of the two proposed quantities; divide one of them by this G. C. M.; and multiply the quotient thus obtained by the other quantity. The product is the Least Common Multiple required.

Ex. Required the Least Common Multiple of a^4-x^4 and $a^3-a^2x-ax^2+x^3$.

The G. C. M. of these two quantities (See Art. 105, Ex. 2), is a^2-x^2; and $(a^4-x^4)\div(a^2-x^2)=a^2+x^2$. Therefore the Least Common Multiple required

$$=(a^2+x^2).(a^3-a^2x-ax^2+x^3),$$
$$=a^5-a^4x-ax^4+x^5.$$

117. *Every other common multiple of* a *and* b *is a multiple of* m.

Let n be any other common multiple of the two quantities; and, if possible, let m be contained r times in n, with a remainder s, which is less than m; then $n-rm=s$; and since a and b measure n and rm, they measure $n-rm$, or s (Art. 104); that is, they have a common multiple less than m, which is contrary to the supposition.

118. *To find the Least Common Multiple of three quantities* a, b, c, *find* m *the Least Common Multiple of* a *and* b, *and* n *the Least Common Multiple of* m *and* c; *then* n *is the Least Common Multiple sought.*

For every common multiple of a and b is a multiple of m (Art. 117); therefore every common multiple of a, b, and c is a multiple of m and c; also every multiple of m and c is a multiple of a, b, and c; consequently the Least Common Multiple of m and c is the Least Common Multiple of a, b, and c.

And similarly if there be four or more quantities of which the Least Common Multiple is required.

Ex. Required the Least Com. Mult. of $x^3-a^2x-ax^2+a^3$, x^4-a^4, and $ax^3+a^3x-a^2x^2-a^4$.

Here $ax^3+a^3x-a^2x^2-a^4=a(x^3+a^2x-ax^2-a^3)$;

$\therefore$ to find the G. C. M. of this quantity and the first, reject the factor a;

$$x^3-a^2x-ax^2+a^3\,)\,x^3+a^2x-ax^2-a^3\,(\,1$$
$$\underline{x^3-a^2x-ax^2+a^3}$$
$$2a^2x-2a^3=2a^2(x-a),$$
$$x-a\,)\,x^3-a^2x-ax^2+a^3\,(\,x^2-a^2$$
$$\underline{x^3-ax^2}$$
$$-a^2x+a^3$$
$$\underline{-a^2x+a^3}$$

∴ $x-a$ is the G. C. M. of the first and last of the proposed quantities; and their least com. mult. is

$$(ax^3+a^3x-a^2x^2-a^4)(x^2-a^2)\ldots\ldots(1)$$

The other quantity is

$$(x^2+a^2)(x^2-a^2)\ldots\ldots\ldots\ldots(2).$$

The G.C.M. of (1) and (2) is $(x^2-a^2)\times$ the G.C.M. of $ax^3+a^3x-a^2x^2-a^4$ and x^2+a^2. Rejecting the factor a in the former quantity,

$$\begin{array}{r|l|l} x^2+a^2 & x^3+a^2x-ax^2-a^3 & x-a \\ & \underline{x^3+a^2x} & \\ & -ax^2-a^3 & \\ & \underline{-ax^2-a^3} & \end{array}$$

∴ the G.C.M. of (1) and (2) is $(x^2-a^2)(x^2+a^2)$;

∴ least com. mult. required is

$$(ax^3+a^3x-a^2x^2-a^4)(x^2-a^2),$$

$$\text{or } ax^5-a^2x^4-a^5x+a^6.$$

119. A more expeditious method of applying the preceding rule to find the Least Com. Mult., when it can readily be done, is that of resolving each quantity into its component factors, as follows:—taking the last Example,

$$\begin{aligned} (1)\quad & x^3-a^2x-ax^2+a^3=x^2(x-a)-a^2(x-a), \\ & \qquad =(x^2-a^2)(x-a). \\ (2)\quad & x^4-a^4=(x^2+a^2)(x^2-a^2)=(x^2+a^2)(x-a)(x+a). \\ (3)\quad & ax^3+a^3x-a^2x^2-a^4=ax^2(x-a)+a^3(x-a), \\ & \qquad =a(x^2+a^2)(x-a). \end{aligned}$$

Now the G.C.M. of (2) and (3) is $(x^2+a^2)(x-a)$;

∴ least com. mult. of (2) and (3) is $a(x^4-a^4)\ldots\ldots\ldots(4)$.

Again, the G.C.M. of (1) and (4) is x^2-a^2;

∴ least com. mult. required is $a(x^4-a^4)(x-a)$,

or $ax^5-a^2x^4-a^5x+a^6$.

119*. *The* G.C.M. *of two or more quantities is the* L.C.M. *of all the common measures.*

For the L.C.M. of all the common measures contains all the factors that appear in them, and therefore contains all the factors common to the proposed quantities: but their G.C.M. contains all these common factors: therefore the L.C.M. in question is either equal to, or a multiple of the G.C.M. But since every common measure of the proposed quantities measures their G.C.M., *i. e.* their G.C.M. is a common multiple of all their common measures, therefore (Art. 117) this G.C.M. is either equal to, or a multiple of the L.C.M. of all the common measures. But these two conditions cannot be both satisfied unless the above G.C.M. and L.C.M. are equal, therefore they are equal.

[*Exercises* H.]

ADDITION AND SUBTRACTION OF FRACTIONS.

120. *If the fractions to be added together have a common denominator, their sum is found by adding the numerators together for a new numerator and retaining the common denominator.*

Thus $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$.

This follows from the principle laid down in Art. 99, Ex. 1.

Or thus; in each of the fractions the unit is divided into the same number, b, of equal parts; and it is plain that a of these parts, or $\frac{a}{b}$, together with c of the same parts, or $\frac{c}{b}$, must be $a+c$ such parts, or $\frac{a+c}{b}$.

121. If the fractions have not a common denominator they must be transformed to others of the same value, which have a common denominator (Art. 112...114), and then the addition may take place as before.

Ex. 1. $\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$.

Ex. 2.
$$\frac{1}{a+b}+\frac{1}{a-b}=\frac{a-b}{a^2-b^2}+\frac{a+b}{a^2-b^2},$$
$$=\frac{a-b+a+b}{a^2-b^2}=\frac{2a}{a^2-b^2}.$$

Ex. 3. $a+\frac{e}{f}=\frac{af}{f}+\frac{e}{f}=\frac{af+e}{f}$.

Here a is considered as a fraction whose denominator is 1.

Ex. 4.
$$2+\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{2a^2-2b^2}{a^2-b^2}+\frac{a^2+2ab+b^2}{a^2-b^2}+$$
$$\frac{a^2-2ab+b^2}{a^2-b^2}=\frac{2a^2-2b^2+a^2+2ab+b^2+a^2-2ab+b^2}{a^2-b^2},$$
$$=\frac{4a^2}{a^2-b^2}.$$

Ex. 5.
$$\frac{a}{mx}+\frac{b}{my}+\frac{c}{mz}=\frac{ayz}{mxyz}+\frac{bxz}{mxyz}+\frac{cxy}{mxyz},$$
$$=\frac{ayz+bxz+cxy}{mxyz}.$$

Ex. 6. Required the sum of $\frac{2}{x^3+x^2+x+1}$ and $\frac{3}{x^3-x^2+x-1}$.

By Art. 116, or Art. 119, the Least Com. Mult. of the denominators is found to be x^4-1; therefore the sum required is

$$\frac{2(x-1)}{x^4-1}+\frac{3(x+1)}{x^4-1},$$

$$=\frac{2x-2+3x+3}{x^4-1},$$

$$=\frac{5x+1}{x^4-1}.$$

122. *If two fractions have a common denominator, their difference is found by taking the difference of the numerators for a new numerator and retaining the common denominator.*

Thus $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$. (See Art. 99, Ex. 1.)

Or, the same reasoning will apply as that in Art. 120.

123. If they have not a common denominator, they must be transformed to others of the same value, which have a common denominator, and then the subtraction may take place as before.

Ex. 1. $\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$.

Ex. 2. $a-\frac{cd}{b}=\frac{ab}{b}-\frac{cd}{b}=\frac{ab-cd}{b}$.

Ex. 3. $\frac{a}{b}-\frac{c+d}{c-d}=\frac{ac-ad}{bc-bd}-\frac{bc+bd}{bc-bd}=\frac{ac-ad-bc-bd}{bc-bd}$.

The sign of bd in the numerator is negative, because every part of the latter fraction is to be taken from the former. (See Art. 87.)

Ex. 4. $\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{a^2+2ab+b^2}{a^2-b^2}-\frac{a^2-2ab+b^2}{a^2-b^2}$,

$$=\frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2-b^2}=\frac{4ab}{a^2-b^2}.$$

Ex. 5. $\frac{3+2x}{1-x^2}-\frac{2}{1-x}=\frac{3+2x}{1-x^2}-\frac{2(1+x)}{1-x^2}$,

$$=\frac{3+2x-2-2x}{1-x^2}=\frac{1}{1-x^2}.$$

[*Exercises* I.]

MULTIPLICATION AND DIVISION OF FRACTIONS.

124. *To multiply a fraction by any quantity multiply the numerator by that quantity and retain the denominator.*

Thus $\frac{a}{b} \times c = \frac{ac}{b}$.

For in both the fractions $\frac{a}{b}$, $\frac{ac}{b}$, the unit is divided into the same number of equal parts (since the *denominators* are the same), and c times as many of these equal parts are taken in the latter as in the former, therefore the latter fraction is c times as great as the former.

125. Cor. 1. $\frac{a}{b} \times b = \frac{ab}{b} = a$; that is, if a fraction be multiplied by its denominator, the product is the numerator.

126. Cor. 2. The result is the same, whether the numerator be multiplied by a given quantity, or the denominator divided by it.

Let the fraction be $\frac{ad}{bc}$, and let its numerator be multiplied by c, the result is $\frac{adc}{bc}$, or $\frac{ad}{b}$ (Art. 101), the quantity which arises from the division of its denominator by c.

127. *To divide a fraction by any quantity multiply the denominator by that quantity, and retain the numerator.*

The fraction $\frac{a}{b}$ divided by c is $\frac{a}{bc}$. Because $\frac{a}{b} = \frac{ac}{bc}$, and a c^{th} part of this is $\frac{a}{bc}$: the quantity to be divided being a c^{th} part of what it was before, and the divisor the same. (Art. 96.)

128. Cor. The result is the same, whether the denominator is multiplied by the quantity, or the numerator divided by it.

Let the fraction be $\frac{ac}{bd}$; if the denominator be multiplied by c, it becomes $\frac{ac}{bdc}$ or $\frac{a}{bd}$, the quantity which arises from the division of the numerator by c.

129. *To prove the Rule for the Multiplication of Fractions,*

Rule. *The product of two fractions is found by multiplying the numerators together for a new numerator, and the denominators for a new denominator.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the two fractions; then $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

Obs. In the strict sense of the word Multiplication, which supposes a quantity to be added to itself *a certain number of times*, to multiply by a fraction would be impossible: the operation must therefore be understood in an extended sense.

Since to multiply a by b is the same thing as to take b of it, we shall easily perceive that the extension of the sense of multiplication will lead us to conclude that to multiply $\frac{a}{b}$ by $\frac{c}{d}$ will be the same thing as to take $\frac{c}{d}$ of it, *i.e.* to take the d^{th} part of it, and then to take c of such parts. But the d^{th} part of $\frac{a}{b}$ is $\frac{a}{bd}$ (Art. 127), and c of such quantities as $\frac{a}{bd}$ being taken will produce $\frac{ac}{bd}$ (Art. 124): this therefore is the result required, that is, $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}=\frac{a\times c}{b\times d}$, which proves the Rule.

130. *To prove the Rule for the Division of Fractions.*

Rule. *To divide one fraction by another, invert the numerator and denominator of the divisor, and use it as a multiplier according to the rule for multiplication.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the two fractions; then $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.

Obs. Since ordinary Division is the *inverse* of ordinary Multiplication, the operation of division here must be extended to mean the *inverse* of that of multiplication in all cases; that is, the *Quotient* must *always* be such a quantity, as when multiplied by the *Divisor* will produce the *Dividend.*

Hence, to divide $\frac{a}{b}$ by $\frac{c}{d}$ is, to find the quantity (the *Quotient*) which, when multiplied by $\frac{c}{d}$, gives $\frac{a}{b}$. But to multiply any quantity by $\frac{c}{d}$ is, to take the d^{th} part of it, and then c of such parts. Therefore the d^{th} part of the *Quotient* taken c times is equal to $\frac{a}{b}$; or the d^{th} part of the *Quotient* $=$ the c^{th} part of $\frac{a}{b}=\frac{a}{bc}$ (Art. 127),

$$\therefore \text{ the } \textit{Quotient} = d \text{ times } \frac{a}{bc}=\frac{ad}{bc}=\frac{a}{b}\times\frac{d}{c}.$$

130*. In the proofs of most of the rules for fractions we have supposed the *letters* to represent whole numbers; but the rules hold good also, when the *letters* stand for *fractions.* Thus,

I. *To shew that* $\frac{a}{b}=a\div b$, *when* a *and* b *are both fractional.*

Let a stand for $\frac{p}{q}$ and b for $\frac{r}{s}$; then, since, when a and b are whole numbers, $\frac{a}{b}$ signifies that the quantity, which taken b times gives 1, is to be taken a times, so $\frac{\frac{p}{q}}{\frac{r}{s}}$ signifies that the quantity, $\frac{r}{s}$ of which gives 1, is to be taken $\frac{p}{q}$ times, in the sense explained in Art. 129. But the quantity, $\frac{r}{s}$ of which gives 1, is $1\div\frac{r}{s}$, and this taken $\frac{p}{q}$ times $=\frac{p}{q}\times 1\div\frac{r}{s}=\frac{p}{q}\div\frac{r}{s}$.

Again,

II. *To shew that* $\frac{a}{b}=\frac{ac}{bc}$, *when* a, b, *and* c, *are fractional.*

Let a stand for $\frac{p}{q}$, b for $\frac{r}{s}$, c for $\frac{m}{n}$; then, by the foregoing,

$$\frac{a}{b}=\frac{p}{q}\div\frac{r}{s}=\frac{p}{q}\times\frac{r}{s}=\frac{pr}{qs}, \text{ (Arts. 129, 130).}$$

$$\text{Also}\qquad ac=\frac{p}{q}\times\frac{m}{n}=\frac{mp}{nq}, \text{ and } bc=\frac{r}{s}\times\frac{m}{n}=\frac{mr}{ns},$$

$$\therefore \frac{ac}{bc}=\frac{mp}{nq}\div\frac{mr}{ns}=\frac{mpns}{nqmr}=\frac{mnps}{mnqr}=\frac{ps}{qr}=\frac{a}{b}.$$

131. *To prove the Rules for Multiplication and Division of* Decimal *Fractions.*

By the Definition of a Decimal Fraction (Art. 39), $\frac{P}{10^m}$ is equivalent to a Decimal having m decimal places, and P being the number which the decimal represents, when the decimal point is erased; similarly $\frac{Q}{10^n}$ is equivalent to a decimal which has n decimal places.

Now $\frac{P}{10^m}\times\frac{Q}{10^n}=\frac{PQ}{10^{m+n}}$ = a decimal fraction formed by multiplying P into Q, and then marking off $\overline{m+n}$ decimal places:—which proves the rule for multiplication, viz. *Multiply as in whole numbers, taking no notice of the decimal points, and point off as many decimal places in the product as there are in the multiplicand and multiplier together.*

To prove the Rule for Division:—Let the dividend be made to have, if it has not already, a few more decimal places than the divisor, by adding ciphers to the right, which does not alter its value, (Art. 41); and let D, d, be the whole numbers which the Dividend so altered, and the divisor, become respectively, when the decimal point is erased. Then, if m be the number of decimal places in the Dividend so altered, and n the number in the divisor, $m>n$, and

$$\frac{\text{Dividend}}{\text{Divisor}}=\frac{D}{10^m}\div\frac{d}{10^n}=\frac{D}{d}\cdot\frac{1}{10^{m-n}},$$

$$=\left(Q+\frac{R}{d}\right)\cdot\frac{1}{10^{m-n}},\ \text{if } d \text{ is contained } Q \text{ times in } D \text{ with Remainder } R,$$

$$=\frac{Q}{10^{m-n}}+\frac{R}{d}\cdot\frac{1}{10^{m-n}}=\frac{Q}{10^{m-n}},\quad \text{if } R=0,$$

$$\text{or } =\frac{Q}{10^{m-n}}+\text{a fraction which}<\frac{1}{10^{m-n}},\quad \because R<d \text{ always};$$

$\therefore$ Quotient is either actually equal to $\frac{Q}{10^{m-n}}$, or approaches to $\frac{Q}{10^{m-n}}$ *as near as we please*, since $\frac{1}{10^{m-n}}$ can be made *as small as we please* by increasing $m-n$, that is, by adding ciphers to the proposed dividend. Hence the Rule, viz. *Affix ciphers to the right of the dividend, if wanted, so that it may have a few more decimal places than the divisor, proceed as in whole numbers, taking no notice of the decimal points, and then mark off as many decimals in the quotient as the number of decimal places in the dividend (including the ciphers used) exceeds the number in the divisor.*

N.B. The fraction $\frac{Q}{10^{m-n}}$ expressed decimally will always be the correct quotient, *as far as it goes*, because no quantity less than $\frac{1}{10^{m-n}}$ can add 1 to the last figure of a decimal with $(m-n)$ decimal places†.

COR. Since a vulgar fraction represents the quotient of the numerator ÷ denominator, the preceding rule may be applied to find its equivalent decimal fraction by *actually performing* the division; in this case $n=0$, or the divisor is an integer, viz. the denominator of the given fraction; and the dividend, *i.e.* the numerator of the fraction, can be made to have as many decimal places as may be necessary, by putting a decimal point at the end of it, and adding ciphers: and the number of decimal places in the quotient will here be m, *i.e.* there will be as many decimal places as we have added ciphers.

131*. If $\frac{a}{b}$ be a fraction in its lowest terms, by the operation of adding ciphers as above after a decimal point, we transform it into $\frac{a\times 10^m}{b\times 10^m}$, m being the number of ciphers added: and if the equivalent decimal is a terminating one, $a\times 10^m$ is divisible by b exactly. But by hypothesis a is not so; therefore 10^m is exactly divisible by b. Now the prime factors of 10 are 2 and 5, and no others. Therefore b must contain no factors but 2 and 5, *i.e.* it must be of the form $2^p\times 5^q$. And when this is the case, if m

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† The proof of the Rule for the Division of Decimals, found in most Treatises on Algebra, is unsatisfactory and imperfect, because it includes only the *particular case* when, neglecting the decimal point, the Dividend is an exact multiple of the Divisor.

be taken equal to the greater of the two p and q, $10^m \div (2^p \times 5^q)$ is an integer, and is not so for any less value of m. Therefore the fractions that will produce terminating decimals are those only whose denominators (when they are reduced to their lowest terms) contain as factors 2 and 5 and no others; and the number of decimal places will be equal to the greatest number of times either *one* of those factors is repeated.

All other fractions will therefore produce non-terminating decimals. If the division in these cases be performed as far as the factors 2 and 5 are concerned, we have a terminating decimal to be divided by the product of the other factors of the denominator, c suppose. As the decimal does not terminate there must always be a remainder, which of course is less than c, *i.e.* is one of the numbers $1, 2, \ldots \overline{c-1}$; after $\overline{c-1}$ places of figures then have been obtained, to take the most unfavourable case, when the remainders are at first all different, the next remainder must be the same as some one of the preceding, and then the whole operation is repeated, and the figures in the quotient recur. There can therefore be no more than $c-1$ recurring figures; and all fractions that do not produce terminating decimals will produce recurring ones. See Notes, Art. 52.

132. The rule for *multiplying* the powers of the same quantity (Art. 91) will hold when one or both of the indices are negative.

Thus $a^m \times a^{-n} = a^{m-n}$; for $a^m \times a^{-n} = a^m \times \dfrac{1}{a^n}$ (Art. 65) $= \dfrac{a^m}{a^n} = a^{m-n}$, (Art. 95, Cor.). In the same manner $x^3 \times x^{-5} = \dfrac{x^3}{x^5} = \dfrac{1}{x^2} = x^{-2}$.

Again, $a^{-m} \times a^{-n} = a^{-(m+n)}$; because

$$a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} \text{ (Art. 65)} = \frac{1}{a^{m+n}} = a^{-(m+n)}.$$

133. Cor. If $m = n$, $a^m \times a^{-m} = a^{m-m} = a^0$; also $a^m \times a^{-m} = \dfrac{a^m}{a^m}$ $= 1$; therefore $a^0 = 1$, according to the notation adopted in Arts. 63, 65.

134. The rule for *dividing* any power of a quantity by any other power of the same quantity (Art. 95, Cor.) holds, whether those powers are positive or negative.

Thus $a^m \div a^{-n} = a^m \div \dfrac{1}{a^n}$ (Art. 65) $= a^m \times a^n = a^{m+n} = a^{m-(-n)}$.

Again, $a^{-m} \div a^{-n} = \dfrac{1}{a^m} \div \dfrac{1}{a^n} = \dfrac{a^n}{a^m}$ (Art. 130) $= a^{n-m} = a^{-m-(-n)}$.

135. Cor. Hence it appears, that a *factor* may be transferred from the numerator of a fraction to the denominator, and *vice versâ*, by changing the sign of its index.

Thus $\dfrac{a^m . a^n}{b^p} = \dfrac{a^m}{b^p a^{-n}}$; and $\dfrac{a^m}{a^n b^p} = \dfrac{a^m . a^{-n}}{b^p} = a^m . a^{-n} . b^{-p}$.

Also $\frac{4ab^{-2}cx^{-3}}{5a^{-2}y^{-1}d^2}=\frac{4a.a^2cy}{5b^2d^2x^3}=\frac{4a^3cy}{5b^2d^2x^3}$.

[*Exercises* J.]

INVOLUTION AND EVOLUTION.

136. If a quantity be continually multiplied by itself, it is said to be involved, or raised; and the power to which it is raised is expressed by the number of times the quantity has been employed, as a factor, in the multiplication.

Thus $a\times a$, or a^2, is called the second power of a; $a\times a\times a$, or a^3, the third power, $a\times a$...to n factors, or a^n, the n^{th} power of a.

137. If the quantity to be involved be negative, the signs of the even powers will be positive, and the signs of the odd powers negative.

For $-a\times -a = a^2$, $-a\times -a\times -a=-a^3$, &c.

138. A *simple* quantity is raised to any power by multiplying the index of every factor in the quantity by the exponent of that power, and prefixing the proper sign determined by the last Article.

Thus a^m raised to the n^{th} power is a^{mn}; because $a^m\times a^m\times a^m$...to n factors, by the rule of multiplication, is $a^{m+m+\text{\&c. to } n \text{ terms}}$, or a^{mn}. Also $(ab)^n = ab\times ab\times ab\times$ &c. to n factors, or $a\times a\times a$... to n factors $\times b\times b\times b$...to n factors (Art. 88) $= a^n\times b^n$. And a^2b^3c raised to the fifth power is $a^{10}b^{15}c^5$. Also $-a^m$ raised to the n^{th} power is $\pm a^{mn}$; where the positive or negative sign is to be prefixed, according as n is an even or odd number.

This last rule is algebraically expressed thus, $(-a^m)^n=(-1)^n.a^{mn}$; the coefficient $(-1)^n$ being sufficient to express that the positive or negative sign is to be prefixed to a^{mn}, according as n is an even or an odd number.

Again, $(a^m)^{-n}=\frac{1}{(a^m)^n}=\frac{1}{a^{mn}}=a^{-mn}=a^{m(-n)}$;

and $(a^{-m})^{-n}=\frac{1}{(a^{-m})^n}=\frac{1}{\left(\frac{1}{a^m}\right)^n}=\frac{1}{\frac{1}{a^{mn}}}=a^{mn}=a^{(-m)(-n)}$; which proves the rule when the indices are *negative* integers.

139. If the quantity to be involved be a *fraction*, both the numerator and denominator must be raised to the proposed power (Art. 127).

For $\left(\frac{a}{b}\right)^n=\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}$... to n factors $=\frac{a.a.a\ldots\text{to } n \text{ factors}}{b.b.b\ldots\text{to } n \text{ factors}}=\frac{a^n}{b^n}$.

140. If the quantity proposed be a *compound* one, the involution may either be represented by the proper index, or it may actually take place.

1. Let $a+b$ be the quantity to be raised to any power.

$$\begin{array}{l}
a+b \\
a+b \\
\hline
a^2+ab \\
\quad +ab+b^2 \\
\hline
\end{array}$$

$(a+b)^2$ or $a^2+2ab+b^2$ the square, or 2d power.

$$\begin{array}{l}
a+b \\
\hline
a^3+2a^2b+ab^2 \\
\quad +a^2b+2ab^2+b^3 \\
\hline
\end{array}$$

$(a+b)^3$ or $a^3+3a^2b+3ab^2+b^3$ the 3d power.

$$\begin{array}{l}
a+b \\
\hline
a^4+3a^3b+3a^2b^2+ab^3 \\
\quad +a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
\end{array}$$

$(a+b)^4$ or $a^4+4a^3b+6a^2b^2+4ab^3+b^4$ the 4th power; and so on.

If b be negative, or the quantity to be involved be $a-b$, wherever an odd power of b enters the sign of the term will be negative (Art. 137).

Hence $(a-b)^2=a^2-2ab+b^2$,

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$,

$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

2. Let $a+b+c$ be the quantity to be raised to any power.

$$\begin{array}{l}
a+b+c \\
a+b+c \\
\hline
a^2+ab+ac \\
\quad +ab+b^2+bc \\
\qquad +ac+bc+c^2 \\
\hline
\end{array}$$

$(a+b+c)^2$ or $a^2+b^2+c^2+2ab+2bc+2ac$ the square, or 2d power.

$$\begin{array}{l}
a+b+c \\
\hline
a^3+ab^2+ac^2+2a^2b+2abc+2a^2c \\
+a^2b+b^3+bc^2+2ab^2+2b^2c+2abc \\
+a^2c+b^2c+c^3+2abc+2bc^2+2ac^2 \\
\hline
\end{array}$$

$(a+b+c)^3$ or $a^3+b^3+c^3+3(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2)+6abc$ the cube, or 3d power; and so on for any higher power.

141. By using *brackets* the involution of a quantity consisting of mor than two terms may be always made to depend upon that of a binomia and thereby the operation be much abridged.

$$\begin{aligned}
\text{Thus } (a+b+c)^2 &= (a+\overline{b+c})^2, \\
&= a^2+(b+c)^2+2a(b+c), \\
&= a^2+b^2+c^2+2bc+2ab+2ac. \\
(a+b+c)^3 &= (\overline{a+b}+c)^3, \\
&= (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3, \\
&= a^3+3a^2b+3ab^2+b^3+3(a^2+b^2+2ab)c \\
&\qquad +3ac^2+3bc^2+c^3, \\
&= a^3+b^3+c^3+3(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)+6abc.
\end{aligned}$$

$$\begin{aligned}
\text{Again, } (a+b+c+d)^2 &= (\overline{a+b}+\overline{c+d})^2, \\
&= (a+b)^2+(c+d)^2+2(a+b)(c+d), \\
&= a^2+b^2+2ab+c^2+d^2+2cd+2ac+2ad+2bc+2bd, \\
&= a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd);
\end{aligned}$$

and so on.

142. Since $(a+b+c+d+\&c.)^2$ may be thus arranged,

$$a^2+2a(b+c+d+\&c.)+b^2+2b(c+d+\&c.)+c^2+2c(d+\&c.)+\&c.$$

the square of any multinomial may be readily found by the following

RULE: *Square each term, and multiply twice that term into the sum of the several terms that come after; the sum of all the results so obtained will be the square of the whole quantity. Thus,*

Ex. 1.

$$\begin{aligned}
(a+b+c+d)^2 = {} & a^2+2a(b+c+d) \\
& +b^2+2b(c+d) \\
& +c^2+2cd \\
& +d^2.
\end{aligned}$$

Ex. 2.

$$\begin{aligned}
(a-b-c+d)^2 = {} & a^2+2a(d-c-b) \\
& +b^2-2b(d-c) \\
& +c^2-2cd \\
& +d^2.
\end{aligned}$$

Ex. 3.

$$\begin{aligned}
\left(1+\frac{1}{2}x+\frac{1}{3}x^2\right)^2 = {} & 1+x+\frac{2}{3}x^2 \\
& +\frac{1}{4}x^2+\frac{1}{3}x^3 \\
& +\frac{1}{9}x^4, \\
= {} & 1+x+\frac{11}{12}x^2+\frac{1}{3}x^3+\frac{1}{9}x^4.
\end{aligned}$$

Ex. 4. $(1+x+2x^2+3x^3+4x^4+5x^5+\ldots)^2$

$$\begin{aligned}&=1+2(x+2x^2+3x^3+4x^4+5x^5+6x^6+\ldots)\\&+x^2+2x(2x^2+3x^3+4x^4+5x^5+\ldots)\\&+4x^4+4x^2(3x^3+4x^4+\ldots)\\&+9x^6+6x^3(4x^4+\ldots)\\&+\ldots\ldots\ldots\ldots\ldots,\\&=1+2x+5x^2+10x^3+18x^4+30x^5+47x^6+\ldots\end{aligned}$$

N.B. It will be found useful *to commit to memory* the following results :—

$$(A+B)^3=A^3+B^3+3AB(A+B)\ldots\ldots(1),$$
$$(A-B)^3=A^3-B^3-3AB(A-B)\ldots\ldots(2),$$

whatever quantities A and B may represent, *simple* or *compound.*

Thus, Ex. 1. Let the cube of $1+x+x^2$ be required.

$$\begin{aligned}(1+x+x^2)^3&=(\overline{1+x}+x^2)^3,\\&=(1+x)^3+(x^2)^3+3(1+x)x^2(1+x+x^2),\text{ by (1)},\\&=1+x^3+3x+3x^2+x^6+(3x^2+3x^3)(1+x+x^2),\\&=1+x^3+3x+3x^2+x^6+3x^2+3x^3+3x^4+3x^3+3x^4+3x^5,\\&=1+3x+6x^2+7x^3+6x^4+3x^5+x^6.\end{aligned}$$

Ex. 2. Required the cube of $\sqrt[3]{a+x}-\sqrt[3]{a--x}$.

$$\begin{aligned}(\sqrt[3]{a+x}-\sqrt[3]{a-x})^3&=a+x-\overline{a-x}-3\sqrt[3]{a+x}.\sqrt[3]{a-x}\times(\sqrt[3]{a+x}-\sqrt[3]{a-x}),\text{ by (2)},\\&=2x-3.\sqrt[3]{a+x}.\sqrt[3]{a-x}.\{\sqrt[3]{a+x}-\sqrt[3]{a-x}\}.\end{aligned}$$

143. EVOLUTION, or the extraction of roots, is the method of determining a quantity which raised to a proposed power will produce a given quantity.

144. Since the n^{th} power of a^m is a^{mn} (Art. 138), the n^{th} root of a^{mn} must be a^m; that is, to extract any root of a simple quantity, we must divide the index of that quantity by the index of the root required.

Thus $\sqrt[3]{a^6}=a^2$, $\sqrt[4]{a^{12}}=a^3$; &c. $\sqrt[n]{a^{mn}}=a^m$.

145. When the index of the quantity is not exactly divisible by the number which expresses the root to be extracted, that root must be represented according to the notation pointed out in Art. 70. Thus the square, cube, fourth, n^{th}, root of a^2+x^2, are respectively represented by

$$(a^2+x^2)^{\frac{1}{2}},\ (a^2+x^2)^{\frac{1}{3}},\ (a^2+x^2)^{\frac{1}{4}},\ (a^2+x^2)^{\frac{1}{n}};$$

the same roots of $\frac{1}{a^2+x^2}$, or $(a^2+x^2)^{-1}$, are represented by

$$(a^2+x^2)^{-\frac{1}{2}},\ (a^2+x^2)^{-\frac{1}{3}},\ (a^2+x^2)^{-\frac{1}{4}},\ (a^2+x^2)^{-\frac{1}{n}}.$$

146. If the root to be extracted be expressed by an odd number, the sign of the root will be the same with the sign of the proposed quantity, as appears by Art. 137.

Thus $\sqrt{8}$ is 2; $\sqrt[3]{-8}$ is -2; &c.

147. If the root to be extracted be expressed by an even number, and the quantity proposed be positive, the root may be either positive or negative. Because either a positive or negative quantity raised to such a power is positive (Art. 137).

Thus $\sqrt{a^2}$ is $\pm a$; $\sqrt[4]{(a+x)^8}$ is $\pm(a+x)^2$; &c.

148. If the root proposed to be extracted be expressed by an even number, and the sign of the proposed quantity be negative, the root cannot be extracted; because no quantity raised to an even power can produce a negative result. Such roots are called *impossible*.

149. Any root of a *product* may be found by taking that root of each factor, and multiplying the roots, so taken, together.

Thus $(ab)^{\frac{1}{n}}=a^{\frac{1}{n}}\times b^{\frac{1}{n}}$; because each of these quantities, raised to the n^{th} power, is ab (Art. 138).

Exs. $\sqrt[2]{a^2b^4}=\sqrt[2]{a^2}.\sqrt[2]{b^4}=ab^2$; and $\sqrt[5]{a^{10}b^{15}c^5}=a^2b^3c$.

Cor. If $a=b$, then $a^{\frac{1}{n}}\times a^{\frac{1}{n}}=a^{\frac{2}{n}}$; and in the same manner $a^{\frac{r}{n}}\times a^{\frac{s}{n}}=a^{\frac{r+s}{n}}$.

This will be proved more fully and clearly in Art. 162.

150. Any root of a *fraction* may be found by taking that root of both the numerator and denominator (Art. 139), that is, the root of the numerator for a new numerator, and the same root of the denominator for a new denominator.

Thus the cube root of $\frac{a^2}{b^2}$ is $\frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}$, or $a^{\frac{2}{3}}\times b^{-\frac{2}{3}}$; and $\left(\frac{a}{b}\right)^{\frac{1}{n}}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$, or $a^{\frac{1}{n}}\times b^{-\frac{1}{n}}$.

Exs. $\sqrt{\frac{9a^2}{4x^2}}=\frac{\sqrt{9a^2}}{\sqrt{4x^2}}=\frac{3a}{2x}$. $\sqrt[3]{\frac{8a^3b^3}{27}}=\frac{\sqrt[3]{8a^3b^3}}{\sqrt[3]{27}}=\frac{2ab}{3}$.

151. *To extract the square root of a* compound *quantity.*

Since the square root of $a^2+2ab+b^2$ is $a+b$ (Art. 140), whatever be the values of a and b, we may obtain a general rule for the extraction of the square root, by observing in what manner a and b may be derived from $a^2+2ab+b^2$.

$$\begin{array}{r|l} & a^2+2ab+b^2\,(a+b \\ & a^2 \\ \hline 2a+b\,) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

Having arranged the terms according to the dimensions of one letter, a, the square root of the first term, a^2, is a, the first term in the root; subtract its square from the whole quantity, and bring down the remainder $2ab+b^2$; divide $2ab$ by $2a$, and the result is b, the other term in the root; then multiply the sum of twice the first term and the second, $2a+b$, by the second, b, and subtract this product, $2ab+b^2$, from the remainder. If there be more terms, consider $a+b$ as a new value of a; and its square, that is, $a^2+2ab+b^2$, having by the first part of the process been subtracted from the proposed quantity, divide the remainder by the double of this new value of a, for a new term in the root; and for a new subtrahend multiply this term by twice the sum of the former terms increased by this term. The process must be repeated till the root, or the necessary approximation to the root, is obtained*.

Ex. 1. To extract the square root of $a^2+2ab+b^2+2ac+2bc+c^2$.

Having arranged the terms of the proposed quantity according to the dimensions of one letter, a, it becomes

$$a^2+2(b+c)a+b^2+2bc+c^2.$$

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* That the Rule may be thus extended will be obvious from comparing the form of the squares of $a+b+c$, $a+b+c+d$, &c. with that of $a+b$, from which the Rule was deduced. For

$$\begin{aligned}(a+b+c)^2&=(a+b)^2+2(a+b)c+c^2, \text{ (Art. 141)}\\ &=a^2+(2a+b)b+\{2(a+b)+c\}c.\\ (a+b+c+d)^2&=(a+b+c)^2+2(a+b+c)d+d^2,\\ &=a^2+(2a+b)b+\{2(a+b)+c\}c+\{2(a+b+c)+d\}d;\end{aligned}$$

and so on; from which method of exhibiting the square of a *multinomial* the rule for extracting the square root is evidently seen to hold whatever be the number of terms in the root.—Ed.

Hence, following the rule, we have

$$\begin{array}{r|l}
 & a^2+2(b+c)a+b^2+2bc+c^2\,\big(\,a+(b+c) \\
 & a^2 \\ \hline
2a+(b+c)\,\big) & 2(b+c)a+b^2+2bc+c^2 \\
 & 2(b+c)a+b^2+2bc+c^2 \\ \hline
\end{array}$$

$\therefore a+b+c$ is the root required.

Ex. 2. To extract the square root of $a^2-ax+\frac{x^2}{4}$.

$$\begin{array}{r|l}
 & a^2-ax+\frac{x^2}{4}\,\big(\,a-\frac{x}{2},\text{ the root required.} \\
 & a^2 \\ \hline
2a-\frac{x}{2}\,\big) & -ax+\frac{x^2}{4} \\
 & -ax+\frac{x^2}{4} \\ \hline
\end{array}$$

Ex. 3. To extract the square root of $1+x$.

$$\begin{array}{r|l}
 & 1+x\,\big(\,1+\frac{x}{2}-\frac{x^2}{8}+\&c. \\
 & 1 \\ \hline
2+\frac{x}{2}\,\big) & x \\
 & x+\frac{x^2}{4} \\ \hline
2+x-\frac{x^2}{8}\,\big) & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\ \hline
 & \frac{x^3}{8}-\frac{x^4}{64}\ \&c.
\end{array}$$

Since $1+\frac{x}{2}=\sqrt{1+x+\frac{x^2}{4}}$; $1+\frac{x}{2}-\frac{x^2}{8}=\sqrt{1+x-\frac{x^3}{8}+\frac{x^4}{64}}$; and so on; if x be of such a magnitude that each of these successive quantities under the root differs from $1+x$ less than the preceding one, the continued series of terms $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-$&c. will be an *approximation* to the true root of $1+x$; that is, the more terms of the series are taken, the less will their sum differ from the square root of $1+x$. And, since it is evident that no expression whatsoever, simple or compound, multiplied into itself, can produce $1+x$, therefore an *approximation* to the root is all that can be required.

But if x be *not* of such a magnitude as is here supposed, then it is clear that the operation performed upon $1+x$ leads to a result which is not even an *approximation* to its square root, because as more terms of that result are taken and squared we obtain a quantity which recedes farther and farther from $1+x$. Its proper interpretation, in this stage of the subject, cannot well be given, and must be deferred.

Cor. If the approximate square root of *any* binomial, as $a+b$, be required; since $a+b=a\left(1+\frac{b}{a}\right)$; and therefore $\sqrt{a+b}=\sqrt{a}\times\sqrt{1+\frac{b}{a}}$ (Art. 149); we have by the last Example, putting $\frac{b}{a}$ in the place of x,

$$\sqrt{1+\frac{b}{a}}=1+\frac{b}{2a}-\frac{b^2}{8a^2}+\&c.$$

$$\therefore \sqrt{a+b}=a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}}+\&c.$$

152. It appears from Ex. 2, that a trinomial, $a^2-ax+\frac{x^2}{4}$, in which four times the product of the first and last terms is equal to the square of the middle term, is a complete square.

The same relation is found to subsist between the parts of all *complete squares* of three terms arranged according to the powers of some one letter.

Thus $4x^2+4cx+c^2$ is a complete square, viz. $(2x+c)^2$, because $4\times4x^2\times c^2=16c^2x^2=(4cx)^2$.

Similarly $x^2-px+\frac{p^2}{4}$ is a complete square, viz. $\left(x-\frac{p}{2}\right)^2$, because

$$4\times x^2\times\frac{p^2}{4}=p^2x^2=(-px)^2.$$

153. The method of extracting the cube root is discovered in the same manner as that for the square root.

The cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$ (Arts. 140, 143); and to obtain $a+b$ from this compound quantity, arrange the terms

as before, and the cube root of the first term, a^3, is a, the first term in the root; subtract its cube from the whole quantity, and divide the first term of the remainder by $3a^2$, the result is b, the second term in the root; then subtract $3a^2b+3ab^2+b^3$ from the remainder, and the whole cube of $a+b$ has been subtracted. If any quantity be left, proceed with $a+b$ as a new a, and divide the last remainder by $3(a+b)^2$ for a third term in the root; and thus any number of terms may be obtained*.

$$\begin{array}{rl}
 & a^3+3a^2b+3ab^2+b^3\,(\,a+b \\
 & a^3 \\
\hline
3a^2\,) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\
\hline
\end{array}$$

Ex. To extract the cube root of $8x^3+6xy^2-12x^2y-y^3$.

$$\begin{array}{rlll}
 & 8x^3-12yx^2+6y^2x-y^3\,(\,2x-y. & & \\
a^3=8x^3 & & & \\
\hline
3a^2=12x^2\,) & -12yx^2+6y^2x-y^3 & & \\
\hline
 & -12yx^2 & & =3a^2b \\
 & \quad +6y^2x & & =3ab^2 \\
 & \qquad -y^3= & & b^3 \\
\hline
\end{array}$$

$\therefore\ 2x-y$ is the cube root required.

[*Exercises* K.]

SCHOLIUM.

154. The rules above laid down for the extraction of the roots of compound quantities are but little used in algebraical operations; but it was necessary to give them at full length, for the purpose of investigating rules for the extraction of the square and cube roots in *numbers*.

The square root of 100 is 10, of 10000 is 100, of 1000000 is 1000, &c. from which consideration it follows, that the square root of a number less than 100 must consist of only one figure, of a number between 100 and 10000 of two places of figures, of any

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* That the rule may be thus extended will be obvious from comparing the form of the cubes of $a+b+c$, $a+b+c+d$, &c., with that of $a+b$ from which the Rule was deduced; For

$$(a+b+c)^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3,$$
$$=a^3+(3a^2+3ab+b^2)b+\{3(a+b)^2+3(a+b)c+c^2\}c.$$

Similarly $(a+b+c+d)^3=a^3+(3a^2+3ab+b^2)b+\{3(a+b)^2+3(a+b)c+c^2\}c$
$$+\{3(a+b+c)^2+3(a+b+c)d+d^2\}d;$$

and so on.—Ed.

number from 10000 to 1000000, of three places of figures, &c. If then a point be made over every second figure in any number, beginning with the units, the number of points will shew the number of figures, or places in the square root. Thus the square root of $4\dot{3}5\dot{7}$ consists of two figures, the square root of $\dot{5}6\dot{4}7\dot{8}$, of three figures, &c.

Ex. 1. Let the square root of 4356 be required.

$$\begin{array}{r l}
 & 4\dot{3}5\dot{6}(60+6 \text{ or } 66 \\
 & \underline{3600} \\
120+6) & 756 \\
\text{or } 126] & \underline{756}
\end{array}$$

Having pointed it according to the direction, it appears that the root consists of two places of figures; let $a+b$ be the root, where a is the value of the figure in the tens' place, and b of that in the units'; then is a the nearest square root of 4300, which does not exceed the true root*; this appears to be 60; subtract the square of 60, (a^2), from the given number, and the remainder is 756; divide this remainder by 120, ($2a$), and the quotient is 6, (the value of b), and the subtrahend, or quantity to be taken from the last remainder 756, is 126×6, $(2a+b)b$, or 756.

Hence 66 is the root required.

It is said that a must be the greatest number whose square does not exceed 4300†: it evidently cannot be a greater number‡ than this; and if possible let it be some quantity‡‡, x, less than this; then since x is in the tens' place and b in the units', $x+b$ is less than a; therefore the square of $x+b$, whatever be the value of b, must be less than a^2, and consequently $x+b$ less than the true root.

If the root consist of three places of figures, let a represent the hundreds, and b the tens; then having obtained a and b as before, let the new value of a be the hundreds and tens together, and find a new value of b for the units: and thus the process may be continued when there are more places of figures in the root.

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* It will be clearer to read "a the greatest multiple of 10 whose square does not exceed 4300."—ED.

† Or, the greatest multiple of 10 whose square does not exceed 4300.—ED.

‡ Multiple.—ED.

‡‡ . Let it be x, some less multiple of 10.—ED.

155. The ciphers being omitted for the sake of expedition, the following rule is obtained from the foregoing process.

```
       43̇56̇(66
       36
       ---
  126) 756
       756
       ---
```

Point every second figure beginning with the units' place, dividing by this process the whole number into several periods; find the greatest number whose square is contained in the first period, this is the first figure in the root; subtract its square from the first period, and to the remainder bring down the next period: divide this quantity, omitting the last figure, by twice the part of the root already obtained, and annex the result to the root and also to the divisor; then multiply the divisor, as it now stands, by the part of the root last obtained, for the subtrahend. If there be more periods to be brought down, the operation must be repeated.*

Ex. 2. Let the square root of 611525 be required.

```
         61̇15̇25̇(782
         49
         ----
     148)1215
         1184
         -----
     1562)3125
          3124
          ----
             1 remainder.
```

The remainder in this example shews that we have not obtained the number which is the *exact* square root of the proposed quantity; but 782 is a near approximation to the square root, being in fact the square root of 611524; and 783 is too great, being the square root of 613089.

156. In extracting the square root of a *decimal*, the pointing must be made the contrary way, beginning with the second place of decimals, and the integral part must be pointed as before, beginning with the units' place: or, if the rule be applied as in whole numbers, care must be taken to have an even number of decimal

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* The integer quotient of 75 divided by 12, viz. 6, will be the same as that of 756 by 120. Thus, when a divisor ends in ciphers, it is a well-known abridgement to cut off the ciphers, and as many figures from the right of the dividend. For these terms thus curtailed give the same integer quotient as they would do entire, and a remainder, which, on annexing to the right the figures cut off from the dividend, will be the *true* remainder.—Ed.

places, by annexing ciphers to the right (Art. 41); because, if the root have 1, 2, 3, 4, &c. decimal places, the square must have 2, 4, 6, 8, &c. places (Art. 46).

Ex. 3. To extract the square root of 64·853.

$$\begin{array}{rl}
 & 6\dot{4}\cdot8\dot{5}3\dot{0}(8\cdot053 \text{ \&c.} \\
 & \underline{64} \\
1605) & 8530 \\
 & \underline{8025} \\
16103) & 50500 \\
 & \underline{48309} \\
 & 2191
\end{array}$$

The remainder in this example appears to be great; but if the decimal point were retained throughout the operation, it would easily be seen that its real value is very small, and that it becomes smaller for every figure that is added to the root.

For every pair of ciphers which we suppose annexed to the decimal another figure is obtained in the root.

And in this and similar cases, when ciphers are added, the root can never terminate, because no figure multiplied by itself can produce a cipher in the units' place.

157. The cube root of 1000 is 10, of 1000000 is 100, &c. therefore the cube root of a number less than 1000 consists of one figure, of any number between 1000 and 1000000, of two places of figures, &c. If then a point be made over every third figure contained in any number, beginning with the units, the number of points will shew the number of places in its cube root.

Ex. 1. Let the cube root of 405224 be required.

$$\begin{array}{rl}
 & 40\dot{5}22\dot{4}(70+4 \\
a^3 = & \underline{343000} \\
3a^2 = 14700) & 62224 \text{ the first remainder.} \\
 & \overline{58800 = 3a^2b} \\
 & 3360 = 3ab^2 \\
 & \underline{64 = b^3} \\
 & \underline{62224} \text{ subtrahend.}
\end{array}$$

By pointing the number according to the direction, it appears that the root consists of two places; let *a* be the value of the figure

in the tens' place, and b of that in the units'. Then a is the greatest number* whose cube is contained in 405000, that is, 70; subtract its cube from the whole quantity, and the remainder is 62224; divide this remainder by $3a^2$, or 14700, and the quotient 4, or b, is the second term in the root: then subtract the cube of 74 from the original number, and as the remainder is nothing, 74 is the cube root required. Observe that the ciphers may be omitted in the operation; and that as a^3 was at first subtracted, if from the first remainder $3a^2b+3ab^2+b^3$ be taken, the whole cube of $a+b$ will be taken from the original quantity.

158. In extracting the cube root of a *decimal* care must be taken that the decimal places be three, or some multiple of three, before the operation is begun, by annexing ciphers to the right (Art. 41); because there are three times as many decimal places in the cube as there are in the root (Art. 46).

Ex. 2. Required the cube root of 311897·91.

$$
\begin{array}{rl}
 & 31\dot{1}89\dot{7}\cdot91\dot{0}(67\cdot8 \\
 & 216\ldots = a^3 \\
3a^2 = 108..) & \underline{95897} \text{ first remainder.} \\
 & 756\,.. = 3a^2b \\
 & 882. = 3ab^2 \\
 & \underline{343 = b^3} \\
 & 84763 \text{ subtrahend.} \\
3a^2 = 13467..) & \underline{11134910} \text{ second remainder.}
\end{array}
$$

The new value of a is 670, or, omitting the cipher, 67; and $3a^2$, the new divisor, is 13467.., hence 8 is the next figure in the root; and

$$
\begin{array}{rl}
107736.. & = 3a^2b \\
12864. & = 3ab^2 \\
\underline{512} & = b^3 \\
\underline{10902752} & \text{subtrahend.} \\
232158 & \text{the third remainder.}
\end{array}
$$

It appears from the pointing, that there is one decimal place in the root; therefore 67·8 is the root required nearly. If three more ciphers be annexed to the decimal, another decimal place is obtained in the root; and thus approximation may be made to the true root of the proposed number to any required degree of accuracy.

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* It will be clearer to read "a is the greatest multiple of 10, &c."—ED.

159. Since the first remainder is $3a^2b+3ab^2+b^3$, the exact value of b is not obtained by dividing by $3a^2$; and if upon trial the subtrahend be found to be greater than the first remainder, the value assumed for b is too great, and a less number must be tried.

The greater a is with respect to b, the more nearly is the true value obtained by division.

For the first remainder divided by $3a^2$ gives $b+\frac{b^2}{a}+\frac{b^3}{3a^2}$ for the quotient; and if this be adopted for b, the error $=\frac{b^2}{a}+\frac{b^3}{3a^2}$; which for a given value of b is evidently less as a is greater.

160. In extracting the square or cube root of a vulgar fraction the rule stated in Art. 150 may be followed; but it is generally preferable to convert the vulgar fraction into a decimal, and then extract the root.

Thus let the cube root of $5\frac{1}{2}$, or $\frac{11}{2}$ be required.

Now, if the rule of Art. 150 be applied to this case, the cube root of 11, and the cube root of 2, must both be found to a certain number of places of decimals, and then the long division of the one root by the other must be effected: whereas, if $5\frac{1}{2}$ be, first of all, converted into a decimal, viz. 5·5, one single extraction of the cube root completes the whole process.

Another method is, to multiply the numerator and denominator by such a quantity as will make the latter a perfect cube, and then apply the rule of Art. 150.

Thus the cube root of $5\frac{1}{2}$, or $\frac{11}{2}$, or $\frac{44}{8}$, $=\frac{\sqrt[3]{44}}{\sqrt[3]{8}}=\frac{1}{2}\sqrt[3]{44}$.

161. In extracting either the square or cube root of any number, when a certain number of figures in the root have been obtained by the common rule, that number may be nearly doubled by division only.

I. The square root of any number may be found by using the common Rule for extracting the square root until *one more than half the number* of digits in the root is obtained; then the rest of the digits in the root may be determined by Division.

For, let N represent the number whose square root is required; and suppose the greatest *integer* in the square root to be of $\overline{2n+1}$ digits;

a .... the first $\overline{n+1}$ digits of the root found by the common Rule, with n ciphers annexed;

x ...... the remaining part; so that $\sqrt{N}=a+x$.

$$\text{Then } N=a^2+2ax+x^2 \text{ (Art. 81)};$$

$$\therefore \frac{N-a^2}{2a}=x+\frac{x^2}{2a} \text{ (Arts. 80, 82)};$$

that is, $N-a^2$, (which is the remainder after $\overline{n+1}$ digits of the root are found) divided by $2a$ will give the rest of the root required, x, increased by $\frac{x^2}{2a}$. Now, since the greatest integer in x is a number of n digits, $x<10^n$, and $x^2<10^{2n}$. But, by the supposition, a is a number of $2n+1$ digits, and $2a$ has $2n+1$ digits at least; therefore $2a$ is not less than 10^{2n}; so that

$$x^2<2a, \text{ or } \frac{x^2}{2a} \text{ is a proper fraction, or } <1;$$

that is, if the quotient of $(N-a^2)\div 2a$ be taken for x, the error is less than 1.

Hence it appears, that if $\overline{n+1}$ digits of a square root are obtained by the common Rule, n digits more may be correctly obtained by *Division* only.

Ex. Required the square root of 2 to 6 places of decimals.

```
           2·0000...(1·414
           1
         -----
      24)100
          96
         -----
     281)400
         281
        -----
    2824)11900
         11296
         ------
     2828)604000(213
          5656
          -----
           3840
           2828
           -----
           10120
            8484
           -----
            1636
```

∴ the root required is 1·414213......

When only one figure in the root has been obtained, a, which represents the part already obtained, may be as small as 10; and x, the next digit, may be as great as 9; the error in the quotient $\frac{x^2}{2a}$, may therefore be easily greater than 1, unless a be as great as 50, *i.e.* the first figure in the root as great as 5, and we should then obtain too large a quotient; this is not unfrequently observed to happen at the first division, but from the foregoing proposition it appears that this error cannot take place in any subsequent division.

II. In the extraction of a cube root, when $\overline{n+1}$ digits have been found by the ordinary rule, n more can be correctly obtained by dividing by the trial divisor.

Let $a+b$ be the cube root,

where a consists of $\overline{n+1}$ digits, and n ciphers,

.... b .... n digits, (*i.e.* the greatest *integer* contained in b),

$a^3+3a^2b+3ab^2+b^3$ the quantity whose root is required.

Then after a has been found, we have

$$\text{remainder} = 3a^2b+3ab^2+b^3;$$

$$\text{trial divisor} = 3a^2;$$

$$\therefore \text{ quotient} = b+\frac{b^2}{a}+\frac{b^3}{3a^2}.$$

If this be adopted as the value of b,

$$\text{the error} = \frac{b^2}{a}+\frac{b^3}{3a^2}.$$

Now a consists of $2n+1$, and greatest integer in b of n, digits;

$\therefore$ the least value of a is 10^{2n},

and the greatest value of b is 10^n-1;

$\therefore$ the greatest possible error will be when a and b have the above values;

$$i.e. \text{ the greatest error} = \frac{(10^n-1)^2}{10^{2n}}+\frac{(10^n-1)^3}{3\times10^{4n}},$$

$$=\frac{10^n-1}{10^n}\cdot\frac{10^n-1}{10^n}\left(1+\frac{10^n-1}{3\times10^{2n}}\right),$$

$$=\left(1-\frac{1}{10^n}\right)\left(1-\frac{1}{10^n}\right)\left(1+\frac{1}{3\times10^n}-\frac{1}{3\times10^{2n}}\right),$$

$$\text{and} \therefore <\left(1-\frac{1}{10^n}\right)\left(1-\frac{1}{10^n}\right)\left(1+\frac{1}{3\times10^n}\right),$$

$$i.e. <\left(1-\frac{1}{10^n}\right)\left(1-\frac{2}{3\times10^n}-\frac{1}{3\times10^{2n}}\right),$$

which evidently < 1.

The error therefore *always* < 1, *i.e.* the n last digits can be correctly obtained by ordinary division.

From this it will be seen that as in square root, it is only at the first division that too large a quotient can be obtained for the next digit in the root.

THEORY OF INDICES.

162. The subject of *Indices* deserves a separate and distinct consideration. It is proposed to bring together here all that has been *defined* or *proved* with respect to them in the preceding pages—to shew that the several Definitions are not *in practice* inconsistent with each other—and to supply the proofs still wanting in order that the Rules may be extended to all possible cases which can occur.

(1) The primary *Definition* was given in Art. 63, whereby we agreed to represent $a.a.a.$ &c. to n factors by a^n, where n expresses the *number of factors*, and therefore can only be *a positive integer.*

(2) The next *Definition* was given in Art. 65, whereby we agreed to represent $\frac{1}{a^n}$ by a^{-n}; but at that stage we could only consider it as a short way of writing $\frac{1}{a^n}$, since a *negative* quantity can in no sense express *a number of factors.*

(3) The last *Definition* was given in Art. 70, whereby we agreed to represent the n^{th} root of a by $a^{\frac{1}{n}}$, and the n^{th} root of the m^{th} power of a by $a^{\frac{m}{n}}$. Here again we felt the restriction that a *fraction* can in no sense express a *number of factors* multiplied together, that is, a *power* of a, in the true sense of the word.

(4) From (1) it is strictly proved in Arts. 91, 95, 138, and 144, that

$$a^m \times a^n = a^{m+n},$$

$$a^m \div a^n = a^{m-n}, \text{ or } \frac{1}{a^{n-m}}, \text{ according as } m > \text{or} < n,$$

$$(a^m)^n = a^{m\times n},$$

$$\text{and } \sqrt[n]{a^{mn}} = a^m = a^{mn-n},$$

m and n being *positive integers.*

These are the fundamental Rules for the Multiplication, Division, Involution, and Evolution of powers and roots.

(5) When a *negative* value is given to either m or n, or both of them, and *restricted to the meaning pointed out in* (2), it is proved in Arts. 132, 134, and 138, that these Rules still hold true. Hence it appears, that no error can arise from using *negative* powers according to the second Definition.

(6) It remains, however, to be proved, that the 3rd Definition is generally admissible, that is, that the above Rules hold true when *positive or negative fractions* are treated as *indices* of the *powers* or *roots* of any quantity, with the meaning assigned to them by the Definition. This being done, the Rules for the treatment of *Indices* will have been shewn to apply generally to all cases whatever, whether the indices be positive or negative, whole or fractional.

(7) **Lemma I.** *To shew that* $\sqrt[n]{a}.\sqrt[n]{b}=\sqrt[n]{ab}$.

By Def. $(\sqrt[n]{a})^n=a$, and $(\sqrt[n]{b})^n=b$,

$\therefore\ ab=(\sqrt[n]{a})^n.(\sqrt[n]{b})^n=(\sqrt[n]{a}.\sqrt[n]{b})^n$, Art. 138,

$\therefore\ \sqrt[n]{ab}=\sqrt[n]{a}.\sqrt[n]{b}$, by Def.

Cor. $\sqrt[n]{\dfrac{a}{b}}=\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

(8) **Lemma II.** *To shew that* $(\sqrt[n]{a})^m=\sqrt[n]{a^m}$.

$\{(\sqrt[n]{a})^m\}^n=(\sqrt[n]{a})^{mn}$ (Art. 138) $=\{(\sqrt[n]{a})^n\}^m=a^m$,

$\therefore\ (\sqrt[n]{a})^m=\sqrt[n]{a^m}$.

Cor. $(\sqrt[n]{a^m})^p=\sqrt[n]{(a^m)^p}=\sqrt[n]{a^{mp}}$.

(9) **Lemma III.** *To shew that* $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$.

$(\sqrt[m]{\sqrt[n]{a}})^m=\sqrt[n]{a},\ \therefore\ a=\{(\sqrt[m]{\sqrt[n]{a}})^m\}^n=(\sqrt[m]{\sqrt[n]{a}})^{mn}$,

$\therefore\ \sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$.

Cor. I. $\sqrt[n]{\sqrt[m]{a}}=\sqrt[nm]{a}=\sqrt[mn]{a}=\sqrt[m]{\sqrt[n]{a}}$.

Cor. II. $\sqrt[m]{(\sqrt[n]{a^p})^q}=\sqrt[m]{\sqrt[n]{a^{pq}}}=\sqrt[mn]{a^{pq}}=\sqrt[nm]{a^{qp}}=\sqrt[n]{(\sqrt[m]{a^q})^p}$,

which shews that the *order* in which the operations of involution and evolution are performed upon any quantity is immaterial.

(10) **Lemma IV.** *To shew that* $\sqrt[n]{a^m}=\sqrt[nq]{a^{mq}}$.

$\sqrt[n]{a^m}=\sqrt[q]{(\sqrt[n]{a^m})^q}=\sqrt[q]{\sqrt[n]{a^{mq}}}$, (Lemma II. Cor.)

$=\sqrt[nq]{a^{mq}}$, by Lemma III.

Cor. $\sqrt[q]{a^p}=\sqrt[nq]{a^{np}},\ \therefore\ \sqrt[n]{a^m}.\sqrt[q]{a^p}=\sqrt[nq]{a^{mq}}.\sqrt[nq]{a^{np}}=\sqrt[nq]{a^{mq+np}}$, by Lemma I.

(11) *To shew that the Rules for the multiplication and division of powers hold true when the indices are fractional.*

1st, $a^{\frac{m}{n}}\times a^{\frac{p}{q}}=\sqrt[n]{a^m}.\sqrt[q]{a^p}=\sqrt[nq]{a^{mq+np}}$, by Lemma IV. Cor.

$=a^{\frac{mq+np}{nq}}$ by Def., $=a^{\frac{m}{n}+\frac{p}{q}}$.

2nd, $a^{\frac{m}{n}}\div a^{\frac{p}{q}}=\dfrac{\sqrt[n]{a^m}}{\sqrt[q]{a^p}}=\dfrac{\sqrt[nq]{a^{mq}}}{\sqrt[nq]{a^{np}}}=\sqrt[nq]{\dfrac{a^{mq}}{a^{np}}}$ by Lemma I. Cor., $=\sqrt[nq]{a^{mq-np}}$,

$=a^{\frac{mq-np}{nq}}=a^{\frac{m}{n}-\frac{p}{q}}$.

3rd, $a^{\frac{m}{n}}\times a^{-\frac{p}{q}}=\dfrac{a^{\frac{m}{n}}}{a^{\frac{p}{q}}}=a^{\frac{m}{n}-\frac{p}{q}}$ by the last case, $=a^{\frac{m}{n}+\left(-\frac{p}{q}\right)}$.

4th, $a^{\frac{m}{n}}\div a^{-\frac{p}{q}}=a^{\frac{m}{n}}\div\dfrac{1}{a^{\frac{p}{q}}}=a^{\frac{m}{n}}.a^{\frac{p}{q}}=a^{\frac{m}{n}+\frac{p}{q}}$ by the 1st case, $=a^{\frac{m}{n}-\left(-\frac{p}{q}\right)}$.

5th, $a^{-\frac{m}{n}}\times a^{-\frac{p}{q}}=\frac{1}{a^{\frac{m}{n}}}.\frac{1}{a^{\frac{p}{q}}}=\frac{1}{a^{\frac{m}{n}+\frac{p}{q}}}$ by 1st case, $=a^{-(\frac{m}{n}+\frac{p}{q})}=a^{-\frac{m}{n}+(-\frac{p}{q})}$.

6th, $a^{-\frac{m}{n}}\div a^{-\frac{p}{q}}=a^{-\frac{m}{n}}\div\frac{1}{a^{\frac{p}{q}}}=a^{-\frac{m}{n}}.a^{\frac{p}{q}}=a^{\frac{p}{q}}.a^{-\frac{m}{n}}=a^{\frac{p}{q}-\frac{m}{n}}$ by 3rd case,

$$=a^{-\frac{m}{n}-(-\frac{p}{q})};$$

which proves the Rules *for all possible cases of fractional Indices.*

(12) *To shew that the Rules for Involution and Evolution of powers hold true for fractional Indices.*

1st, $(a^{\frac{m}{n}})^{\frac{p}{q}}=\sqrt[q]{(a^{\frac{m}{n}})^p}$ by Def., $=\sqrt[q]{(\sqrt[n]{a^m})^p}=\sqrt[q]{\sqrt[n]{a^{mp}}}$, (8)

$$=\sqrt[nq]{a^{mp}},\ (9),\ =a^{\frac{mp}{nq}}=a^{\frac{m}{n}\cdot\frac{p}{q}}.$$

2nd, $\left(\sqrt[\frac{p}{q}]{a^{\frac{m}{n}}}\right)^{\frac{p}{q}}=a^{\frac{m}{n}}=\sqrt[n]{a^m}$, or $\sqrt[q]{\left(\sqrt[\frac{p}{q}]{a^{\frac{m}{n}}}\right)^p}=\sqrt[n]{a^m}$,

$$\therefore\ \left(\sqrt[\frac{p}{q}]{a^{\frac{m}{n}}}\right)^p=(\sqrt[n]{a^m})^q=\sqrt[n]{a^{mq}},\ (8),\ \text{Cor.}$$

$$\therefore\ \sqrt[\frac{p}{q}]{a^{\frac{m}{n}}}=\sqrt[p]{\sqrt[n]{a^{mq}}}=\sqrt[np]{a^{mq}}=a^{\frac{mq}{np}}=a^{\frac{m}{n}\div\frac{p}{q}}.$$

3rd, $(a^{\frac{m}{n}})^{-\frac{p}{q}}=\frac{1}{(a^{\frac{m}{n}})^{\frac{p}{q}}}=\frac{1}{a^{\frac{mp}{nq}}}$ by 1st case, $=a^{-\frac{mp}{nq}}=a^{\frac{m}{n}\cdot(-\frac{p}{q})}$.

4th, Let $\sqrt[-\frac{p}{q}]{a^{\frac{m}{n}}}=x$, then $x^{-\frac{p}{q}}=a^{\frac{m}{n}}$, $\sqrt[q]{x^{-p}}=\sqrt[n]{a^m}$,

$$x^{-p}=(\sqrt[n]{a^m})^q=\sqrt[n]{a^{mq}},\quad \frac{1}{x^p}=\sqrt[n]{a^{mq}},\quad x^p=\frac{1}{\sqrt[n]{a^{mq}}},$$

$$\therefore\ x=\sqrt[p]{\frac{1}{\sqrt[n]{a^{mq}}}}=\frac{1}{\sqrt[p]{\sqrt[n]{a^{mq}}}}=\frac{1}{\sqrt[np]{a^{mq}}}=\frac{1}{a^{\frac{mq}{np}}}=a^{-\frac{mq}{np}}=a^{\frac{m}{n}\div(-\frac{p}{q})}.$$

5th, $(a^{-\frac{m}{n}})^{-\frac{p}{q}}=\sqrt[q]{(a^{-\frac{m}{n}})^{-p}}=\sqrt[q]{\frac{1}{(a^{-\frac{m}{n}})^p}}=\frac{1}{\sqrt[q]{a^{-\frac{mp}{n}}}}=\frac{1}{\sqrt[q]{\sqrt[n]{a^{-mp}}}}$,

$$=\frac{1}{\sqrt[nq]{a^{-mp}}}=\frac{1}{\sqrt[nq]{\frac{1}{a^{mp}}}}=\frac{1}{\frac{1}{a^{\frac{mp}{nq}}}}=a^{\frac{mp}{nq}}=a^{(-\frac{m}{n})\cdot(-\frac{p}{q})}.$$

6th, Let $\sqrt[-\frac{p}{q}]{a^{-\frac{m}{n}}}=x$, then $x^{-\frac{p}{q}}=a^{-\frac{m}{n}}$, $\frac{1}{x^{\frac{p}{q}}}=\frac{1}{a^{\frac{m}{n}}}$, $\therefore\ x^{\frac{p}{q}}=a^{\frac{m}{n}}$,

$$\sqrt[\frac{p}{q}]{x^{\frac{p}{q}}}=x=\sqrt[\frac{p}{q}]{a^{\frac{m}{n}}}=a^{\frac{m}{n}\div\frac{p}{q}}\ \text{by 2nd case},\ =a^{(-\frac{m}{n})\div(-\frac{p}{q})};$$

which proves the Rules *for all possible cases of fractional Indices.*

Thus we have proved that the Definitions and Fundamental Rules are perfectly compatible, and that no error can arise from giving to the Rules the most general application.

It is not meant that negative and fractional *indices* can really represent the *powers* of any quantity, but simply that they may be treated *as such* in all algebraic operations without error.

REDUCTION OF SURDS.

163. *A rational quantity may be reduced to the form of a given surd by raising it to the power whose root the surd expresses and affixing the radical sign.*

Thus $a = \sqrt{a^2} = \sqrt[3]{a^3}$, &c. and $a + x = (a+x)^{\frac{m}{m}} = \sqrt[m]{(a+x)^m}$.

In the same manner, the form of any surd may be altered; thus $(a+x)^{\frac{1}{2}} = (a+x)^{\frac{2}{4}} = (a+x)^{\frac{3}{6}}$, &c. The quantities are here raised to certain powers, and the roots of those powers are again taken; therefore the values of the quantities are not altered.

164. *The coefficient of a surd may be introduced under the radical sign by first reducing it to the form of the surd, by the last* Art., *and then multiplying according to* Art. 149.

Exs. $a\sqrt{x} = \sqrt{a^2} \times \sqrt{x} = \sqrt{a^2 x}$;

$$ay^{\frac{3}{2}} = (a^2 y^3)^{\frac{1}{2}}; \qquad x\sqrt{2a - x} = \sqrt{2ax^2 - x^3};$$

$$a \times (a - x)^{\frac{3}{2}} = \{a^2 \times (a - x)^3\}^{\frac{1}{2}};$$

$$4\sqrt{2} = \sqrt{16 \times 2} = \sqrt{32}.$$

165. *Conversely, any quantity may be made the coefficient of a surd, if every part under the sign be divided by this quantity raised to the power whose root the sign expresses.*

Thus $\sqrt{a^2 - ax} = a^{\frac{1}{2}}\sqrt{a - x}$; $\sqrt{a^3 - a^2 x} = a\sqrt{a - x}$;

$$(a^2 - x^2)^{\frac{1}{n}} = a^{\frac{2}{n}}\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{n}}; \quad \sqrt{60} = \sqrt{4 \times 15} = 2\sqrt{15};$$

$$\left(\frac{1}{b^2} - \frac{1}{x^2}\right)^{\frac{1}{2}} = \frac{1}{b}\sqrt{1 - \frac{b^2}{x^2}}.$$

ADDITION AND SUBTRACTION OF SURDS.

166. *When surds have the same irrational part, their sum or difference is found by affixing to that irrational part the sum or difference of their coefficients.*

Thus $a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$;

$$10\sqrt{3} \pm 5\sqrt{3} = 15\sqrt{3}, \text{ or } 5\sqrt{3}.$$

If the proposed surds have not the same irrational part, they may sometimes be reduced to others which have, by Art. 165. Thus,

Ex. 1. Let the sum of $\sqrt{3a^2b}$ and $\sqrt{3b}$ be required.

Since $\sqrt{3a^2b} = \sqrt{a^2} \times \sqrt{3b} = a\sqrt{3b}$,

$\therefore \sqrt{3a^2b} + \sqrt{3b} = a\sqrt{3b} + \sqrt{3b} = (a+1)\sqrt{3b}$.

Ex. 2. Find the sum of $4a\sqrt[3]{a^3b^4}$, $b\sqrt[3]{8a^6b}$, and $-\sqrt[3]{125a^6b^4}$.

$$\text{Here } 4a\sqrt[3]{a^3b^4} = 4a\sqrt[3]{a^3b^3}\,.\,\sqrt[3]{b} = 4a^2b\sqrt[3]{b},$$
$$b\sqrt[3]{8a^6b} = b\sqrt[3]{8a^6}\,.\,\sqrt[3]{b} = 2a^2b\sqrt[3]{b},$$
$$-\sqrt[3]{125a^6b^4} = -\sqrt[3]{125a^6b^3}\,.\,\sqrt[3]{b} = -5a^2b\sqrt[3]{b};$$
$$\therefore \text{ the sum required} = a^2b\sqrt[3]{b}.$$

If the proposed surds cannot be reduced to others which have the same irrational part, then they must be connected together merely by the signs + and −.

[*Exercises* K*.]

MULTIPLICATION OF SURDS.

167. *If two surds have the same index, their product is found by taking the product of the quantities under the signs and retaining the common index.*

Thus $\sqrt[n]{a} \times \sqrt[n]{b} = a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$ (Art. 149) $= \sqrt[n]{ab}$.

$$\sqrt{2} \times \sqrt{3} = \sqrt{6}; \qquad (a+b)^{\frac{1}{2}} \times (a-b)^{\frac{1}{2}} = (a^2-b^2)^{\frac{1}{2}};$$
$$\sqrt[3]{a+x} \times \sqrt[3]{a-x} = \sqrt[3]{a^2-x^2}.$$

If the surds have coefficients, the product of these coefficients must be prefixed.

Thus $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$.

168. *If the indices of two surds have a common denominator, let the quantities be raised to the powers expressed by their respective numerators, and their product may be found as before.*

Ex. $2^{\frac{3}{2}}\times 3^{\frac{1}{2}} = (2^3)^{\frac{1}{2}}\times 3^{\frac{1}{2}} = 8^{\frac{1}{2}}\times 3^{\frac{1}{2}} = (24)^{\frac{1}{2}}$;

also $(a+x)^{\frac{1}{2}}\times(a-x)^{\frac{3}{2}} = \{(a+x)(a-x)^3\}^{\frac{1}{2}}$.

169. *If the indices have not a common denominator, they may be transformed to others of the same value with a common denominator, and their product found as in* Art. 168.

Ex. $(a^2-x^2)^{\frac{1}{4}}\times(a-x)^{\frac{1}{2}} = (a^2-x^2)^{\frac{1}{4}}\times(a-x)^{\frac{2}{4}}$,

$= \{(a^2-x^2)\times(a-x)^2\}^{\frac{1}{4}}$;

again $2^{\frac{1}{2}}\times 3^{\frac{1}{3}} = 2^{\frac{3}{6}}\times 3^{\frac{2}{6}} = (8\times 9)^{\frac{1}{6}} = (72)^{\frac{1}{6}}$.

170. *If two surds have the same rational quantity under the radical signs, their product is found by making the sum of the indices the index of that quantity.*

Thus $\sqrt[n]{a}\times\sqrt[m]{a} = a^{\frac{1}{n}}\times a^{\frac{1}{m}} = a^{\frac{m}{mn}}\times a^{\frac{n}{mn}} = a^{\frac{m+n}{mn}}$; (see Art. 162).

Ex. $\sqrt{2}\times\sqrt[3]{2} = 2^{\frac{1}{2}}\times 2^{\frac{1}{3}} = 2^{\frac{3}{6}+\frac{2}{6}} = 2^{\frac{5}{6}}$.

[*Exercises* L.]

DIVISION OF SURDS.

171. *If the indices of two quantities have a common denominator, the quotient of one divided by the other is obtained by raising them respectively to the powers expressed by the numerators of their indices, and extracting that root of the quotient which is expressed by the common denominator.*

For $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}$; and $\dfrac{a^{\frac{m}{n}}}{b^{\frac{p}{n}}} = \dfrac{(a^m)^{\frac{1}{n}}}{(b^p)^{\frac{1}{n}}} = \left(\dfrac{a^m}{b^p}\right)^{\frac{1}{n}}$ (Art. 150).

Ex. $4^{\frac{1}{2}}\div 2^{\frac{3}{2}} = \left(\dfrac{4}{8}\right)^{\frac{1}{2}} = \dfrac{1}{\sqrt{2}}$; $\left(\dfrac{p}{q}\right)^{\frac{1}{m}}\div\left(\dfrac{r}{s}\right)^{\frac{2}{m}} = \left(\dfrac{ps^2}{qr^2}\right)^{\frac{1}{m}}$.

172. *If the indices have not a common denominator, reduce them to others of the same value with a common denominator, and proceed as before.*

Ex. $(a^2-x^2)^{\frac{1}{2}}\div(a^3-x^3)^{\frac{1}{3}} = (a^2-x^2)^{\frac{3}{6}}\div(a^3-x^3)^{\frac{2}{6}}$,

$= \left\{\dfrac{(a^2-x^2)^3}{(a^3-x^3)^2}\right\}^{\frac{1}{6}}$.

173. ***If two surds have the same rational quantity under the radical signs, their quotient is obtained by making the difference of the indices the index of that quantity.***

Thus $\sqrt[n]{a} \div \sqrt[m]{a}$, or $a^{\frac{1}{n}}$ divided by $a^{\frac{1}{m}}$, or $a^{\frac{m}{mn}}$ divided by $a^{\frac{n}{mn}}$, that is $\dfrac{a^{\frac{m}{mn}}}{a^{\frac{n}{mn}}}$, is equal to $a^{\frac{m-n}{mn}}$; because these quantities, raised to the power mn, produce equal results $\dfrac{a^m}{a^n}$ and a^{m-n}.

Ex. $2^{\frac{1}{2}} \div 2^{\frac{1}{3}} = 2^{\frac{3}{6}} \div 2^{\frac{2}{6}} = 2^{\frac{1}{6}}$.

[*Exercises* M.]

INVOLUTION AND EVOLUTION OF SURDS.

174. *Any power of a surd is found by multiplying the fractional index of the surd by the number which expresses the power.*

For $(\sqrt[n]{a})^m = (a^{\frac{1}{n}})^m = a^{\frac{1}{n}+\frac{1}{n}+\dots\text{to } m \text{ terms}} = a^{\frac{m}{n}}$.

175. *Any root of a surd is found by dividing the fractional index of the surd by the number which expresses the root.*

Thus $\sqrt[m]{(\sqrt[n]{a})} = \sqrt[m]{a^{\frac{1}{n}}} = a^{\frac{1}{mn}}$; because each of these quantities raised to the m^{th} power will produce $a^{\frac{1}{n}}$.

It will be seen that the rules hitherto required for the management of surds are simply those which apply to quantities raised to powers expressed by *Fractional Indices.*

[*Exercises* N.]

TRANSFORMATION OF SURDS.

176. *Having given a quantity containing quadratic surds, to find another quantity which, multiplied into the former, shall produce a rational result.*

1. If the given quantity be a simple surd, as $3\sqrt{a}$, the multiplier required is $\sqrt{a}$, which gives the product $3a$, a rational quantity.

2. If the given quantity be a binomial surd, as $\sqrt{a}+\sqrt{b}$, then the multiplier required is $\sqrt{a}-\sqrt{b}$, and the product is $a-b$.

3. If the quantity be a trinomial, as $\sqrt{a}+\sqrt{b}+\sqrt{c}$, first multiply by $\sqrt{a}+\sqrt{b}-\sqrt{c}$, which gives $(\sqrt{a}+\sqrt{b})^2-(\sqrt{c})^2$, or $a+b-c+2\sqrt{ab}$. Next multiply by $a+b-c-2\sqrt{ab}$, and the product is $(a+b-c)^2-4ab$. Therefore the multiplier required is $(\sqrt{a}+\sqrt{b}-\sqrt{c})\times(a+b-c-2\sqrt{ab})$.

The use of this proposition is to enable us without much labour to find the values of fractions which have irrational *denominators.* Thus, suppose the actual value of $\frac{1}{\sqrt{3}+\sqrt{2}}$ were required to 7 places of decimals; if we were to proceed to extract the square roots of 2 and 3, and divide 1 by the sum of those roots, the operation would be long and troublesome. But if we first multiply the numerator and denominator by $\sqrt{3}-\sqrt{2}$, the fraction becomes $\frac{\sqrt{3}-\sqrt{2}}{3-2}$, or $\sqrt{3}-\sqrt{2}$, and its value is not altered; we have simply then to extract the square roots of 3 and 2 and subtract the one root from the other, by which the long division is entirely avoided.

$$\text{So also } \frac{\sqrt[3]{2}}{\sqrt[3]{3}} = \frac{\sqrt[3]{2}\times\sqrt[3]{3}\times\sqrt[3]{3}}{(\sqrt[3]{3})^3} = \frac{\sqrt[3]{18}}{3},$$

$$\text{and } \frac{\sqrt[r]{a}}{\sqrt[r]{b}} = \frac{a^{\frac{1}{r}}\times b^{\frac{r-1}{r}}}{b^{\frac{1}{r}}\times b^{\frac{r-1}{r}}} = \frac{\sqrt[r]{ab^{r-1}}}{b};$$

each of which fractions is thus much simplified for purposes of calculation.

Again, $\frac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a} = \frac{1}{\sqrt[3]{x^2}+\sqrt[3]{ax}+\sqrt[3]{a^2}}$; but the former quantity is in a simpler form than the latter.

More generally,

177. *To find the multiplier which will rationalize* any *binomial, having one or both of its terms irrational.*

1st. Let $x+y$ represent the binomial, x and y being, one or both, irrational, and let m be such a number that x^m and y^m are both rational, that is, let m be the Least Com. Mult. of the denominators of the fractional indices of the binomial; then since

$$x^m \pm y^m = (x+y).(x^{m-1}-x^{m-2}y+\ldots\mp xy^{m-2}\pm y^{m-1}),$$

where the upper or lower sign is to be taken, according as m is odd or even, the rationalizing multiplier required is

$$x^{m-1}-x^{m-2}y+\ldots\mp xy^{m-2}\pm y^{m-1}.$$

2nd. Let $x-y$ be the binomial, and m as before, then since $x^m-y^m=(x-y).(x^{m-1}+x^{m-2}y+\ldots+xy^{m-2}+y^{m-1})$, (Art. 99, Ex. 6) the rationalizing multiplier is

$$x^{m-1}+x^{m-2}y+\ldots+xy^{m-2}+y^{m-1}.$$

Ex. Find the multiplier which will rationalize $\sqrt{5}-\sqrt[3]{6}$, or $5^{\frac{1}{2}}-6^{\frac{1}{3}}$.

Here $m=6$, the Least Com. Mult. of 2 and 3,

$$\therefore \text{mult}^{\text{r}}.\ \text{req}^{\text{d}}. = (\sqrt{5})^5 + (\sqrt{5})^4 \times \sqrt[3]{6} + (\sqrt{5})^3 \times (\sqrt[3]{6})^2 + (\sqrt{5})^2 \times (\sqrt[3]{6})^3 + \sqrt{5} \times (\sqrt[3]{6})^4 + (\sqrt[3]{6})^5,$$

$$= 25\sqrt{5} + 25\sqrt[3]{6} + 5\sqrt{5} \times (\sqrt[3]{6})^2 + 30 + 6\sqrt{5} \times \sqrt[3]{6} + 6(\sqrt[3]{6})^2.$$

[*Exercises* O.]

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178. *The square root of a quantity cannot be partly rational and partly a quadratic surd.*

If possible, let $\sqrt{n} = a + \sqrt{m}$; then, by squaring these equal quantities, $n = a^2 + 2a\sqrt{m} + m$, (Art. 81); and $2a\sqrt{m} = n - a^2 - m$, (Art. 80); therefore $\sqrt{m} = \frac{n - a^2 - m}{2a}$, (Art. 82), a rational quantity, which is contrary to the supposition.

179. *If any two quantities, partly rational and partly quadratic surds, be equal to one another, the rational parts of the two are equal, and also the irrational parts.*

Let $x + \sqrt{y} = a + \sqrt{b}$, then $x = a$, and $\sqrt{y} = \sqrt{b}$; for if x be not equal to a, let $x = a + m$; then $a + m + \sqrt{y} = a + \sqrt{b}$, or $m + \sqrt{y} = \sqrt{b}$; that is, $\sqrt{b}$ is partly rational and partly a quadratic surd, which is impossible (Art. 178); therefore $x = a$, and consequently also $\sqrt{y} = \sqrt{b}$.

180. *If two quadratic surds* $\sqrt{x}$ *and* $\sqrt{y}$ *cannot be reduced to others which have the same irrational part, their product is irrational.*

If possible, let $\sqrt{xy} = rx$, where r is a whole number or a fraction. Then $xy = r^2x^2$ (Art. 81), and $y = r^2x$ (Art. 82); therefore $\sqrt{y} = r\sqrt{x}$, that is, $\sqrt{y}$ and $\sqrt{x}$ may be so reduced as to have the same irrational part, which is contrary to the supposition.

181. *One quadratic surd,* $\sqrt{x}$, *cannot be made up of two others,* $\sqrt{m}$ *and* $\sqrt{n}$, *which have not the same irrational part.*

If possible, let $\sqrt{x} = \sqrt{m} + \sqrt{n}$; then by squaring these equal quantities, $x = m + n + 2\sqrt{mn}$, and $x - m - n = 2\sqrt{mn}$, a rational quantity equal to an irrational one; which is absurd.

182. ***The square root of a binomial, one of whose terms is a quadratic surd, and the other rational, may sometimes be expressed by a binomial, one or both of whose terms are quadratic surds.***

Since $(\sqrt{x}\pm\sqrt{y})^2=x+y\pm2\sqrt{xy}$, $\sqrt{x+y\pm2\sqrt{xy}}=\sqrt{x}\pm\sqrt{y}$; hence if any proposed binomial surd can be put under the form $(x+y)\pm2\sqrt{xy}$ its square root is at once found by inspection to be $\sqrt{x}\pm\sqrt{y}$. Now, to proceed with any proposed case, take the term which contains the surd, and if it can be put into factors of the form $2\sqrt{x}\times\sqrt{y}$ in one or more ways, take that pair of factors for which the sum of x and y is equal to the whole of the term in the proposed binomial which is rational. Having thus found x and y, the square root required is $\sqrt{x}\pm\sqrt{y}$, + or − according as the sign of the *surd* in the proposed binomial is + or −.

Ex. 1. Required the square root of $3+2\sqrt{2}$.

Here $2\sqrt{2}=2\sqrt{2}\times\sqrt{1}$;

also $2+1=3$, the rational term;

$\therefore$ the root required is $\sqrt{2}+1$.

Ex. 2. Required the square root of $7-2\sqrt{10}$.

Here $2\sqrt{10}=2\sqrt{5}\times\sqrt{2}$,

also $5+2=7$, the rational term;

$\therefore$ root required is $\sqrt{5}-\sqrt{2}$.

Ex. 3. Required the square root of $11-6\sqrt{2}$.

Here $6\sqrt{2}=2\sqrt{18}=2\sqrt{9}\times\sqrt{2}$, or $2\sqrt{6}\times\sqrt{3}$; of which the former answers the condition required, viz. $9+2=11$, the rational term;

$\therefore$ the root required is $\sqrt{9}-\sqrt{2}$, that is, $3-\sqrt{2}$.

Ex. 4. Required the square root of $2x+2\sqrt{x^2-1}$.

Here $2\sqrt{x^2-1}=2\sqrt{x+1}\times\sqrt{x-1}$,

also $x+1+x-1=2x$, the rational term;

$\therefore$ root required is $\sqrt{x+1}+\sqrt{x-1}$.

Ex. 5. Required the square root of $7+\sqrt{13}$.

Here $\sqrt{13}=2\sqrt{\frac{13}{4}}=2\sqrt{\frac{13}{2}}\times\sqrt{\frac{1}{2}}$, also $\frac{13}{2}+\frac{1}{2}=7$;

$\therefore$ root required is $\frac{\sqrt{13}+1}{\sqrt{2}}$.

A more general method of extracting the roots of irrational binomials will be given hereafter.

[*Exercises* P.]

IMAGINARY OR IMPOSSIBLE QUANTITIES.

183. If an expression appear under the form $\sqrt{-a}$, this indicates an impossibility; for it signifies the square root of a negative quantity, which has no existence, since there is no quantity, positive or negative, which, being multiplied by itself, gives a negative product. It is evident, therefore, that if such symbols are admitted into calculations, they may require special Rules for themselves, and some care in the application of those Rules. And it must be borne in mind, that they are mere symbols, and not expressions of quantity.

We must observe especially, if we do meet with such *imaginary* quantities, that the product of $\sqrt{-a}\times\sqrt{-a}$ does not follow the rule given in Art. 167, but that it is $-a$, because it is that quantity whose square root is $\sqrt{-a}$. Also similarly $\sqrt{-a}\times\sqrt{-b}$ is not $\sqrt{-a\times-b}$, or $\sqrt{ab}$, but it is $\sqrt{a}\times\sqrt{-1}\times\sqrt{b}\times\sqrt{-1}$, or $\sqrt{ab}\times(\sqrt{-1})^2$, that is, $-\sqrt{ab}$.

To avoid mistakes in operating upon *imaginary* quantities, as $\sqrt{-a}$, $\sqrt{-b}$, &c., it will be best in all cases to substitute for them their equivalents $\sqrt{a}.\sqrt{-1}$, $\sqrt{b}.\sqrt{-1}$, &c., and to bear in mind that $(\sqrt{-1})^2=-1$, $(\sqrt{-1})^3=-\sqrt{-1}$, $(\sqrt{-1})^4=+1$, &c.; generally $(\sqrt{-1})^{4n}=1$, $(\sqrt{-1})^{4n+1}=\sqrt{-1}$, $(\sqrt{-1})^{4n+2}=-1$, $(\sqrt{-1})^{4n+3}=-\sqrt{-1}$.

Ex. 1.
$$\begin{aligned}(x-a+\sqrt{-b^2})(x-a-\sqrt{-b^2})&=(x-a+b\sqrt{-1})(x-a-b\sqrt{-1}),\\&=(x-a)^2-(b\sqrt{-1})^2,\\&=x^2-2ax+a^2+b^2.\end{aligned}$$

Ex. 2.
$$\frac{a-b\sqrt{-1}}{a+b\sqrt{-1}}=\frac{(a-b\sqrt{-1})^2}{a^2+b^2}=\frac{a^2-b^2}{a^2+b^2}-\frac{2ab}{a^2+b^2}\sqrt{-1}.$$

Ex. 3.
$$\sqrt[4]{-1}=\sqrt{0+\sqrt{-1}}=\sqrt{0+2\sqrt{-\frac{1}{4}}}=\sqrt{0+2\sqrt{\frac{1}{2}\times-\frac{1}{2}}}.$$

And
$$\frac{1}{2}-\frac{1}{2}=0;\ \therefore\ \sqrt{0+\sqrt{-1}},\ \text{or}\ \sqrt[4]{-1}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\sqrt{-1}.$$

EQUATIONS.

184. If one quantity be equal to another, or to nothing, and this equality be expressed algebraically, it constitutes an *Equation*. Thus $x-a=b-x$ is an *Equation*, of which $x-a$ forms one *side*, and $b-x$ the other.

In this equation it is asserted, that a certain *unknown quantity* (x) is so connected with two known quantities (a and b), that it exceeds the one (a) by as much as it falls short of the other (b).

Again, $x^2+x-20=0$ is an *Equation*, which asserts that, if a certain unknown number be added to its square, and the sum be diminished by 20, the result is 0.

An equality which admits of no question, as $x+a=x+a$, or $2x+3x=5x$, is not an "*Equation*" strictly speaking, but is called an "*Identity.*" An *Identity* is therefore satisfied by *any value whatever* of the unknown quantity; whereas in "*Equations*" the unknown quantities have *particular values*, which alone, and none other, will permit the expressed equality to subsist.—To find these values is to "*Solve*" the Equations, and forms an important part of the business of Algebra. These values are sometimes called the "*Roots*" of the Equations, and are said to *satisfy* them. Thus, if $2x=6$ be the Equation, $x=3$, and can be nothing else; and $x=3$ is called its *solution*. Again, if $x^2=4$, we know that $x=2$, or -2; and $2, -2$, are called the *Roots* of the equation $x^2=4$, or $x^2-4=0$.

185. *When an equation is cleared of fractions and surds,* if it contain the first power only of an unknown quantity, it is called a *Simple Equation,* or an equation of one dimension; if the *square* of the unknown quantity be in any term, (and there be no higher power,) it is called a *Quadratic,* or an equation of two dimensions; if the *Cube* of the unknown quantity appear, (and no higher power,) it is called a *Cubic Equation;* if the fourth power, a *Biquadratic;* and in general, if the index of the highest power of the unknown quantity be n, it is called *an equation of* n *dimensions.*

SIMPLE EQUATIONS.

186. RULE I. *In any equation quantities may be transposed from one* side *to the other, if their signs be changed, and the two sides will still be equal.*

For let $x+10=15$; then by subtracting 10 from each side, (Art. 80), $x+10-10=15-10$, or $x=15-10$.

Let $x-4=6$; by adding 4 to each side, (Art. 79),

$$x-4+4=6+4, \text{ or } x=6+4.$$

If $x-a+b=y$; adding $a-b$ to each side,

$$x-a+b+a-b=y+a-b; \text{ or } x=y+a-b.$$

187. COR. Hence, if the signs of *all* the terms on each side be changed, the two sides will still be equal.

$$\text{Let } x-a=b-2x;$$

$$\text{by transposition, } -b+2x=-x+a;$$

$$\text{or } a-x=2x-b.$$

188. **Rule II.** *If every term on each side be multiplied by the same quantity, the results will be equal.* (Art. 81.)

189. **Cor.** An equation may be cleared of fractions, by multiplying every term successively by the denominators of those fractions.

$$\text{Let } 3x + \frac{5x}{4} = 34;$$

multiplying by 4, $12x + 5x = 136$.

An equation may be cleared of fractions *at once*, by multiplying both sides by the product of all the denominators, or by any quantity which is a multiple of them all.

$$\text{Let } \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 13;$$

multiplying by $2 \times 3 \times 4$, $3 \times 4 \times x + 2 \times 4 \times x + 2 \times 3 \times x = 2 \times 3 \times 4 \times 13$,

or $12x + 8x + 6x = 312$; that is, $26x = 312$.

If the Least Common Multiple of the denominators be made use of, the equation will be *in the lowest terms*.

Thus, if each side of the last equation be multiplied by 12, which is the Least Com. Mult. of 2, 3, and 4, the equation will become

$$\frac{12x}{2} + \frac{12x}{3} + \frac{12x}{4} = 156;$$

or $6x + 4x + 3x = 156$; that is, $13x = 156$.

190. **Rule III.** *If each side of an equation be divided by the same quantity, the results will be equal.* (Art 82.)

Let $17x = 136$; then $x = \frac{136}{17} = 8$.

191. **Rule IV.** *If each side of an equation be raised to the same power, the results will be equal.* (Art. 81.)

Let $x^{\frac{1}{2}} = 9$; then $x = 9 \times 9 = 81$.

Also, if the same root be extracted on both sides, the results will be equal.

Let $x = 81$; then $x^{\frac{1}{2}} = \pm 9$ (Art. 143).

192. **Rule V.** *To clear an equation of surds.*

An equation may be cleared of a surd by transposing the terms so that the surd shall form one side, and the rational quantities the other, and then raising both sides to that power which will rationalize the surd.

Thus, if $\sqrt{a+x}-b=c$, by transposition $\sqrt{a+x}=b+c$, and $a+x=(b+c)^2$. (Art. 81.)

If the equation contain *two* surds, connected by + or −, then the same operation must be repeated for the second surd.

$$
\begin{aligned}
\text{Thus, if}\quad & \sqrt{a+x}+\sqrt{x}=b,\\
\text{by transp.}\quad & \sqrt{a+x}=b-\sqrt{x},\\
\text{squaring,}\quad & a+x=b^2-2b\sqrt{x}+x,\\
\text{by transp.}\quad & 2b\sqrt{x}=b^2-a,\\
\text{squaring,}\quad & 4b^2x=(b^2-a)^2,
\end{aligned}
$$

an equation in which the surds do not appear.

193. *A "simple equation" can have only one solution; that is, there can be but one value of the unknown quantity which satisfies it.*

For every "simple equation" with respect to the unknown quantity x can be reduced to the form $ax+b=0$. Now, if possible, let there be two values of x which satisfy this equation, viz. α and β;

$$
\begin{aligned}
\text{then}\quad & a\alpha+b=0,\\
\text{and}\quad & a\beta+b=0;\\
\therefore\ \text{subtracting,}\quad & a\alpha-a\beta=0,\\
\text{or}\quad & a(\alpha-\beta)=0.
\end{aligned}
$$

But a cannot be equal to 0, for then the proposed equation would be no equation at all with respect to x, therefore $\alpha-\beta=0$, or $\alpha=\beta$; that is, α and β cannot be different values; or there is only one value of x which satisfies the equation. If, however, it be known, that α is not equal to β, *i.e.* that the proposed equation has two different roots, the equation $a(\alpha-\beta)=0$ cannot subsist unless $a=0$, and then also $b=0$; *i.e.* the equality $ax+b=0$ ceases to be an *equation*, and becomes an *identity*, the coefficient of x and the other term becoming *separately* equal to 0.

194. *To find the value of the unknown quantity in a simple equation.*

Let the equation first be *cleared* of fractions and surds* (Arts. 189, 192), then *transpose* all the terms which involve the unknown

---

* It should be borne in mind that this is required to be done only when *the unknown quantity* is found in a fraction or surd. Thus it will not be necessary in such equations as the following:—

$$2x+\frac{4}{5}=x+\frac{6}{7}.$$

$$nx+\sqrt{a}=mx+\sqrt{b}.\text{—Ed.}$$

quantity to one side of the equation, and the known quantities to the other (Art. 186); *divide* both sides by the coefficient, or sum of the coefficients, of the unknown quantity (Art. 190), and the value required is obtained.

Ex. 1. To find the value of x in the equation $3x - 5 = 23 - x$.

By transp. $3x + x = 23 + 5$, (Art. 186),

or $4x = 28$;

by division $x = \frac{28}{4} = 7$. (Art. 190.)

Ex. 2. Let $x + \frac{x}{2} - \frac{x}{3} = 4x - 17$; required x.

Mult. by 2, $2x + x - \frac{2x}{3} = 8x - 34$,

mult. by 3, $6x + 3x - 2x = 24x - 102$, (Art. 189),

by transp. $6x + 3x - 2x - 24x = -102$,

or $-17x = -102$,

$17x = 102$, (Art. 187);

$\therefore\ x = \frac{102}{17} = 6$.

Ex. 3. $\frac{1}{a} + \frac{b}{x} = c$; required x.

Mult. by a, $1 + \frac{ba}{x} = ca$,

mult. by x, $x + ba = cax$,

by transp. $x - cax = -ba$,

$cax - x = ba$, (Art. 187);

$(ca - 1)x = ba$;

$\therefore\ x = \frac{ba}{a - 1}$.

Ex. 4. $5 - \frac{x + 4}{11} = x - 3$; required x.

$55 - x - 4^{*} = 11x - 33$,

$55 - 4 + 33 = 11x + x$,

$84 = 12x$;

$\therefore\ x = \frac{84}{12} = 7$.

Ex. 5. $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$; required x.

$$2x+3x-5=24-\frac{4x-8}{3},$$

$$6x+9x-15=72-4x+8^*,$$

$$6x+9x+4x=72+8+15,$$

$$19x=95;$$

$$\therefore\ x=\frac{95}{19}=5.$$

Ex. 6. $\frac{7x+8}{8}-\frac{9x-12}{16}=\frac{3x+1}{10}-\frac{29-8x}{20}$; required x.

Here the Least Com. Mult. of the denominators is 80, (Art. 23); therefore, multiplying both sides by this number, (Art. 189),

$$70x+80-45x+60=24x+8-116+32x^*,$$

$$70x-45x-24x-32x=8-116-80-60,$$

$$-31x=-248,$$

$$\therefore\ x=\frac{-248}{-31}=8.$$

Ex. 7. $\frac{1}{14}\left(3x+\frac{2}{3}\right)-\frac{1}{7}(4x-6\tfrac{2}{3})=\frac{1}{2}(5x-6)$; required x.

Mult. by 14, $3x+\frac{2}{3}-8x+12\tfrac{4}{3}=35x-42,$

$$42+12\tfrac{4}{3}+\frac{2}{3}=35x+8x-3x,$$

$$42+12+2=40x,\ \because\ \frac{4}{3}+\frac{2}{3}=2,$$

$$40x=56;$$

$$\therefore\ x=\frac{56}{40}=\frac{7}{5}=1\tfrac{2}{5}.$$

---

* See Art. 87, bearing in mind that the line which separates the numerator and denominator of a fraction serves as a *vinculum* for both.—Ed.

Ex. 8. $\frac{x-4\frac{2}{3}}{3}-\frac{2x-3\frac{2}{3}}{4}=\frac{3}{2}\left\{x-\frac{x-1\frac{1}{2}}{2}\right\}+\frac{4x}{3}\left\{x-3-\frac{(x-1)(x-2)}{x}\right\}$; required x.

Mult. by 12, the L.C.M. of 3, 4, and 2,

$$4x-16\tfrac{8}{3}-6x+11=18\,.\,\frac{x+1\frac{1}{2}}{2}+16x\,.\,\frac{x^2-3x-(x^2-3x+2)}{x},$$

$$=9x+13\tfrac{1}{2}-32,$$

$$32+11-16\tfrac{8}{3}-13\tfrac{1}{2}=9x+6x-4x,$$

$$11x=14-\frac{8}{3}-\frac{1}{2}=12-\frac{2}{3}-\frac{1}{2}=\frac{65}{6};$$

$$\therefore\ x=\frac{65}{66}.$$

Ex. 9. $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{16x+15}{28}+\frac{2\frac{1}{4}}{7}$; required x*.

Mult. by 28, $16x+10+\frac{196x-84}{6x+2}=16x+15+9,$

$$\frac{196x-84}{6x+2}=24-10=14,$$

$$196x-84=84x+28,$$

$$112x=112;$$

$$\therefore\ x=1.$$

Ex. 10. $\frac{ad-bc}{d(c+dx)}+\frac{b}{d}=\frac{2a-bx}{c+dx}$; required x.

Mult. by d, $\frac{ad-bc}{c+dx}+b=\frac{2ad-bdx}{c+dx},$

$$b=\frac{2ad-bdx-(ad-bc)}{c+dx},$$

$$=\frac{ad-bdx+bc}{c+dx},$$

$$bc+bdx=ad-bdx+bc,$$

$$2bdx=ad;$$

$$\therefore\ x=\frac{ad}{2bd}=\frac{a}{2b}.$$

[*Exercises* Q.]

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* In cases like this, which have one or more *compound* denominators involving the unknown quantity, it will usually be found convenient to clear the equation of the *simple* denominators first, leaving the fractions with compound denominators to be dealt with afterwards, when the equation has been reduced to fewer terms.

Ex. 11. $\sqrt{a+x}+\sqrt{a-x}=2\sqrt{x}$; required x.

$$\sqrt{a+x}=2\sqrt{x}-\sqrt{a-x},$$
$$a+x=4x-4\sqrt{ax-x^2}+a-x,$$
$$4\sqrt{ax-x^2}=2x,$$
$$2\sqrt{ax-x^2}=x,$$
$$4ax-4x^2=x^2,$$
$$4ax=5x^2,$$
$$4a=5x;$$
$$\therefore\ x=\frac{4a}{5}.$$

Ex. 12. $\sqrt[m]{a+x}=\sqrt[2m]{x^2+5ax+b^2}$; required x.

Raising both sides to the $2m^{\text{th}}$ power, we have (Art. 174)

$$(a+x)^2=x^2+5ax+b^2,$$
$$\text{or } a^2+2ax+x^2=x^2+5ax+b^2,$$
$$3ax=a^2-b^2;$$
$$\therefore\ x=\frac{a^2-b^2}{3a}.$$

195. A very useful formula in solving equations is the following:—

If a, b, c, d be any quantities whatever, and if $\frac{a}{b}=\frac{c}{d}$,

$$\text{then } \frac{a+b}{a-b}=\frac{c+d}{c-d}; \text{ and } \frac{a-b}{a+b}=\frac{c-d}{c+d}.$$

To prove this, $\frac{a}{b}=\frac{c}{d}$;

$$\therefore\ \frac{a}{b}+1=\frac{c}{d}+1, \text{ or } \frac{a+b}{b}=\frac{c+d}{d}.$$

$$\text{Again } \frac{a}{b}-1=\frac{c}{d}-1, \text{ or } \frac{a-b}{b}=\frac{c-d}{d};$$

$$\therefore\ \frac{a+b}{b}\div\frac{a-b}{b}=\frac{c+d}{d}\div\frac{c-d}{d} \text{ (Art. 82)}$$

$$\text{or } \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Cor. Hence also $\frac{a-b}{a+b}=\frac{c-d}{c+d}$.

Obs. In the application of this formula to cases in which one side of the equation is *a whole number*, the whole number must be considered as a fraction with 1 for its denominator.

It is also useful to express the formula in language, viz. that, *if one fraction be equal to another, the sum of the numerator and denominator divided by their difference, or the difference divided by their sum, for one fraction, is equal to the same expression for the other fraction.*

Ex. 1. $\frac{x+2}{x-2}=\frac{7}{5}$; required the value of x.

By the formula, $$\frac{2x}{4}=\frac{12}{2},$$

$$\text{or } \frac{x}{2}=6;$$

$$\therefore\ x=12.$$

Ex. 2. Given $$\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{a}-\sqrt{a-x}}=\frac{1}{a};\ \text{required } x.$$

By formula, $$\frac{2\sqrt{a}}{2\sqrt{a-x}}=\frac{1+a}{1-a},$$

$$\text{or } \sqrt{\frac{a}{a-x}}=\frac{1+a}{1-a};$$

$$\text{squaring, } \frac{a}{a-x}=\left(\frac{1+a}{1-a}\right)^2,$$

$$\text{or } \frac{a-x}{a}=\left(\frac{1-a}{1+a}\right)^2,$$

$$\text{or } 1-\frac{x}{a}=\left(\frac{1-a}{1+a}\right)^2,$$

$$\frac{x}{a}=1-\frac{1-2a+a^2}{1+2a+a^2}=\frac{4a}{(1+a)^2};$$

$$\therefore\ x=\frac{4a^2}{(1+a)^2}=\left(\frac{2a}{1+a}\right)^2.$$

Ex. 3. Given $$\frac{\sqrt[3]{x+1}-\sqrt[3]{x-1}}{\sqrt[3]{x+1}+\sqrt[3]{x-1}}=\frac{1}{2};\ \text{required } x.$$

By formula, $$\frac{\sqrt[3]{x+1}}{\sqrt[3]{x-1}}=\frac{3}{1};$$

$$\text{cubing, } \frac{x+1}{x-1}=27,$$

$$\text{by formula again, } x=\frac{28}{26}=1\tfrac{1}{13}.$$

N.B. As a general Rule it is not advisable to apply the formula when the unknown quantity occurs on *both* sides of the equation; for if the one side be simplified by it, the other side will often be rendered more complex, and so nothing be gained. Thus it is obvious, that no good purpose will be served by applying the formula to the equation

$$\frac{a+x}{a-x}=\frac{x}{a}.$$

195*. Another useful method of solving equations can be deduced from the following property of fractions:—

If any number of fractions be equal to one another,

$$\text{each}=\frac{\text{sum of } \textit{any multiples whatever} \text{ of their numerators}}{\text{sum of } \textit{the same multiples} \text{ of their denominators}}.$$

Let $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, &c. be the fractions, which are equal to one another; and let each be equal to r; then

$$a=br,\ c=dr,\ e=fr,\ \&\text{c.}$$

$$ma=mbr,\ nc=ndr,\ pe=pfr,\ \&\text{c.}$$

$$\therefore\ ma+nc+pe+\&\text{c.}=(mb+nd+pf+\&\text{c.})r,$$

$$\therefore\ r=\text{each fraction}=\frac{ma+nc+pe+\&\text{c.}}{mb+nd+pf+\&\text{c.}}.$$

And it will be observed that m, n, p, &c. may be any quantities whatever, whole or fractional, positive or negative.

Cor. Similarly, if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\&\text{c.}$, each fraction is equal to

$$\left\{\frac{a^n+c^n+e^n+\&\text{c.}}{b^n+d^n+f^n+\&\text{c.}}\right\}^{\frac{1}{n}},\ \text{for all values of } n.$$

Ex. $$\frac{26x+6\frac{1}{9}}{25}=\frac{27x+10\frac{1}{9}}{26};\ \text{required } x.$$

By formula, $\frac{26x+6\frac{1}{9}}{25}=\frac{27x+10\frac{1}{9}-(26x+6\frac{1}{9})}{26-25}$, (where $n=1$, and $m=-1$,)

$$=x+4,$$

$$26x+6\tfrac{1}{9}=25x+100,$$

$$\therefore\ x=93\tfrac{8}{9}.$$

If this method be applied, a troublesome multiplication may often be avoided, and the solution will be effected with greater ease and elegance.

The Student is referred to the *Appendix* for various other methods which may be usefully employed in particular cases.

[*Exercises* R.]

196. If there be *two independent* simple equations involving two unknown quantities, they may be reduced to one which involves only one of the unknown quantities, by any of the following methods:—

First method. In either equation find the value of one of the unknown quantities in terms of the other and known quantities, and for it substitute this value in the other equation, which will then only contain one unknown quantity, whose value may be found by the rules before laid down.

Ex. $$\left.\begin{aligned} x+y &= 10, \\ 2x-3y &= 5, \end{aligned}\right\} \text{ to find } x \text{ and } y.$$

From the first equation $x = 10 - y$, hence $2x = 20 - 2y$; by substitution in the second $20 - 2y - 3y = 5$,

$$20 - 5 = 2y + 3y,$$

$$15 = 5y;$$

$$\therefore\ y = \frac{15}{5} = 3.$$

$$\text{Hence also}\quad x = 10 - y = 10 - 3 = 7.$$

Second method. Find an expression for one of the unknown quantities in each equation; put these expressions equal to each other, and from the resulting equation the other unknown quantity may be found.

Ex. $$\left.\begin{aligned} x+y &= a, \\ bx+cy &= de, \end{aligned}\right\} \text{ to find } x \text{ and } y.$$

From the first equation $x = a - y$,

from the second $bx = de - cy$, and $x = \dfrac{de-cy}{b}$;

$$\therefore\ a - y = \frac{de-cy}{b},$$

$$ba - by = de - cy,$$

$$cy - by = de - ba,$$

$$(c-b)y = de - ba;$$

$$\therefore y = \frac{de - ba}{c - b}.$$

$$\begin{aligned} \text{Also} \quad x &= a - y; \\ &= a - \frac{de - ba}{c - b}, \\ &= \frac{ca - ba - de + ba}{c - b}, \\ &= \frac{ca - de}{c - b}. \end{aligned}$$

Third method. If either of the unknown quantities have the same coefficient in both equations, it may be exterminated by subtracting or adding the equations, according as the sign of the unknown quantity, in the two cases, is the same or different.

Ex. $\left.\begin{aligned} x + y &= 15, \\ x - y &= 7, \end{aligned}\right\}$ to find x and y.

By addition, $2x = 22$, $\therefore x = 11$.

By subtraction, $2y = 8$, $\therefore y = 4$.

If the coefficients of the unknown quantity to be exterminated be *different*, multiply the terms of the first equation by the coefficient of the unknown quantity in the second, and the terms of the second equation by the coefficient of the same unknown quantity in the first; then add, or subtract, the resulting equations, as in the former case.

This is the method in most general use.

Ex. 1. $\left.\begin{aligned} 3x - 5y &= 13, \\ 2x + 7y &= 81, \end{aligned}\right\}$ to find x and y.

Multiply the terms of the first equation by 2, and the terms of the second by 3, then

$$\left.\begin{aligned} 6x - 10y &= 26, \\ \text{and} \quad 6x + 21y &= 243, \end{aligned}\right\}$$

$$\text{by subtraction,} \quad 31y = 217;$$

$$\therefore y = \frac{217}{31} = 7.$$

Also $3x - 5y = 13,$

or $3x - 35 = 13,$

$$3x = 13 + 35 = 48;$$

$$\therefore x = \frac{48}{3} = 16.$$

Ex. 2. $\left.\begin{aligned} ax + by &= c, \\ mx - ny &= d, \end{aligned}\right\}$ to find x and y.

From the first $max + mby = mc,$

from the second $max - nay = ad,$

by subtraction, $(mb + na)y = mc - ad;$

$$\therefore y = \frac{mc - ad}{mb + na}.$$

Again $nax + nby = nc,$

$$mbx - nby = bd;$$

by addition, $(na + mb)x = nc + bd;$

$$\therefore x = \frac{nc + bd}{na + mb}.$$

Ex. 3. $\left.\begin{aligned} \frac{3x - 5y}{2} + 3 &= \frac{2x + y}{5}, \\ 8 - \frac{x - 2y}{4} &= \frac{x}{2} + \frac{y}{3}, \end{aligned}\right\}$ to find x and y.

From the first $15x - 25y + 30 = 4x + 2y,$

$$15x - 4x - 25y - 2y = -30,$$

$$11x - 27y = -30.$$

From the second $96 - 3x + 6y = 6x + 4y,$

$$96 = 6x + 3x + 4y - 6y;$$

$$\left.\begin{aligned} 9x - 2y &= 96, \\ \text{and} \quad 11x - 27y &= -30, \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{hence} \quad 99x - 22y &= 1056, \\ \text{and} \quad 99x - 243y &= -270, \end{aligned}\right\}$$

$$221y = 1326;$$

$$\therefore y = \frac{1326}{221} = 6.$$

$$\text{Also}\quad 9x - 2y = 96,$$
$$9x - 12 = 96,$$
$$9x = 96 + 12 = 108;$$
$$\therefore x = \frac{108}{9} = 12.$$

Fourth method. Sometimes by adding the two equations together, or subtracting one from the other, an equation is obtained, which linked to *one* of the given equations, leads to a speedy solution. Thus,

Ex. 1. $\left.\begin{aligned}54x-121y&=15,\\ 36x-\ \ 77y&=21,\end{aligned}\right\}$ to find x and y.

Subtract, $18x-44y=-6,$

$\left.\begin{aligned}\text{multiply by 2,}\quad 36x-88y&=-12,\\ \text{from 2nd equation,}\quad 36x-77y&=21,\end{aligned}\right\}$

subtract, $11y=33, \therefore y=3.$

Also $18x=44y-6=132-6=126, \therefore x=7.$

Ex. 2. $\left.\begin{aligned}101x-24y&=63,\\ 103x-28y&=29,\end{aligned}\right\}$ to find x and y. Ans. $\left.\begin{aligned}x&=3,\\ y&=10.\end{aligned}\right\}$

[*Exercises* S.]

197. If there be three *independent* simple equations, and three unknown quantities, reduce two of the equations to one, containing only two of the unknown quantities, by the preceding rules; then reduce the third equation and either of the former to one, containing the same two unknown quantities; and from the two equations thus obtained the unknown quantities which they involve may be found. The third quantity may be found by substituting their values in any of the proposed equations.

Ex. $\left.\begin{aligned}2x+3y+4z&=16,\\ 3x+2y-5z&=8,\\ 5x-6y+3z&=6,\end{aligned}\right\}$ to find x, y and z.

From the 1^{st} two equ^{ns}. $\left.\begin{aligned}6x+9y+12z&=48,\\ 6x+4y-10z&=16,\end{aligned}\right\}$

by subtr. $5y+22z=32.$

From the 1^{st} and 3^{rd} $\left.\begin{aligned}10x+15y+20z&=80,\\ 10x-12y+\ \ 6z&=12,\end{aligned}\right\}$

$$\text{by subtr.}\quad \left.\begin{aligned} 27y+14z&=68,\\ \text{and}\quad 5y+22z&=32,\end{aligned}\right\}$$

$$\left.\begin{aligned} \text{hence}\quad 135y+70z&=340,\\ \text{and}\quad 135y+594z&=864,\end{aligned}\right\}$$

$$\text{by subtr.}\quad 524z=524;$$

$$\therefore z=1.$$

$$\text{Also}\quad 5y+22z=32,$$

$$\text{that is,}\quad 5y+22=32,$$

$$5y=32-22=10;$$

$$\therefore y=\frac{10}{5}=2.$$

$$\text{Also}\quad 2x+3y+4z=16,$$

$$\text{that is,}\quad 2x+6+4=16,$$

$$2x=16-6-4=6;$$

$$\therefore x=3.$$

The same method may be applied to any number of *independent* simple equations, in which the number of unknown quantities is the same as the number of equations.

[*Exercises* T.]

Another method.

$$\left.\begin{aligned} a_1x+b_1y+c_1z&=d_1{}^*\ldots\ldots(1)\\ a_2x+b_2y+c_2z&=d_2\ \ \ldots\ldots(2)\\ a_3x+b_3y+c_3z&=d_3\ \ \ldots\ldots(3)\end{aligned}\right\};\ \text{to find } x,\ y, \text{ and } z.$$

Multiply (2) by m, (3) by n, and to the resulting equations add (1); then we have

$$(a_1+ma_2+na_3)x+(b_1+mb_2+nb_3)y+(c_1+mc_2+nc_3)z=d_1+md_2+nd_3.$$

Now to find x, let the *arbitrary* multipliers m and n be such that the coefficients of y and z in this last equation are separately equal to 0; that is,

$$\left.\begin{aligned} b_1+mb_2+nb_3&=0,\\ \text{and}\quad c_1+mc_2+nc_3&=0,\end{aligned}\right\},$$

---

* The small figures here give no particular *values* to the quantities to which they are annexed, a_1 and a_2 being as different as a and b; but it is often convenient to use the same letter thus slightly varied to mark some common meaning of such letters, and thereby assist the memory. Thus in this instance, a_1, a_2, a_3, have this common property, viz. that all are coefficients of x, a_1 in the 1st, a_2 in the 2nd, and a_3 in the 3rd, equation. Similarly for the coefficients of y and z.

$$\left.\begin{array}{r}\text{or } b_1c_3+mb_2c_3+nb_3c_3=0,\\ \text{and } b_3c_1+mb_3c_2+nb_3c_3=0,\end{array}\right\},$$

$$\text{and} \therefore b_1c_3-b_3c_1+(b_2c_3-b_3c_2)m=0,$$

$$\text{or } m=\frac{b_3c_1-b_1c_3}{b_2c_3-b_3c_2};$$

$$\text{and similarly } n=\frac{b_1c_2-b_2c_1}{b_2c_3-b_3c_2}.$$

Then we have $x=\dfrac{d_1+md_2+nd_3}{a_1+ma_2+na_3}$, in which the above values may be substituted for m and n.

Similarly, by making the coefficients of x and z, or of x and y, separately equal to 0, the value of y, or of z, may be found.

As the denominators of m and n are the same, the following Rule may hence be deduced, and will be found easy of application :—

To find x, multiply the 1st equation by $b_2c_3-b_3c_2$, the 2nd by $b_3c_1-b_1c_3$, and the 3rd by $b_1c_2-b_2c_1$; then add together the resulting equations, and a simple equation will be obtained in which y and z do not appear.

A similar rule may be stated for finding either y or z; or having found the value of x, the equations are reduced to simple equations of two unknown quantities y and z, so that y and z may be found by any of the methods of Art. 196.

Ex. Given $$\left.\begin{array}{l}2x+3y+4z=16\ldots(1)\\ 3x+2y-5z=\ \ 8\ldots(2)\\ 5x-6y+3z=\ \ 6\ldots(3)\end{array}\right\}; \text{ required } x.$$

Here $b_2c_3-b_3c_2=6-30=-24$,

$b_3c_1-b_1c_3=-24-9=-33$,

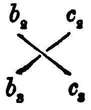
$b_1c_2-b_2c_1=-15-8=-23$;

$$\left.\begin{array}{l}\therefore \text{ from } (1)\ -48x-\ 72y-\ 96z=-384,\\ \ldots\ldots\ (2)\ -99x-\ 66y-165z=-264,\\ \ldots\ldots\ (3)-115x+138y-\ 69z=-138,\end{array}\right\}$$

$\therefore$ adding, and changing signs, $262x=786$,

$$\text{or } x=\frac{786}{262}=3.$$

This Rule is called Cross Multiplication, because the multipliers are formed by taking the coefficients in a cross order, thus:



in applying it, care must be taken that the order of the suffixes is properly kept. It is very useful in several branches of the higher mathematics, Solid Geometry for example, where it is extensively employed.

197*. In some cases the method of Art. 195* may be successfully employed; especially in the case where the ratios of x, y, z, can be obtained: for example, if

$$\left.\begin{aligned} 5x&=4y,\\ 3x&=4z,\\ 2x-3y+6z&=22,\end{aligned}\right\}$$

$$\text{we have then } \frac{x}{4}=\frac{y}{5}=\frac{z}{3},$$

$$\text{and } \therefore \text{ each } =\frac{2x-3y+6z}{2\times4-3\times5+6\times3}$$

$$=\frac{22}{11}=2.$$

$$\therefore x=8,\ y=10,\ z=6.$$

The advantage of this method consists in finding the values of all the unknown quantities at once.

198. That the unknown quantities may have definite values, there must be as many *independent* equations as unknown quantities. When there are *more* equations than unknown quantities, the value of any one of these quantities may be determined from different equations; and should the values thus found differ, the equations are incongruous; should they be the same, one at least of the equations is unnecessary. When there are *fewer* equations than unknown quantities, one of these quantities cannot be found, but in terms which involve some of the rest, whose values may be assumed *at pleasure;* and in such cases the number of answers is indefinite. Thus, if $x+y=a$, then $x=a-y$; and assuming y *at pleasure,* we obtain a value of x such, that $x+y=a$.

These equations must also be *independent,* that is, not deducible one from another.

Let $x+y=a$, and $2x+2y=2a$; these are not *independent* equations, since the latter equation being deducible from the former, it involves no different suppositions, nor requires any thing more for its truth, than that $x+y=a$ should be a just equation.

It is sometimes, however, not easy to discover *at once* whether proposed equations be *independent* or not. Thus in the equations

$$\left.\begin{aligned} x+3y+4z&=9,\\ 3x-2y+17z&=25,\\ x+14y-z&=11,\end{aligned}\right\}$$

it is not obvious at first sight that the third equation is derived from the other two. But by multiplying the first equation by 4, and subtracting the second, the result is the third equation; and accordingly the usual process being applied to find x, y, z, would certainly fail.

As examples of *incongruous* equations, the following may be instanced, $x+y=7$, and $3x+3y=30$, from which we get $7=10$; or, again $x+y=7$, $3x-y=1$, and $x+2y=10$, from which we get $14=12$.

PROBLEMS WHICH PRODUCE SIMPLE EQUATIONS.

199. From certain quantities which are known to investigate others which have a given relation to them is the business of Algebra.

When a question is proposed to be resolved, we must first consider fully its meaning and conditions. Then substituting one or more of the symbols, x, y, z &c. for such unknown quantities as appear most convenient, we must proceed as if they were already determined, and we wished to try whether they answer all the proposed conditions or not, till as many independent equations arise as we have assumed unknown quantities, which will always be the case if the question be properly limited (Art. 198); and by the solution of these equations the quantities sought will be determined.

PROB. 1. A bankrupt owes *A* twice as much as he owes *B*, and *C* as much as he owes *A* and *B* together; out of £300, which is to be divided amongst them, what must each receive?

Let x represent what *B* must receive, in pounds;

then $2x$ = what *A* must receive,

and $x+2x$, or $3x$ = what *C* must receive;

amongst them they receive £300; therefore

$$x+2x+3x=300,$$

$$6x=300;$$

$$\therefore x=\frac{300}{6}=\text{£}50.\text{ what } B \text{ must receive.}$$

$$2x=\text{£}100.\text{ what } A \text{ must receive.}$$

$$3x=\text{£}150.\text{ what } C \text{ must receive.}$$

Prob. 2. To divide a line of 15 inches into two such parts, that one may be three-fourths of the other.

Let x be the number of inches in one part,

$$\text{then } \frac{3x}{4} = \dots\dots\dots\dots\dots\dots \text{ the other,}$$

$$x + \frac{3x}{4} = 15, \text{ by the question,}$$

$$4x + 3x = 60,$$

$$7x = 60;$$

$$\therefore\ x = \frac{60}{7} = 8\tfrac{4}{7}, \text{ one part;}$$

$$\text{and } \frac{3x}{4} = \frac{3}{4} \times \frac{60}{7} = \frac{45}{7} = 6\tfrac{3}{7}, \text{ the other part.}$$

Prob. 3. If A can perform a piece of work in 8 days, and B the same in 10 days, in what time will they finish it together?

Let x be the time required, in days; and w the work.

In one day A performs $\frac{1}{8}$th part of the work, or $\frac{w}{8}$; therefore in x days he performs $\frac{xw}{8}$. And in the same time B performs $\frac{xw}{10}$.

$$\text{Therefore } \frac{xw}{8} + \frac{xw}{10} = w, \text{ by the question,}$$

$$\text{or } \frac{x}{8} + \frac{x}{10} = 1,$$

$$10x + 8x = 80,$$

$$18x = 80;$$

$$\therefore\ x = \frac{80}{18} = 4\tfrac{8}{18} = 4\tfrac{4}{9} \text{ days.}$$

Prob. 4. A workman was employed for 60 days, on condition that for every day he worked he should receive 15 pence, and for every day he played he should forfeit 5 pence; at the end of the time he had 20 shillings to receive; required the number of days he worked.

Let x be the number of days he worked,
then $60 - x$ is the number he played,
$15x =$ his pay, in pence,
$(60 - x) \times 5 = 300 - 5x =$ sum forfeited;
$15x - 300 + 5x = 240$, by the question,
$20x = 240 + 300 = 540$;
$\therefore x = 27$, the number of days he worked,
$60 - x = 33$, the number of days he played.

Prob. 5. How much rye, at four shillings and sixpence a bushel, must be mixed with 50 bushels of wheat, at six shillings a bushel, that the mixture may be worth five shillings a bushel?

Let x be the number of bushels required;
then $9x =$ the price of the rye in sixpences,
$600 =$ the price of the wheat in sixpences,
$(50 + x) \times 10 =$ the price of the mixture..............
$\therefore 9x + 600 = 500 + 10x$,
$100 = x$, the number of bushels required.

Prob. 6. A and B engage together in play; in the first game A wins as much as he had and four shillings more, and finds he has twice as much as B; in the second game B wins half as much as he had at first and one shilling more, and then it appears that he has three times as much as A; what sum had each at first?

Let x be what A had, in shillings,
y what B had.
Then $2x + 4 =$ what A has after the first game;
$y - x - 4 =$ what B has;
$\therefore$ by the question, $2x + 4 = 2y - 2x - 8$,
or $2y - 4x = 12$,
$y - 2x = 6$.

Also $y - x - 4 + \frac{y}{2} + 1 =$ what B has after the second game;

$$2x + 4 - \frac{y}{2} - 1 = \text{what } A \text{ has};$$

$\therefore$ by the question, $y - x - 4 + \frac{y}{2} + 1 = 6x + 12 - \frac{3y}{2} - 3$,

$$\text{or } y + \frac{y}{2} + \frac{3y}{2} - x - 6x = 12 - 3 + 4 - 1,$$

$$\left.\begin{array}{rr} \text{or} & 3y - 7x = 12, \\ \text{also} & y - 2x = 6, \end{array}\right\}$$

$$\left.\begin{array}{rr} \therefore & 3y - 6x = 18, \\ \text{and} & 3y - 7x = 12, \end{array}\right\}$$

$\therefore$ by subtraction, $x = 6$, what A had at first;

and $y - 2x = 6$, or $y - 12 = 6$;

$\therefore y = 18$, what B had.

Prob. 7. A smuggler had a quantity of brandy which he expected would raise £9. 18*s.*; after he had sold 10 gallons, a revenue officer seized one third of the remainder, in consequence of which he makes only £8. 2*s.*; required the number of gallons he had, and the price per gallon.

Let x be the number of gallons;

then $\frac{198}{x}$ is the price per gallon, in shillings,

$\frac{x-10}{3}$ the quantity seized,

and $\frac{x-10}{3} \times \frac{198}{x}$ the value of the quantity seized, which appears, by the question, to be 36 shillings;

$$\therefore \frac{x-10}{3} \times \frac{198}{x} = 36,$$

$$(x - 10) \times 66 = 36x,$$

$$66x - 660 = 36x,$$

$$30x = 660;$$

$$\therefore x = 22, \text{ the number of gallons;}$$

$$\text{and } \frac{198}{x} = \frac{198}{22} = 9 \text{ shillings, the price per gallon.}$$

PROB. 8. *A* and *B* play at bowls, and *A* bets *B* three shillings to two upon every game; after a certain number of games it appears that *A* has won three shillings; but had he ventured to bet five shillings to two, and lost one game more out of the same number, he would have lost thirty shillings: how many games did they play?

Let x be the number of games A won,

y the number B won,

then $2x$ is what A won of B, in shillings,

and $3y$ what B won of A;

$\therefore 2x - 3y = 3$, by the question.

Also $(x-1)\times 2$ is what A would win on the 2^{nd} supposition,

and $(y+1)\times 5$ what B would win ....

$\therefore 5y + 5 - 2x + 2 = 30$, by the question,

or $5y - 2x = 30 - 5 - 2 = 23$;

$$\left.\begin{aligned} \therefore 5y - 2x &= 23, \\ \text{and } 2x - 3y &= 3, \end{aligned}\right\}$$

by addition, $5y - 3y = 26$,

$2y = 26$;

$\therefore y = 13$.

And $2x = 3 + 3y = 3 + 39 = 42$;

$\therefore x = 21$;

and $x + y = 34$, the number of games required.

PROB. 9. A sum of money was divided equally amongst a certain number of persons; had there been three more, each would have received one shilling less, and had there been two fewer, each would have received one shilling more, than he did: required the number of persons, and what each received.

Let x be the number of persons,

y the sum each received, in shillings;

then xy is the sum divided,

$$\left.\begin{aligned} \text{and } (x+3)\times(y-1) &= xy \\ \text{also } (x-2)\times(y+1) &= xy \end{aligned}\right\} \text{by the question;}$$

$$\therefore xy - x + 3y - 3 = xy, \text{ or } -x + 3y = 3;$$

$$\text{and } xy + x - 2y - 2 = xy, \text{ or } \quad x - 2y = 2;$$

$$\therefore y = 5 \text{ shillings, the sum received by each.}$$

$$\text{And } x - 2y = x - 10 = 2,$$

$$\therefore x = 12, \text{ the number of persons.}$$

[*Exercises* U.]

QUADRATIC EQUATIONS.

200. When the terms of an equation involve the square of the unknown quantity, but the first power does not appear, the value of the square is obtained by the preceding rules*; and by extracting the square root on both sides, the quantity itself is found

Such equations are called *Pure* Quadratics.

Ex. 1. $5x^2 - 45 = 0$; to find x.

$$\text{By transp.} \quad 5x^2 = 45,$$

$$x^2 = 9;$$

$$\therefore (\text{Art. } 191),\ x = \sqrt{9} = \pm 3.$$

The signs + and − are both prefixed to the root, because the square root of a quantity may be either positive or negative (Art. 147). The sign of x may also be negative; but still x will be either equal to $+3$ or -3†.

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* It is obvious that the rules proved in Arts. 186...192, apply to *all* equations, *quadratic*, *cubic*, &c. as well as *simple*, because they are founded simply upon the *Axioms* (Arts. 79...82).—Ed.

† This may be shewn as follows:—suppose $x^2 = a^2$, then extracting the square root of both sides, since $\sqrt{x^2} = \pm x$, and $\sqrt{a^2} = \pm a$, we have

$$+x = +a \ldots\ldots (1),$$
$$+x = -a \ldots\ldots (2),$$
$$-x = +a \ldots\ldots (3),$$
$$-x = -a \ldots\ldots (4).$$

But it is evident that (1) and (4) are in fact the same equation and also (2) and (3); so that $x = \pm a$ includes all the four equations.—Ed.

Ex. 2. $ax^2 = bcd$; to find x.

$$x^2 = \frac{bcd}{a}; \qquad \therefore x = \pm\sqrt{\frac{bcd}{a}}.$$

201. If both the first and second powers of the unknown quantity be found in an equation, arrange the terms according to the dimensions of the unknown quantity, beginning with the highest, and transpose the known quantities to the other side; then, if the square of the unknown quantity be affected with a coefficient, divide all the terms by this coefficient, and if its sign be negative, change the signs of all the terms (Art. 187), that the equation may be reduced to one of the forms, $x^2 \pm px = \pm q$. Then add to both sides the square of half the coefficient of the first power of the unknown quantity, by which means the first side of the equation is made a complete square, (Art. 152), and the other consists of known quantities; and by extracting the square root of both sides, a simple equation is obtained, from which the value of the unknown quantity may be found.

Such equations are called *Adfected*[*] Quadratics.

Ex. 1. Let $x^2 + px = q$; now we know that $x^2 + px + \frac{p^2}{4}$ is the square of $x + \frac{p}{2}$ (Art. 152); add therefore $\frac{p^2}{4}$ to both sides, and we have

$$x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4};$$

then by extracting the square root of both sides,

$$x + \frac{p}{2} = \pm\sqrt{q + \frac{p^2}{4}};$$

and by transposition, $x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}$.

In the same manner, if $x^2 - px = q$,

$$x = \frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

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[*] The term *adfected*, or *affected*, was introduced by Vieta, about the year 1600. It is used to distinguish equations, which involve, or are *affected* with, different powers of the unknown quantity, from those which contain one power only, and which are therefore called *pure*.—ED.

Ex. 2. $x^2-12x+35=0$; to find x.

By transposition, $x^2-12x=-35$; and adding the square of $\frac{12}{2}$ or 6, to both sides of the equation,

$$x^2-12x+36=36-35=1;$$

then extracting the square root of both sides,

$$x-6=\pm 1;$$

$$\therefore x=6\pm 1=7, \text{ or } 5;$$

either of which, substituted for x in the original equation, answers the condition, that is, makes the whole expression, $x^2-12x+35$, equal to nothing.

Ex. 3. $$\frac{6}{x+1}+\frac{2}{x}=3; \text{ to find } x.$$

$$6+\frac{2x+2}{x}=3x+3,$$

$$6x+2x+2=3x^2+3x,$$

$$3x^2-5x=2,$$

$$x^2-\frac{5x}{3}=\frac{2}{3},$$

$$x^2-\frac{5x}{3}+\left(\frac{5}{6}\right)^2=\frac{2}{3}+\frac{25}{36},$$

$$=\frac{24}{36}+\frac{25}{36}=\frac{49}{36},$$

$$x-\frac{5}{6}=\pm\frac{7}{6};$$

$$\therefore x=\frac{5\pm 7}{6},$$

$$=2, \text{ or } -\frac{1}{3}.$$

202. Ex. 4. $x+\sqrt{5x+10}=8$; to find x.

By transp. $\sqrt{5x+10}=8-x$,

squaring, $5x+10=64-16x+x^2$,

$$x^2-21x=-54;$$

$$x^2 - 21x + \frac{441}{4} = \frac{441}{4} - 54,$$
$$= \frac{225}{4};$$
$$x - \frac{21}{2} = \pm\frac{15}{2};$$
$$\therefore x = \frac{21 \pm 15}{2} = 18, \text{ or } 3.$$

By this process two values of x are found; but on trial it appears, that 18 does not answer the conditions of the equation, if we suppose that $\sqrt{5x+10}$ represents the positive square root of $5x+10$. The reason is, that $5x+10$ is the square of $-\sqrt{5x+10}$ as well as of $+\sqrt{5x+10}$; thus by squaring both sides of the equation $\sqrt{5x+10} = 8 - x$, a new condition is introduced, and a new value of the unknown quantity corresponding to it, which had no place before. Here 18 is the value which corresponds to the supposition that

$$x - \sqrt{5x+10} = 8.$$

It should be particularly observed, that since $+x\times+y$ is equal to $-x\times-y$, in the multiplication and involution of quantities new values are always introduced, which, if not again excluded by the nature of the question, will appear in the final equation.

OBS. In the above Example *one* of the values obtained for x satisfies the proposed equation, whilst the other does not. In some cases *both* values of x fail to satisfy the equation from which they are derived. The subject is fully discussed in the Scholium, p. 134.

[*Exercises* V.]

203. If a quadratic equation appear under any of the forms included in $ax^2 \pm bx = \pm c$, the left-hand side may be made a complete square, *without fractions*, and the equation solved, by another method, as follows :—

Multiply the whole equation by $4a$, that is, four times the coefficient of x^2, then we have

$$4a^2x^2 \pm 4abx = \pm 4ac;$$

add b^2, the square of the coefficient of x, then $4a^2x^2 \pm 4abx + b^2 = b^2 \pm 4ac$,

extract the square root, $2ax \pm b = \pm\sqrt{b^2 \pm 4ac}$;

$$\therefore x = \frac{\pm\sqrt{b^2 \pm 4ac} \mp b}{2a}.$$

Ex. 1. $\frac{6}{x-1}+\frac{2}{x}=3$; to find x.

$$6+\frac{2x+2}{x}=3x+3,$$

$$6x+2x+2=3x^2+3x,$$

$$3x^2-5x=2.$$

Multiplying by 4×3, or 12,

$$36x^2-60x=24,$$

adding 5^2, or 25, $36x^2-60x+25=24+25=49$;

$$\therefore\ 6x-5=\pm 7,$$

$$6x=5\pm 7=12, \text{ or } -2;$$

$$\therefore\ x=2, \text{ or} -\frac{1}{3}.$$

Ex. 2. $acx^2-bcx+adx=bd$; to find x.

Here $acx^2-(bc-ad)x=bd$;

multiply by $4ac$, $4a^2c^2x^2-4ac(bc-ad)x=4abcd$,

add $(bc-ad)^2$, $4a^2c^2x^2-4ac(bc-ad)x+(bc-ad)^2=(bc-ad)^2+4abcd$,

$$=(bc+ad)^2,$$

extract square root, $2acx-(bc-ad)=\pm(bc+ad)$,

$$2acx=bc-ad\pm(bc+ad),$$

$$=2bc, \text{ or } -2ad;$$

$$\therefore\ x=\frac{b}{a}, \text{ or } -\frac{d}{c}.$$

204. *A quadratic equation has no more than two distinct values of the unknown quantity which will satisfy it.*

For, if possible, let the equation $ax^2+bx+c=0$ have three distinct values of x, viz. α, β, γ.

$$\text{Then}\quad a\alpha^2+b\alpha+c=0\ldots(1),$$

$$a\beta^2+b\beta+c=0\ldots(2),$$

$$a\gamma^2+b\gamma+c=0\ldots(3).$$

Subtracting (2) from (1), $a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$;

$$\therefore\ a(\alpha+\beta)+b=0\ldots\ldots\ldots\ldots\text{(i.)}$$

Subtracting (3) from (1), $a(\alpha^2-\gamma^2)+b(\alpha-\gamma)=0$;

$$\therefore\ a(\alpha+\gamma)+b=0\ldots\ldots\ldots\ldots\text{(ii.)}$$

Subtracting (ii.) from (i.), $a(\beta-\gamma)=0\ldots\ldots\ldots\ldots$(iii.)

But a is not equal to 0, for otherwise the proposed equation would not be a quadratic equation;

$$\therefore \beta-\gamma=0,$$
$$\text{or } \beta=\gamma.$$

Hence a quadratic equation has not *three* distinct values of x, but it may have two.

COR. If however it be known that β is not equal to γ, that is, that the given equality is satisfied by more than two values of x, it appears from (iii.) that $a=0$; therefore by (i.) or (ii.), $b=0$; and by (1), (2), or (3), $c=0$; that is, if a quadratic equation be known to be satisfied by more than two values of the unknown quantity, the coefficients of the square, and of the first power of the unknown quantity, and the term which does not involve it, are separately equal to 0, and the equation becomes an *identity*, being satisfied by *any values whatever* of the unknown quantity.

205. *In any quadratic equation of the form* $x^2+px+q=0$, $-p=$ *the sum of the two values of* x, *and* $q=$ *their product.*

Let α, β, be the two values of x, then

$$\alpha^2+p\alpha+q=0,$$
$$\text{and } \beta^2+p\beta+q=0;$$
$$\therefore \alpha^2-\beta^2+p(\alpha-\beta)=0;$$
$$\therefore \alpha+\beta+p=0,$$
$$\text{or } -p=\alpha+\beta \ldots\ldots (1).$$
$$\text{Again, } q=-p\alpha-\alpha^2,$$
$$=(\alpha+\beta)\alpha-\alpha^2,$$
$$=\alpha\beta \ldots\ldots\ldots (2)^*.$$

COR. 1. If the equation be of the form $ax^2+bx+c=0$, then $x^2+\frac{b}{a}x+\frac{c}{a}=0$. Therefore, by what has been proved,

$$-\frac{b}{a}=\text{the sum of the two values of } x,$$

$$\text{and } \frac{c}{a}=\text{the product} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

COR. 2. Hence also, if α, β, be the *roots* of the equation $x^2+px+q=0$, it is proved that

$$x^2+px+q=x^2-(\alpha+\beta)x+\alpha\beta=0,$$
$$=(x-\alpha)(x-\beta)=0;$$

from which it appears that if one 'root' of the equation be known, the other may be found by *division;* for, α being known,

$$\frac{x^2+px+q}{x-\alpha}=x-\beta=0, \text{ which gives } x=\beta.$$

---

[*] The above proof is open to objection in the case where $\alpha=\beta$; but the relations in question can then be proved to be true by actually solving the equation, as is done in Art. 207. In such a case the equation is said to have *two equal roots.*

Again, conversely, if the *roots* of a quadratic equation be given, the *equation* can be found. For, if α and β be the roots, the equation must be

$$(x-\alpha)(x-\beta)=0,$$
$$\text{or } x^2-(\alpha+\beta)x+\alpha\beta=0.$$

Ex. Required the equation whose roots are 2 and 3.

$$\text{The equation is } (x-2)(x-3)=0,$$
$$\text{or} \quad x^2-5x+6=0.$$

COR. 3. If α, β, be the roots of the equation $x^2+px+q=0$, then $x^2+px+q=(x-\alpha)(x-\beta)$, whatever be the value of x.

For it has been proved that $p=-(\alpha+\beta)$, and $q=\alpha\beta$; therefore, whatever be the value of x,

$$x^2+px+q=x^2-(\alpha+\beta)x+\alpha\beta=(x-\alpha)(x-\beta)^*.$$

206. The results proved in the last Art. shewing the relation between the values of x and the coefficients in the equation, are of use in several ways; first, in enabling us to *verify* the solution of any quadratic equation; secondly, in determining the values of the unknown quantities when an equation is proposed in which certain relations are already known respecting those values; and lastly, in solving various problems, reduced to quadratic equations, of which it is necessary for our purpose to know no more than the sum or the product of the values of the unknown quantity. Thus,

Ex. 1. $3x^2-5x=2$. The values of x (See Art. 201, Ex. 3) are 2, and $-\frac{1}{3}$.

Now, without repeating the work, to see if these values are correct, we put the equation under the form $x^2-\frac{5}{3}x-\frac{2}{3}=0$, and since $2+\left(-\frac{1}{3}\right)=\frac{5}{3}$, and $2\times\left(-\frac{1}{3}\right)=-\frac{2}{3}$, we conclude, at once, that these and no other are the values required.

Ex. 2. $x^2-21x+54=0$; required the values of x, it being known that one of them is six times as great as the other.

Suppose α to represent one of the values,

then 6α is the other,

$$\text{and their sum, or } 7\alpha=21, \text{ (Art. 205)},$$
$$\therefore \alpha=3,$$

and the values required are 3 and 18.

---

[*] It must be borne in mind that we are here concerned not with the *equation* $x^2+px+q=0$, but with the *expression* x^2+px+q, irrespectively of anything that it may be equal to; but if any difficulty should arise, we might state the proposition thus: If α, β, be the roots of the equation $x^2+px+q=0$, then $y^2+py+q=(y-\alpha)(y-\beta)$ whatever be the value of y.

Ex. 3. If α and β be the values of x in the equation $ax^2+bx+c=0$, find the value of $\frac{1}{\alpha}+\frac{1}{\beta}$.

Dividing the equation by x^2, $a+\frac{b}{x}+\frac{c}{x^2}=0$,

$$\text{or } \frac{c}{x^2}+\frac{b}{x}+a=0. \quad \text{Assume } y=\frac{1}{x},$$

$$\text{then } cy^2+by+a=0,$$

$$\text{or } y^2+\frac{b}{c}y+\frac{a}{c}=0;$$

and the values of y are $\frac{1}{\alpha}$, $\frac{1}{\beta}$, since $y=\frac{1}{x}$,

$$\therefore \frac{1}{\alpha}+\frac{1}{\beta}=-\frac{b}{c}. \quad \text{(Art. 205.)}$$

207. Since every quadratic equation may be reduced to the form $x^2+px+q=0$, in which p and q may be positive or negative, we assume this as the *general* equation including every other.

Then, since $x^2+px=-q$,

$$x^2+px+\frac{p^2}{4}=\frac{p^2-4q}{4},$$

$$x+\frac{p}{2}=\pm\frac{1}{2}\sqrt{p^2-4q},$$

$$\therefore x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q};$$

$$\text{so then } -\frac{p}{2}+\frac{1}{2}\sqrt{p^2-4q}, \text{ and } -\frac{p}{2}-\frac{1}{2}\sqrt{p^2-4q},$$

are the only two values of x that will satisfy the equation.

Now, from this result, it follows,

1st. That there is no possible value of x, if $p^2<4q$.

2ndly. That the values of x are equal to each other, and each equal to $-\frac{p}{2}$, if $p^2=4q$.

3rdly. That there are two unequal values of x, whose sum is $-p$, if $p^2>4q$.

Again, it appears that

1st. If q be negative, since $\sqrt{p^2-4q}$ will then be greater than p and always possible, there can be but one *positive* value of x.

2ndly. If p be negative, q positive, and $p^2>4q$, there will be two *positive* values of x.

The last two conclusions may also be deduced from Art. 205.

Cor. 1. Similar conclusions may be drawn with regard to the equation $ax^2+bx+c=0$, by substituting $\frac{b}{a}$ for p, and $\frac{c}{a}$ for q. Thus, if a proposed equation be of the form $bx-ax^2=c$, in which a, b, c, are positive quantities, there will be two *positive* values of x, when $4ac<b^2$; and no possible value of x, when $4ac>b^2$.

Cor. 2. Hence also, if the roots of $ax^2+bx+c=0$ be equal,

$$\left(\frac{b}{a}\right)^2=4\frac{c}{a}, \text{ or } b^2=4ac;$$

$$\text{and } ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)=a\left(x+\frac{b}{2a}\right)^2;$$

and is, therefore, a perfect square for all values of x.

This might have been easily deduced from Cor. 3, Art. 205; for

$$ax^2+bx+c=a(x-\alpha)(x-\beta)=a(x-\alpha)^2, \because \alpha=\beta.$$

208. If an equation appear under the form

$$(x+a)X=0,$$

in which X represents an expression involving x, the unknown quantity; it is evident that either $x+a=0$, or $X=0$, that is, $x=-a$ is one solution of the equation, as well as those which are found by proceeding with $X=0$. So that, *whenever an equation is simplified by division, or the omission of a factor, common to all the terms, if the divisor or factor contain the unknown quantity, one solution at least of the equation will be found by putting that divisor or factor equal to* 0.

Thus, let $x^2+3x=7x$; the whole equation is divisible by x, therefore $x=0$ is one solution.

Again, let $x^2-5x+6=0$. This may be put under the form

$$(x-2)(x-3)=0;$$

$$\left.\begin{aligned} \therefore x-2=0, \text{ or } x=2 \\ \text{and } x-3=0, \text{ or } x=3 \end{aligned}\right\}; \text{ which are the only values of } x.$$

By this method, therefore, the necessity for solving a quadratic in the usual way may sometimes be superseded.

Ex. Given $(x-c)\sqrt{ab}-(a-b)\sqrt{cx}=0$; required x.

$$\text{Here } x\sqrt{ab}-c\sqrt{ab}-a\sqrt{cx}+b\sqrt{cx}=0;$$

$$\therefore \sqrt{bx}(\sqrt{ax}+\sqrt{bc})-\sqrt{ac}(\sqrt{ax}+\sqrt{bc})=0,$$

$$\text{or } (\sqrt{bx}-\sqrt{ac})(\sqrt{ax}+\sqrt{bc})=0;$$

$$\therefore \sqrt{bx}-\sqrt{ac}=0, \text{ and } \sqrt{ax}+\sqrt{bc}=0;$$

$$bx=ac, \text{ and } ax=bc,$$

$$\therefore x=\frac{ac}{b}, \text{ and } x=\frac{bc}{a}.$$

209. Every equation, where the unknown quantity is found in two terms, and its index in one is twice as great as in the other, may be resolved in the same manner as a quadratic.

Ex. 1. $z+4z^{\frac{1}{2}}=21$; required z.

$$z+4z^{\frac{1}{2}}+4=21+4=25,$$

$$z^{\frac{1}{2}}+2=\pm 5,$$

$$z^{\frac{1}{2}}=\pm 5-2=3, \text{ or } -7;$$

$$\therefore z=9, \text{ or } 49.$$

Ex. 2. $x^{-1}+x^{-\frac{1}{2}}=6$; required x.

$$x^{-1}+x^{-\frac{1}{2}}+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4},$$

$$x^{-\frac{1}{2}}+\frac{1}{2}=\pm\frac{5}{2},$$

$$x^{-\frac{1}{2}}=\frac{-1\pm 5}{2}=2, \text{ or } -3,$$

$$x^{\frac{1}{2}}=\frac{1}{2}, \text{ or } -\frac{1}{3};$$

$$\therefore x=\frac{1}{4}, \text{ or } \frac{1}{9}.$$

Ex. 3. $y^4-6y^2-27=0$; required y.

$$y^4-6y^2=27,$$

$$y^4-6y^2+9=27+9=36,$$

$$y^2-3=\pm 6,$$

$$y^2=3\pm 6=9, \text{ or } -3;$$

$$\therefore y=\pm 3, \text{ or } \pm\sqrt{-3}.$$

Ex. 4. $y^6 + ry^3 + \frac{q^3}{27} = 0$; required x.

$$y^6 + ry^3 = -\frac{q^3}{27},$$

$$y^6 + ry^3 + \frac{r^2}{4} = \frac{r^2}{4} - \frac{q^3}{27},$$

$$y^3 + \frac{r}{2} = \pm\sqrt{\frac{r^2}{4} - \frac{q^3}{27}},$$

$$y^3 = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}};$$

$$\therefore y = \sqrt[3]{-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}}.$$

210. Some other equations may be conveniently solved as quadratics, that is, by completing the square, when they can be made to assume the form

$$X^2 + pX = q,$$

X representing a compound expression involving the unknown quantity.

Ex. 1. $ax^2 + \sqrt{ax^2 - bx + c} = bx$; required x.

By transp. $ax^2 - bx + \sqrt{ax^2 - bx + c} = 0$,

adding c, $ax^2 - bx + c + \sqrt{ax^2 - bx + c} = c$,

completing the square,

$$(ax^2 - bx + c) + \sqrt{ax^2 - bx + c} + \frac{1}{4} = c + \frac{1}{4},$$

$$\sqrt{ax^2 - bx + c} + \frac{1}{2} = -\sqrt{c + \frac{1}{4}},$$

$$\sqrt{ax^2 - bx + c} = \frac{\pm\sqrt{4c+1} - 1}{2},$$

$$ax^2 - bx + c = \left\{\frac{\pm\sqrt{4c+1} - 1}{2}\right\}^2;$$

the equation is thus reduced to a common quadratic, from which x may be found by the usual method.

Ex. 2. $x^2 - x + 5\sqrt{2x^2 - 5x + 6} = \frac{1}{2}(3x + 33)$; required x.

Here $2x^2 - 2x + 10\sqrt{2x^2 - 5x + 6} = 3x + 33$,

$$2x^2-5x+6+10\sqrt{2x^2-5x+6}=39,$$

$$(2x^2-5x+6)+10\sqrt{2x^2-5x+6}+25=25+39=64,$$

$$\sqrt{2x^2-5x+6}+5=\pm 8,$$

$$\sqrt{2x^2-5x+6}=\pm 8-5=3, \text{ or } -13,$$

$$2x^2-5x+6=9, \text{ or } 169;$$

which leaves two common quadratics for solution, viz. $2x^2-5x=3$, and $2x^2-5x=160$.

[*Exercises* W.]

211. When there are more equations and unknown quantities than one, a single equation involving only one of the unknown quantities may sometimes be obtained by the rules laid down for the solution of simple equations; and one of the unknown quantities being discovered, the others may be obtained by substituting its value in the other equations.

Ex. $\left.\begin{aligned} x-\frac{x-y}{2}&=4, \\ y-\frac{x+3y}{x+2}&=1,\end{aligned}\right\}$ to find x and y.

$$\text{From the first equation, } 2x-x+y=8,$$

$$x+y=8,$$

$$x=8-y.$$

$$\text{From the 2nd equation, } xy+2y-x-3y=x+2,$$

$$\text{or } xy-2x-y=2,$$

$$\text{by substitution, } (8-y)y-2(8-y)-y=2,$$

$$8y-y^2-16+2y-y=2,$$

$$9y-y^2=16+2=18,$$

$$y^2-9y=-18,$$

$$y^2-9y+\frac{81}{4}=\frac{81}{4}-18=\frac{9}{4},$$

$$y-\frac{9}{2}=\pm\frac{3}{2};$$

$$\therefore y=\frac{9\pm 3}{2}=6, \text{ or } 3.$$

$$\text{And } x=8-y=2, \text{ or } 5.$$

The solution of equations will often be rendered more simple by particular artifices, the proper application of which is best learned by experience*.

Ex. 1. $\left.\begin{aligned} x^2+y^2&=65, \\ xy&=28, \end{aligned}\right\}$ to find x and y.

From the second equation, $2xy = 56$,

$$\left.\begin{aligned} \text{adding this to the } 1^{\text{st}},\quad x^2+2xy+y^2&=121, \\ \text{subtracting it,}\quad x^2-2xy+y^2&=9, \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{extracting the square roots,}\quad x+y&=\pm 11, \\ \text{and}\quad x-y&=\pm 3, \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{adding,}\quad 2x&=\pm 14, \\ \text{subtracting,}\quad 2y&=\pm 8, \end{aligned}\right\}$$

$$\therefore x = 7, \text{ or } -7,$$

$$\text{and } y = 4, \text{ or } -4.$$

Ex. 2. $\left.\begin{aligned} \frac{x^2}{y}+\frac{y^2}{x}&=18, \\ x+y&=12, \end{aligned}\right\}$ to find x and y.

$$\left.\begin{aligned} \text{From first equation,}\quad x^3+y^3&=18xy, \\ \text{from second,}\quad x^3+y^3+3xy(x+y)&=1728, \end{aligned}\right\}$$

$$\text{by substitution, } 18xy+36xy=1728,$$

$$54xy=1728;$$

$$\therefore xy=32.$$

$$\left.\begin{aligned} \text{Also } x^2+2xy+y^2&=144, \\ \text{and } 4xy\quad&=128, \end{aligned}\right\}$$

$$\therefore x^2-2xy+y^2=16,$$

$$\left.\begin{aligned} x-y&=\pm 4, \\ \text{and } x+y&=12, \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore 2x&=16, \text{ or } 8, \\ \text{and } 2y&=8, \text{ or } 16, \end{aligned}\right\}$$

$$\therefore x= \ , \text{ or } 4,$$

$$\text{and } y=4, \text{ or } 8.$$

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* Many of these artifices are pointed out in the Appendix.—Ed.

212. It may sometimes be of use to substitute for one of the unknown quantities the product of the other and a third unknown quantity.

This substitution may be successfully applied whenever the sum of the *dimensions* (Art. 63) of the *unknown quantities* in every term of each equation is the same.

Ex. $\left.\begin{aligned} x^2 + xy &= 12, \\ xy - 2y^2 &= 1, \end{aligned}\right\}$ to find x and y.

Let $vy = x$,

$$\left.\begin{aligned} &\text{then} \quad v^2y^2 + vy^2 = 12, \\ &\text{and} \quad vy^2 - 2y^2 = 1, \end{aligned}\right\}$$

$$\text{from the former, } y^2 = \frac{12}{v^2 + v},$$

$$\text{from the latter, } y^2 = \frac{1}{v - 2};$$

$$\therefore \frac{12}{v^2 + v} = \frac{1}{v - 2},$$

$$\text{or } v^2 + v = 12v - 24,$$

$$v^2 - 11v = -24,$$

$$v^2 - 11v + \frac{121}{4} = \frac{121}{4} - 24 = \frac{25}{4},$$

$$v - \frac{11}{2} = \pm\frac{5}{2};$$

$$\therefore v = \frac{11 \pm 5}{2} = 8, \text{ or } 3.$$

$$\text{And } y^2 = \frac{1}{v - 2} = \frac{1}{6}, \text{ or } 1;$$

$$\therefore y = \pm\frac{1}{\sqrt{6}}, \text{ or } \pm 1.$$

$$\text{And } x = vy = \pm\frac{8}{\sqrt{6}}, \text{ or } \pm 3.$$

213. The operation may sometimes be facilitated by substituting for the unknown quantities the sum and difference of two others.

This artifice may be used, when the unknown quantities in each equation are similarly involved.

Ex. $\left.\begin{array}{r} x+y=4, \\ (x^2+y^2)(x^3+y^3)=280, \end{array}\right\}$ to find x and y.

$$\begin{aligned}
\text{Assume } x&=z+v,\\
\text{and } y&=z-v,\\
\text{then } x+y&=2z=4;\\
\therefore\ z&=2.\\
\text{Also } x^2+y^2&=(2+v)^2+(2-v)^2,\\
&=8+2v^2,\\
\text{and } x^3+y^3&=(2+v)^3+(2-v)^3,\\
&=8+12v+6v^2+v^3+8-12v+6v^2-v^3,\\
&=16+12v^2;
\end{aligned}$$

$$\begin{gathered}
\therefore\ (8+2v^2)(16+12v^2)=280,\\
\text{or } (4+v^2)(4+3v^2)=35,\\
16+16v^2+3v^4=35,\\
v^4+\frac{16}{3}v^2=\frac{19}{3},\\
v^4+\frac{16}{3}v^2+\frac{64}{9}=\frac{19}{3}+\frac{64}{9}=\frac{121}{9},\\
v^2+\frac{8}{3}=\pm\frac{11}{3};\\
\therefore\ v^2=\frac{\pm 11-8}{3}=1, \text{ or } -\frac{19}{3};\\
v=\pm 1;\\
\therefore\ x=z+v=3, \text{ or } 1,\\
\text{and } y=z-v=1, \text{ or } 3.
\end{gathered}$$

Obs. In algebraical analysis it is frequently useful to observe whether the algebraical expressions under consideration are *homogeneous* or not, that is, whether the 'dimensions' of every term be the same or not; for, if this homogeneity be found at first, no legitimate operation can destroy it; or, if it be not found at first, it cannot be introduced; and thus an easy test is afforded, to a certain extent, of the accuracy of each succeeding step in the analysis.

For example, if the equation

$$ax^2+b^2x+c^3=0,$$

be proposed for solution, in which every term is of three dimensions, that is, which is homogeneous, every step in the process will present an homogeneous equation, if it be correct.

As a simple case it may be well to observe that, if the proposed equation be *homogeneous*, the final result must be so. A proper attention to this observation will frequently detect an error in the process of solving an equation.

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

214. PROB. I. A person bought a certain number of oxen for 80 guineas, and if he had bought 4 more for the same sum, they would have cost a guinea a piece less; required the number of oxen and price of each.

Let x be the number of oxen,

then $\frac{80}{x}$ is the price of each, in guineas,

and $\frac{80}{x+4}$ the price of each on the second supposition;

$$\therefore \ \frac{80}{x+4}=\frac{80}{x}-1, \text{ by the question,}$$

$$80=\frac{80x+320}{x}-x-4,$$

$$80x=80x+320-x^2-4x,$$

$$x^2+4x=320,$$

$$x^2+4x+4=324,$$

$$x+2=\pm 18;$$

$$\therefore \ x=\pm 18-2=16, \text{ or } -20, \text{ the number of oxen;}$$

$$\text{and } \frac{80}{x}=\frac{80}{16}=5 \text{ guineas, the price of each.}$$

In this, and in many other cases, especially in the solution of philosophical questions, we deduce from the algebraical process answers which do not correspond with the conditions. The reason seems to be, that the algebraical expression is more general than the common language; and the equation, which is a proper representation of the conditions, will also express other conditions, and answer other suppositions. In the foregoing instance x may either represent a positive or a negative quantity, and cannot in the operation represent a positive quantity alone (Art. 202); and the equation

$$\frac{80}{x+4}=\frac{80}{x}-1,$$

when x is negative, or represents the diminution of stock, will be a proper expression for the solution of the following problem: A person *sells* a certain number of oxen for 80 guineas; and had he sold 4 fewer for the same sum, he would have received a guinea a piece more for them; required the number sold.

215. PROB. II. To divide a line of 20 inches into two such parts, that the rectangle under the whole and one part may be equal to the square of the other part.

Let x be the greater part, then will $20-x$ be the less,

and $x^2=(20-x)\times 20=400-20x$, by the question,

$$x^2+20x=400,$$

$$x^2+20x+100=400+100=500,$$

$$x+10=\pm\sqrt{500};$$

$$\therefore\ x=\sqrt{500}-10, \text{ or } -\sqrt{500}-10.$$

The observation contained in the preceding article may be applied here; and it is to be remarked, that the negative values thus deduced are not insignificant, or useless. Here the negative value shews, that if the line be *produced* $\sqrt{500}+10$ inches, the square of the part produced is equal to the rectangle under the line given and the line made up of the whole and part produced.

215*. In order to ascertain the problem, the solution of which is given by the negative roots, take the equations which constitute the algebraical interpretation of the problem: write therein $-x$ for x, &c.: then the new equations will suggest the required question. This of course must be done by considering from what conditions of the given problem the several terms of the equations arise, and making the necessary alterations in agreement with the remarks of Art. 463. Thus, if we take the first of the preceding problems, we see that $\frac{80}{x}$ represents the price of each ox, $\frac{80}{x+4}$ the price of each if 4 more were bought, and the -1 arises from the relation between those prices given by the question. Now, if we write in the equation of Prob. I. $-x$ for x, we obtain $\frac{80}{-x+4}=\frac{80}{-x}-1$, *i.e.* $\frac{80}{x-4}=\frac{80}{x}+1$; and here $\frac{80}{x}$ represents the price of each of x oxen, $\frac{80}{x-4}$ the price of each if there were 4 less, and the $+1$ gives a relation between those prices. But as x is negative, it represents the *selling* of oxen (Art. 463), whence the problem above stated may evidently be seen to be that of which the altered equation is the algebraical interpretation. From this example it can easily be perceived how the interpretation of the negative result is to be made.

216. PROB. III. To find two numbers, whose sum, product, and the sum of whose squares, are equal to each other.

Let $x+y$ and $x-y$ be the numbers,

their sum is $2x$,

their product x^2-y^2,

the sum of their squares $2x^2+2y^2$,

and, by the question, $2x=2x^2+2y^2$,

$$\text{or } x=x^2+y^2.$$

$$\text{Also } 2x=x^2-y^2,$$

$$\text{adding, } 3x=2x^2;$$

$$\therefore x=\frac{3}{2}.$$

$$2x=x^2-y^2,$$

$$\text{or } 3=\frac{9}{4}-y^2,$$

$$y^2=\frac{9}{4}-3=\frac{9-12}{4}=\frac{-3}{4};$$

$$\therefore y=\pm\frac{\sqrt{-3}}{2}.$$

$$\text{Hence } x+y=\frac{3+\sqrt{-3}}{2},$$

$$\text{and } x-y=\frac{3-\sqrt{-3}}{2};$$

both of which are "impossible" quantities (Art. 183), a conclusion which shews that there are no such numbers as the question supposes.

[A collection of Problems *with their Solutions* will be found in the Appendix.]

[*Exercises* X.]

SCHOLIUM.

By the method of solution pursued in Art. 202 it is clear, that *both* the resulting values of the unknown quantity may be those of a different equation and not of the proposed one; for if the proposed equation be of the form

$$ax+\sqrt{bx+c}=d,$$

the solution effected may be that of the equation

$$ax-\sqrt{bx+c}=d,$$

and it is impossible to say, without trial, to which equation either of the resulting values of x belongs.

That there is no value of x which will satisfy *both* equations (except in a particular case) is easily proved. For, if possible, let there be such value; then, for that value,

$$\sqrt{bx+c}=0, \qquad \text{or } x=-\frac{c}{b},$$

a value of x which will satisfy *neither* equation, except in the particular case when $-\frac{c}{b}=\frac{d}{a}$, or $ac+bd=0$.

Hence it appears, that after solving an equation of the above form by the usual method, it still remains doubtful whether *either* of the values of the unknown quantity obtained will satisfy the equation; and if one of the two be the value sought, it remains doubtful which it is.

Thus from the equation

$$3x+\sqrt{30x-71}=5,$$

the values of x obtained are 4, and $2\frac{2}{3}$, *neither* of which will satisfy the equation.

And from the equation

$$3x+\sqrt{2x-2}=7,$$

the values of x obtained are 3, and $1\frac{8}{9}$, of which only the fractional, and not the integral, value will satisfy the equation.

The fact is, that in the former instance both values of x are the values belonging to the equation

$$3x-\sqrt{30x-71}=5;$$

in the latter $x=3$ belongs to the equation

$$3x-\sqrt{2x-2}=7,$$

and the other solution $x=1\frac{8}{9}$ to the equation as proposed.

Since, then, the values of x in $3x+\sqrt{30x-71}=5$, found by the common method of solution, do not belong to the equation at all, as is also the case in many others of like form, where, it may be asked, lies the fallacy in the process whereby we obtain a false result? It is here. We assume, as an axiom, that if the same root of equal quantities be extracted, those roots are equal to each other in all cases; whereas we know, that they *may* be unequal. For instance, retracing the steps in the following operation,

$$ax-\sqrt{bx+c}=d,$$
$$ax-d=\sqrt{bx+c},$$
$$(ax-d)^2=bx+c,$$

we assume, that the same value of x which satisfies the last of these equations must also satisfy the preceding one; but this may not be the case, since it may be the value which satisfies

$$-(ax-d)=\sqrt{bx+c},$$
$$\text{or}\quad ax+\sqrt{bx+c}=d,$$
$$\text{instead of}\quad ax-\sqrt{bx+c}=d,$$

the equation we commonly assume to be satisfied by that value, when it is the proposed equation, whose solution is required.

The fact is, that the equation really solved is not the proposed one

$$ax-\sqrt{bx+c}=d,$$
$$\text{but}\quad (ax-d+\sqrt{bx+c})(ax-d-\sqrt{bx+c})=0,$$
$$\text{or}\quad (ax-d)^2-(bx+c)=0\ ;$$

and it is quite a chance whether both or either of the values of x obtained belong to the proposed equation.

It is certain, however, that the values obtained belong to one or other of the two distinct equations

$$ax+\sqrt{bx+c}=d,$$
$$ax-\sqrt{bx+c}=d.$$

Also, when there are *two* values of x which will satisfy *one* of this pair of equations, there is *no* value of x which will satisfy the other; because, if there could be such a value, then the quadratic equation

$$(ax-d)^2-(bx+c)=0,$$

would have more than *two* distinct values of x, which was shewn to be impossible in Art. 204.

From what has been said it appears, that the results which are commonly obtained as solutions of quadratic equations, *when those equations* *are given in an irrational form*, require to be verified, before they can be depended upon.

Hence also, *when a proposed problem depends upon the solution of an irrational quadratic equation, the problem may, or may not, have those solutions which appear as solutions of the quadratic equation.* This conclusion, it must be evident, is of too important a nature to be safely overlooked.

INEQUALITIES.

✓ If one quantity be greater or less than another, or than nothing, and this be expressed algebraically, it is called an *Inequality.*

Thus, $x-a>b-x$ is an *Inequality*, of which $x-a$ forms one *side*, and $b-x$ the other.

✓ 217. *Any quantity may be added to, or subtracted from, each side of an inequality, and the sign of inequality remain as before.*

Thus, if $a>b$, $a\pm x>b\pm x$; for if $a>b$, it is evident that $a+x>b+x$. Similarly, if $a<b$, $a+x<b+x$.

Again, if $a>b$, it is evident that $a-x>b-x$, as long as x is not greater than a or b. If $x>b$, but not greater than a, then it is evident that $a-x>b-x$, for a positive quantity must be greater than a negative one. If $x>a$, and $>b$, then both sides $a-x$, and $b-x$, are negative; but a is nearer to x than b is to x, therefore $a-x<b-x$ independently of the signs; and of two negative quantities that is the greater, which is the smaller when the signs are omitted; therefore, in this case also, $a-x>b-x$.

Similarly, if $a<b$, in all cases $a-x<b-x$.

COR. Hence any quantity may be transposed (as in equations) from one side of an inequality to the other by changing its sign. Thus, if

$$a^2+b^2>2ab+c^2,$$

$$a^2+b^2-2ab>2ab-2ab+c^2,$$

$$\text{or } (a-b)^2>c^2.$$

218. If $a>b$, and $c>d$, and $e>f$, &c., then it is evident that

$$a+c+e+\&c.>b+d+f+\&c.;$$

But if $a>b$, and $c>d$, it does not always follow that $a-c>b-d$; for a may be more nearly equal to c, than b is to d. Thus $9>6$, and $7>2$, but $9-7$, or 2, is not greater than $6-2$, or 4.

219. *If every term on each side of an inequality be multiplied or divided by any the same positive quantity, the sign of inequality will remain as before.*

Thus, if $a>b$, $2a>2b$, $6a>6b$, &c.; or, if $-a>-b$, $-2a>-2b$, $-6a>-6b$; &c. as is sufficiently manifest.

COR. Hence an inequality may be cleared of fractions by multiplying both sides by the product of the denominators of all the fractions, or by the least common multiple of them all; provided the multiplier is a *positive* quantity.

Ex. If $\frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{a}+\frac{1}{b}$, multiplying by a^2b^2, which is necessarily *positive*,

$$a^3+b^3>ab^2+a^2b.$$

But, if all the terms of an inequality be multiplied or divided by a *negative* quantity, the sign of inequality is reversed, that is, $>$ is changed into $<$, or $<$ into $>$.

Thus, for example, $6>4$, but 6×-2 or $-12<4\times-2$ or -8. Also $\frac{6}{-2}$ or $-3<\frac{4}{-2}$ or -2.

N.B. Hence, if we multiply or divide the terms of an inequality by any *algebraical* quantity, it will be requisite to know whether the quantity is positive or negative.

COR. Also, if the signs of all the terms of an inequality be changed, the sign of inequality is *reversed*, for this is equivalent to multiplying each side by -1.

220. *Both sides of an inequality may be raised to any power or any root of them extracted, and the sign of inequality remain as before, provided that each side is a positive quantity.*

Thus, $7>5$ and 7^2 or $49>5^2$ or 25, 7^3 or $343>5^3$ or 125, and so on.

But, if either side be negative, then no general conclusion can be stated as to the resulting inequality. For $-3<4$, and $(-3)^2$ or $9<4^2$ or 16. Also $-2>-3$; but $(-2)^2$ or $4<(-3)^2$ or 9.

Similarly $16<25$, and $\sqrt{16}$ or $4<\sqrt{25}$ or 5. But, if either or both the sides be negative, no conclusion can be drawn as to their square roots.

Hence, if the sides of an *algebraical* inequality be raised to any power, or any root of them be extracted, we must first know whether the sides of the proposed inequality are positive or negative: otherwise the conclusion may not be correct in all cases.

221. *If the same quantity or two equal quantities be divided by each side of an inequality, the inequality will be reversed.*

For $3<6$, but $\frac{10}{3}>\frac{10}{6}$; and so also in any similar case.

Hence it appears, from this and the preceding articles, that the rule which belong to Equations must not be inconsiderately applied to Inequalities, since these latter have distinct rules of their own materially differing from those of the former.

Ex. 1. Shew that $a^2+b^2+c^2>ab+ac+bc$, unless $a=b=c$.

Here $(a-b)^2+(a-c)^2+(b-c)^2=2a^2+2b^2+2c^2$
$$-2ab-2ac-2bc,$$

$$\therefore\ a^2+b^2+c^2-(ab+ac+bc)=\tfrac{1}{2}\{(a-b)^2+(a-c)^2+(b-c)^2\}.$$

Now every quantity, when *squared*, is always positive; therefore the right-hand member of this equality, being half the sum of three squares, will be always positive, and greater than 0, except when $a=b=c$.

Hence $a^2+b^2+c^2-(ab+ac+bc)$ is always positive, and greater than 0, unless $a=b=c$,

$$\text{or } a^2+b^2+c^2 > ab+ac+bc, \text{ unless } a=b=c.$$

✓ **Ex. 2.** Shew that $x^5+y^5 > x^4y+y^4x$.

$$\text{Here } x^5+y^5-(x^4y+y^4x)=x^4(x-y)-y^4(x-y),$$
$$=(x^4-y^4)(x-y).$$

Now, whether $x >$ or $< y$, the two factors x^4-y^4, and $x-y$, have the *same* sign, and therefore their product is always positive; hence

$$x^5+y^5-(x^4y+y^4x) \text{ is always positive,}$$
$$\therefore\ x^5+y^5 > x^4y+y^4x.$$

Ex. 3. Given that $\dfrac{x+2}{4}+\dfrac{x}{3} < \dfrac{x-4}{2}+3$, and $> \dfrac{x+1}{2}+\dfrac{1}{3}$, find x.

$$\frac{x+2}{4}+\frac{x}{3} < \frac{x-4}{2}+3, \text{ and} > \frac{x+1}{2}+\frac{1}{3};$$

mult. by 12, $3x+6+4x < 6x-24+36$, and $> 6x+6+4$, (Art. 219),

$$\text{or } 7x+6 < 6x+12, \text{ and } > 6x+10,$$
$$x < 6, \text{ and } > 4. \text{ (Art. 217.)}$$

Hence x is any number between 4 and 6; and, if it be a whole number, $x=5$.

Ex. 4. Which is greater $\sqrt{10}+\sqrt{7}$, or $\sqrt{19}+\sqrt{3}$?

$$\sqrt{10}+\sqrt{7} > \text{ or } < \sqrt{19}+\sqrt{3},$$

according as $17+2\sqrt{70} >$ or $< 22+2\sqrt{57}$, squaring, by Art. 220,

.... $2\sqrt{70} >$ or $< 5+2\sqrt{57}$, (Art. 217),

.... $280 >$ or $< 253+20\sqrt{57}$, (Art. 220),

.... $27 >$ or $< 20\sqrt{57}$.

Now 27 is clearly less than $20\sqrt{57}$, therefore $\sqrt{19}+\sqrt{3}$ is the greater of the two proposed quantities.

Ex. 5. Shew that $\dfrac{a+b+c+d}{p+q+r+s} >$ the least and $<$ the greatest of the fractions $\dfrac{a}{p}$, $\dfrac{b}{q}$, $\dfrac{c}{r}$, $\dfrac{d}{s}$, each letter representing a positive quantity.

Of the fractions $\frac{a}{p}, \frac{b}{q}, \frac{c}{r}, \frac{d}{s}$, suppose $\frac{a}{p}$ to be the greatest (G), and $\frac{d}{s}$ the least (g). Then

$$\frac{a}{p}=G,\ \frac{b}{q}<G,\ \frac{c}{r}<G,\ \frac{d}{s}<G,$$

$$\frac{a}{p}>g,\ \frac{b}{q}>g,\ \frac{c}{r}>g,\ \frac{d}{s}=g.$$

$$\left.\begin{array}{l}\therefore\ a=pG,\ b<qG,\ c<rG,\ d<sG,\\ \quad\ a>pg,\ b>qg,\ c>rg,\ d=sg,\end{array}\right\}\ \text{Art. 219.}$$

$$\left.\begin{array}{l}\therefore\ a+b+c+d<(p+q+r+s)G,\\ \text{and}\ a+b+c+d>(p+q+r+s)g,\end{array}\right\}\ \text{Art. 218.}$$

$$\therefore\ \frac{a+b+c+d}{p+q+r+s}<G \text{ and } >g. \quad \text{(Art. 219.)} \quad \text{Q.E.D.}$$

This proposition may without difficulty be extended to more than four fractions. And it will be equally true whether a, b, c, d, &c. be positive or negative, provided that p, q, r, s, &c., the denominators, be all positive; since of negative quantities we suppose that one to be the greatest which is *numerically* the least. Moreover, if the denominators be all negative, $-p$, $-q$, &c....; since $\frac{a}{-p}=\frac{-a}{p}$, &c., the above proposition will still hold. Therefore, if a number of fractions have their denominators all of the same sign, $\frac{\text{sum of numerators}}{\text{sum of denominators}}$ is intermediate in magnitude to the least and greatest of the fractions.

Ex. 6. Shew that if $a_1, a_2, a_3, \dots a_n$ be any positive quantities,

$$\frac{a_1+a_2+\dots+a_n}{n}>\sqrt[n]{a_1.a_2.a_3\dots a_n},$$

unless the n quantities are all equal.

The left-hand member of the above inequality remains unaltered, however $a_1, a_2, \dots$ may change in magnitude, provided only that the sum of them all is not changed. It will therefore be sufficient to prove that, if the sum of $a_1, a_2, \dots a_n$ be given, their product is the greatest possible when they are all equal. For if not, let it be the greatest possible for certain unequal values of the symbols, and let a_p, a_q be any two of them that are not equal. Then if for each of a_p, a_q we substitute $\frac{a_p+a_q}{2}$, (which we are at liberty to do, as the *sum* of all the quantities is not thereby altered,) the product becomes

$$\sqrt[n]{a_1.a_2\dots\frac{a_p+a_q}{2}.\frac{a_p+a_q}{2}\dots a_n},$$

$$\text{instead of}\quad \sqrt[n]{a_1.a_2\dots a_p.a_q\dots a_n}.$$

But $\frac{a_p+a_q}{2}.\frac{a_p+a_q}{2}>a_p.a_q$, ∵ $(a_p+a_q)^2$, which is equal to $a_p^2+a_q^2+2a_pa_q$, $>4a_pa_q$, for $(a_p-a_q)^2$ is always positive.

Therefore the new product we have obtained is greater than the former one; *i.e.* our supposition of the former being the greatest possible is not tenable.

Therefore the product is the greatest possible when they are all equal; but then it is equal to a_1, or a_2, or &c., and therefore it is equal to the left-hand member of the proposed inequality. Consequently, in all other cases, it is less, *i.e.*

$$\frac{a_1+a_2+\ldots+a_n}{n} > \sqrt[n]{a_1 . a_2 \ldots a_n},$$

unless the n quantities are all equal.

This is often expressed by stating that the arithmetical mean between any number of positive quantities is greater than their geometrical mean.

Another proof.—By the Binomial Theorem, (Art. 308), we have

$$\left(1+\frac{x}{n}\right)^n = 1+x+\frac{1-\frac{1}{n}}{1.2}x^2+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1.2.3}x^3+\&c.,$$

$$\text{and } \left(1+\frac{x}{n-1}\right)^{n-1} = 1+x+\frac{1-\frac{1}{n-1}}{1.2}x^2+\frac{\left(1-\frac{1}{n-1}\right)\left(1-\frac{2}{n-1}\right)}{1.2.3}x^3+\&c.,$$

n being a positive integer.

Now $\frac{1}{n}$, $\frac{2}{n}$, &c. are respectively less than $\frac{1}{n-1}$, $\frac{2}{n-1}$, &c., and, therefore, after the first two terms of each series, which are equal, each term in the first series is greater than the corresponding term in the second.

$$\therefore \left(1+\frac{x}{n}\right)^n > \left(1+\frac{x}{n-1}\right)^{n-1}.$$

Now let $a_1, a_2, \ldots a_n$ be n quantities that are not all equal,

$$\text{then } \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^n = a_1^n\left(1+\frac{a_1+a_2+\ldots+a_n-na_1}{na_1}\right)^n,$$

$$\text{and } \therefore > a_1^n\left\{1+\frac{a_1+a_2+\ldots+a_n-na_1}{(n-1)a_1}\right\}^{n-1},$$

$$\textit{i.e.} > a_1\left(\frac{a_2+a_3+\ldots+a_n}{n-1}\right)^{n-1},$$

$$\text{and } \therefore \text{ is, } \textit{a fortiori}, > a_1 . a_2\left(\frac{a_3+a_4+\ldots+a_n}{n-2}\right)^{n-2},$$

$$\therefore > a_1 . a_2 . a_3\left(\frac{a_4+\ldots+a_n}{n-3}\right)^{n-3},$$

&c.

$$> a_1 . a_2 . a_3 \ldots a_n;$$

$$\therefore \frac{a_1+a_2+a_3+\ldots+a_n}{n} > \sqrt[n]{a_1 . a_2 . a_3 \ldots a_n}.$$

[*Exercises* Y.]

RATIOS.

222. **Ratio is the relation which one quantity bears to another in respect of magnitude, the comparison being made by considering what multiple, part, or parts, one is of the other*.**

Thus, in comparing 6 with 3, we observe that it has a certain magnitude with respect to 3, which it contains twice; again, in comparing it with 2, we see that it has a different relative magnitude, for it contains 2 three times, or it is greater when compared with 2 than it is when compared with 3. The ratio of a to b is usually expressed by two points placed between them thus, $a : b$; and the former, a, is called the *antecedent* of the ratio, the latter, b, the *consequent.*

Since in the ratio $a : b$ the comparison is made in regard to quantuplicity, (*κατὰ πηλικότητα*), the ratio may evidently be expressed by whatever expresses the degree of that quantuplicity, *i.e.* by what is necessary to multiply b by to obtain a. But this multiplier is the fraction $\frac{a}{b}$; the *fraction* $\frac{a}{b}$ therefore is equivalent to the *ratio* $a : b$.

223. COR. 1. When one antecedent is the same multiple, part, or parts, of its consequent, that another antecedent is of its consequent, the ratios are equal. Thus, the ratio of $4 : 6$ is equal to the ratio of $2 : 3$, that is, 4 has the same magnitude when compared with 6, that 2 has when compared with 3, since $\frac{4}{6} = \frac{2}{3}$. The ratio of $a : b$ is equal to the ratio of $c : d$, if $\frac{a}{b} = \frac{c}{d}$, because $\frac{a}{b}$ and $\frac{c}{d}$ represent the multiple, part, or parts, that a is of b, and c of d.

224. COR. 2. If the terms of a ratio be multiplied or divided by the same quantity, the value of the ratio is not altered.

For $\frac{a}{b} = \frac{ma}{mb}$ (Art. 101); $\therefore a : b = ma : mb$.

225. COR. 3. That ratio is greater than another, whose antecedent is the greater multiple, part, or parts, of its consequent.

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* *Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητα πρὸς ἄλληλα ποιὰ σχέσις.* (EUCLID, Book V. Def. III.)

Thus the ratio 7 : 4 is greater than the ratio 8 : 5; because $\frac{7}{4}$, or $\frac{35}{20}$, is greater than $\frac{8}{5}$, or $\frac{32}{20}$. These conclusions follow immediately from our idea of ratio.

Ex. Which is greater $a+x : a-x$, or $a^2+x^2 : a^2-x^2$?

$$a+x : a-x > \text{ or } < a^2+x^2 : a^2-x^2,$$

$$\text{according as } \frac{a+x}{a-x} > \text{ or } < \frac{a^2+x^2}{a^2-x^2},$$

$$\text{or, } \frac{a^2+x^2+2ax}{a^2-x^2} > \text{ or } < \frac{a^2+x^2}{a^2-x^2};$$

and since the former is the greater by the quantity $\frac{2ax}{a^2-x^2}$;

$$\therefore\ a+x : a-x > a^2+x^2 : a^2-x^2.$$

226. Def. A ratio is called a ratio *of greater inequality*, *of less inequality*, or *of equality*, according as the antecedent is *greater* than, *less* than, or *equal* to, the consequent.

227. *A ratio of greater inequality is diminished, and of less inequality increased, by adding any quantity to both its terms.*

If 1 be added to the terms of the ratio 7 : 4, it becomes the ratio 8 : 5, which is less than the former, (Art 225). And in general, let x be added to the terms of the ratio $a : b$, and it becomes $a+x : b+x$, which is greater or less than the former, according as $\frac{a+x}{b+x} >$ or $< \frac{a}{b}$, or, by reducing them to a common denominator, according as $\frac{ab+bx}{b(b+x)} >$ or $< \frac{ab+ax}{b(b+x)}$; that is, as $bx >$ or $< ax$, or as $b >$ or $< a$.

228. Cor. Hence a ratio of greater inequality is increased, and of less inequality diminished, by taking from the terms *a quantity less than either of them.*

229. Def. If the antecedents of any ratios be multiplied together, and also the consequents, a new ratio results, which is said to be *compounded* of the former. Thus $ac : bd$ is said to be compounded of the two $a : b$ and $c : d$. It is also sometimes called

the *sum* of the ratios; and when the ratio $a : b$ is compounded with itself, the resulting ratio, $a^2 : b^2$, is called the *double* of the ratio $a : b$; and if three of these ratios be compounded together, the result, $a^3 : b^3$, is called the *triple* of the first, &c. Also the ratio $a : b$ is said to be *one third* of the ratio $a^3 : b^3$; and $a^{\frac{1}{m}} : b^{\frac{1}{m}}$ is said to be an m^{th} part of the ratio $a : b$.

$a^2 : b^2$ is also said to be in the duplicate ratio of $a : b$, and $a^3 : b^3$ in the triplicate ratio of $a : b$. Moreover, $a^{\frac{1}{2}} : b^{\frac{1}{2}}$, $a^{\frac{1}{3}} : b^{\frac{1}{3}}$, $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ are respectively said to be in the subduplicate, subtriplicate, and sesquiplicate, ratio of $a : b$.

230. COR. Let the first ratio be $a : 1$; then $a^2 : 1$, $a^3 : 1$, $a^n : 1$, are twice, three times, n times, the first ratio; where n, the index of a, shews what multiple, or part, of the ratio $a^n : 1$, the first ratio $a : 1$ is. On this account the indices 1, 2, 3, n, are called *measures* of the ratios $a^1 : 1$, $a^2 : 1$, $a^3 : 1$, $a^n : 1$.

231. *If the consequent of the preceding ratio be the antecedent of the succeeding one, and any number of such ratios be taken, the ratio which arises from their composition is that of the first antecedent to the last consequent.*

Let $a : b$, $b : c$, $c : d$, be the ratios, the compound ratio is $a \times b \times c : b \times c \times d$, (Art. 229), or, dividing by $b \times c$ (Art. 224), $a : d$; and similarly for any number of ratios.

232. *A ratio of greater inequality, compounded with another, increases it; and a ratio of less inequality diminishes it.*

Let the ratio $x : y$ be compounded with the ratio $a : b$, then the resulting ratio $ax : by >$ or $<$ the ratio $a : b$, according as $\frac{ax}{by} >$ or $< \frac{a}{b}$ (Art. 225); that is, according as $x >$ or $< y$.

233. *If the difference between the antecedent and consequent of a ratio be small when compared with either of them, the double of the ratio, or the ratio of their squares, is nearly obtained by doubling this difference.*

Let $a + x : a$ be the proposed ratio, where x is small when compared with a; then $a^2 + 2ax + x^2 : a^2$ is the ratio of the squares of the antecedent and consequent; but since x is small when compared with a, x^2, or $x\times x$, is small when compared with $2a\times x$, and much smaller than $a\times a$; therefore $a^2+2ax : a^2$, or $a+2x : a$ (Art. 224), will nearly express the ratio of $a^2+2ax+x^2 : a^2$.

Thus the ratio of the square of 1001 to the square of 1000 is nearly 1002 : 1000; the real ratio is 1002·001 : 1000, in which the antecedent differs from its approximate value only by one thousandth part of an unit.

234. COR. Hence the ratio of the square root of $a+2x$ to the square root of a is the ratio $a+x : a$, nearly; that is, if the difference of two quantities be small with respect to either of them, the ratio of their square roots is nearly obtained by halving that difference.

235. In the same manner it may be shewn, when x is very small compared with a, that

$$a\pm 3x : a;\quad a\pm 4x : a;\quad a\pm 5x : a;\ \&c.$$

are nearly equal respectively to the ratios

$$(a\pm x)^3 : a^3,\quad (a\pm x)^4 : a^4,\quad (a\pm x)^5 : a^5,\ \&c.$$

Also $a\pm\frac{1}{3}x : a$, $a\pm\frac{1}{4}x : a$, &c. are nearly equal to the ratios $\sqrt[3]{a\pm x} : \sqrt[3]{a}$, $\sqrt[4]{a\pm x} : \sqrt[4]{a}$, &c.

For $(a\pm x)^3 : a^3 = a^3\pm 3a^2x+3ax^2\pm x^3 : a^3$,

$$=1\pm\frac{3x}{a}+\frac{3x^2}{a^2}\pm\frac{x^3}{a^3} : 1 \text{ (Art. 224).}$$

Now if x be small when compared with a, $\frac{x}{a}$ is a small fraction; and $\frac{3x^2}{a^2} : \frac{3x}{a} = \frac{x}{a} : 1$, therefore since $\frac{x}{a}$ is small compared with 1, $\frac{3x^2}{a^2}$ is small compared with $\frac{3x}{a}$; *a fortiori* $\frac{x^3}{a^3}$ is very small compared with 1 or with $\frac{3x}{a}$.

Hence, neglecting $\frac{3x^2}{a^2}$ and $\frac{x^3}{a^3}$, which are very small fractions, $1\pm\frac{3x}{a} : 1$, or $a\pm 3x : a$, is a near approximation to $(a\pm x)^3 : a^3$, if x be small when compared with a.

Similarly it may be shewn that $a\pm 4x : a$, $a\pm 5x : a$, &c. are approximations respectively to $(a\pm x)^4 : a^4$, Again, since $\sqrt{a \pm x} : \sqrt{a} = \sqrt{1 \pm \frac{x}{a}} : 1$, (Art. 224),

$$= 1 \pm \tfrac{1}{2}\frac{x}{a} - \tfrac{1}{8}\frac{x^2}{a^2} \pm \&c. : 1, \text{ (Art. 151, Ex. 3)},$$

by the same reasoning this ratio is reduced to

$$1 \pm \tfrac{1}{2}\frac{x}{a} : 1, \text{ or } a \pm \tfrac{1}{2}x : a.$$

Also $\quad \sqrt[3]{a \pm x} : \sqrt[3]{a} = a \pm \tfrac{1}{3}x : a$, nearly; and so on.

The utility of the rules here proved will be sufficiently manifest from the following Examples, when it is observed by what a troublesome process the several proposed ratios would be found without the rules.

Ex. 1. $\quad (1{\cdot}5241)^4 : (1{\cdot}524)^4 = 1{\cdot}5240 + 4 \times 0{\cdot}0001 : 1{\cdot}524$ nearly $= 1{\cdot}5244 : 1{\cdot}524$ nearly.

Ex. 2. $\quad \sqrt[3]{729} : \sqrt[3]{728} = 728\tfrac{1}{3} : 728$ nearly.

Ex. 3. $\quad \sqrt[5]{2134} : \sqrt[5]{2131} = 2131\tfrac{3}{5} : 2131$ nearly.

[*Exercises* Z.]

PROPORTION.

236. **Def.** Four quantities are said to be *proportionals*, when the first is the same multiple, part, or parts, of the second, that the third is of the fourth; that is, when $\frac{a}{b} = \frac{c}{d}$, the four quantities a, b, c, d, are called proportionals. This is usually expressed by saying a is to b as c is to d, and is thus represented, $a : b :: c : d$; or sometimes, $a : b = c : d$.

The terms a and d are called the *extremes*, and b and c the *means*.

237. *When four quantities are proportionals, the product of the extremes is equal to the product of the means.*

Let a, b, c, d, be the four quantities; then, since they are proportionals, $\frac{a}{b} = \frac{c}{d}$ (Art. 236); and by multiplying both sides of the equation by bd, $ad = bc$.

238. **Cor. 1.** If the first be to the second as the second to the third, the product of the extremes is equal to the square of the mean.

239. Cor. 2. Any three terms in a proportion being given, the fourth may be determined from the equation $ad=bc$.

For $d=\frac{bc}{a}$, $c=\frac{ad}{b}$, $b=\frac{ad}{c}$, $a=\frac{bc}{d}$. Hence we have the Single Rule of Three in Arithmetic.

240. *If the product of two quantities be equal to the product of two others, the four are proportionals, making the factors of one product the means, and the factors of the other the extremes.*

Let $xy=ab$, then dividing by ay, $\frac{x}{a}=\frac{b}{y}$,

$$\text{or } x : a :: b : y. \quad \text{(Art. 236.)}$$

241. *If* a : b :: c : d, *and* c : d :: e : f, *then also* a : b :: e : f. (Euclid, B. v. Prop. xi.)

Because $\frac{a}{b}=\frac{c}{d}$, and $\frac{c}{d}=\frac{e}{f}$; therefore $\frac{a}{b}=\frac{e}{f}$;

$$\text{or } a : b :: e : f.$$

242. *If four quantities be proportionals, they are also proportionals when taken* inversely. (Euclid, B. v. Prop. B.)

If $a : b :: c : d$, then $b : a :: d : c$. For $\frac{a}{b}=\frac{c}{d}$, and dividing unity by each of these equal quantities,

$$\frac{b}{a}=\frac{d}{c}; \text{ that is, } b : a :: d : c.$$

243. *If four quantities be proportionals, they are proportionals when taken* alternately. (Euclid, B. v. Prop. xvi.)

If $a : b :: c : d$, then $a : c :: b : d$.

Because the quantities are proportionals, $\frac{a}{b}=\frac{c}{d}$, and multiplying by $\frac{b}{c}$, $\frac{a}{c}=\frac{b}{d}$, or $a : c :: b : d$.

Unless the four quantities are of the *same kind*, the alternation cannot take place; because this operation supposes the first to be some multiple, part, or parts, of the third.

One line may have to another line the same ratio that one weight has to another weight, but a line has no relation in respect of magnitude to a weight. In cases of this kind, if the four quantities be represented by numbers, or other quantities which are similar, the alternation may take place, and the conclusions drawn from it will be just.

244. *When four quantities are proportionals, the first together with the second is to the second, as the third together with the fourth is to the fourth. This operation is called* componendo. (EUCLID, B. v. Prop. XVIII.)

If $a : b :: c : d$, then also

$$a+b : b :: c+d : d.$$

Because $\frac{a}{b}=\frac{c}{d}$, by adding 1 to each side,

$$\frac{a}{b}+1=\frac{c}{d}+1;$$

$$\text{that is, } \frac{a+b}{b}=\frac{c+d}{d};$$

$$\text{or } a+b : b :: c+d : d.$$

245. *Also,* dividendo, *the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth.* (EUCLID, B. v. Prop. XVII.)

Because $\frac{a}{b}=\frac{c}{d}$, by subtracting 1 from each side,

$$\frac{a}{b}-1=\frac{c}{d}-1;$$

$$\text{that is, } \frac{a-b}{b}=\frac{c-d}{d};$$

$$\text{or } a-b : b :: c-d : d.$$

246. *Again,* convertendo, *the first is to its excess above the second, as the third is to its excess above the fourth.* (EUCLID, B. v. Prop. E.)

By the last Article, $\frac{a-b}{b} = \frac{c-d}{d}$;

and $\frac{b}{a} = \frac{d}{c}$ (Art. 242);

$$\therefore \frac{a-b}{b} \times \frac{b}{a} = \frac{c-d}{d} \times \frac{d}{c}; \text{ or, } \frac{a-b}{a} = \frac{c-d}{c};$$

that is, $a-b : a :: c-d : c$;

and inversely, $a : a-b :: c : c-d$.

247. *When four quantities are proportionals, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.*

If $a : b :: c : d$; then $a+b : a-b :: c+d : c-d$

By Art. 244, $\frac{a+b}{b} = \frac{c+d}{d}$;

and by Art. 245, $\frac{a-b}{b} = \frac{c-d}{d}$;

$$\therefore \frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d} \text{ (Art. 82)};$$

$$\text{or } \frac{a+b}{a-b} = \frac{c+d}{c-d};$$

that is, $a+b : a-b :: c+d : c-d$.

248. *When any number of quantities are proportionals, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.* (EUCLID, B. V. Prop. XII.)

If $a : b :: c : d :: e : f$, &c.

then $a : b :: a+c+e+$ &c. $: b+d+f+$ &c.

Because $\frac{a}{b} = \frac{c}{d}$, $ad = bc$; in the same manner, $af = be$; also $ab = ba$; hence $ab + ad + af = ba + bc + be$,

or $a(b+d+f) = b(a+c+e)$;

$\therefore$ by Art. 240, $a : b :: a+c+e : b+d+f$;

and similarly when more quantities are taken.

249. *When four quantities are proportionals, if the first and second be multiplied, or divided, by any quantity, as also the third and fourth, the resulting quantities will be proportionals.*

If $a : b :: c : d$, then $ma : mb :: \frac{c}{n} : \frac{d}{n}$.

$$\text{For } \frac{a}{b} = \frac{c}{d}; \quad \therefore \frac{ma}{mb} = \frac{\frac{1}{n}.c}{\frac{1}{n}.d} \text{ (Art. 101)};$$

$$\text{or } \quad ma : mb :: \frac{c}{n} : \frac{d}{n}.$$

250. *If the first and third be multiplied, or divided, by any quantity, and also the second and fourth, the resulting quantities will be proportionals.*

$$\text{For } \frac{a}{b} = \frac{c}{d}; \quad \therefore \frac{ma}{b} = \frac{mc}{d}; \text{ and } \frac{ma}{\frac{1}{n}.b} = \frac{mc}{\frac{1}{n}.d};$$

$$\text{or } \quad ma : \frac{b}{n} :: mc : \frac{d}{n}.$$

251. COR. Hence, in any proportion, if instead of the second and fourth terms quantities proportional to them be substituted, we have still a proportion. For $\frac{b}{n}$ and $\frac{d}{n}$ are in the same proportion with b and d (Art. 249).

252. *In two ranks of proportionals, if the corresponding terms be multiplied together, the products will be proportionals.*

$$\begin{aligned} \text{If } \quad & a : b :: c : d, \\ \text{and } \quad & e : f :: g : h, \\ \text{then also } \quad & ae : bf :: cg : dh. \end{aligned}$$

Because $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$; therefore $\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h}$, or $\frac{ae}{bf} = \frac{cg}{dh}$; that is, $ae : bf :: cg : dh$.

This is called *compounding* the proportions.

The proposition is true if applied to *any number* of proportions.

253. *If four quantities be proportionals, the like powers, or roots, of these quantities, will be proportionals.*

If $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$, and $\frac{a^n}{b^n} = \frac{c^n}{d^n}$; or $a^n : b^n :: c^n : d^n$; where n may be either whole or fractional.

254. If $a : b :: c : d$, to prove that

$$ma \pm nb : pa \pm qb :: mc \pm nd : pc \pm qd.$$

$$\frac{a}{b} = \frac{c}{d},$$

$$\frac{ma}{b} = \frac{mc}{d},$$

$$\frac{ma}{b} \pm n = \frac{mc}{d} \pm n,$$

$$\text{or} \quad \frac{ma \pm nb}{b} = \frac{mc \pm nd}{d}.$$

$$\text{Similarly} \quad \frac{pa \pm qb}{b} = \frac{pc \pm qd}{d};$$

$$\therefore \frac{ma \pm nb}{b} \div \frac{pa \pm qb}{b} = \frac{mc \pm nd}{d} \div \frac{pc \pm qd}{d},$$

$$\frac{ma \pm nb}{pa \pm qb} = \frac{mc \pm nd}{pc \pm qd},$$

$$\text{or} \quad ma \pm nb : pa \pm qb :: mc \pm nd : pc \pm qd.$$

255. *If three quantities,* a, b, c *be in continued proportion, that is,* $a : b :: b : c$, *then* $a : c :: a^2 : b^2$, *that is, the first has to the third the duplicate ratio of that which it has to the second.* (EUCLID, B. v. Def. x.)

$$\text{For } \frac{a}{b} = \frac{b}{c};$$

$$\therefore \frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b},$$

$$= \frac{a^2}{b^2};$$

$$\text{or} \quad a : c :: a^2 : b^2.$$

256. ***If four quantities are in continued proportion, that is,*** $a : b :: b : c :: c : d$, ***then*** $a : d :: a^3 : b^3$, ***that is, the first has to the fourth the triplicate ratio of that which it has to the second.*** **(EUCLID, B. v. Def. XI.)**

$$\text{For } \frac{a}{d}=\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}, \text{ and } \frac{a}{b}=\frac{b}{c}=\frac{c}{d};$$

$$\therefore\ \frac{a}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b},$$

$$=\frac{a^3}{b^3};$$

$$\text{or } a : d :: a^3 : b^3.$$

257. The Definition of Proportion here used will not serve for a definition in Geometry, because there is no Geometrical method of representing the quotient of $a \div b$, a and b being any Geometrical magnitudes whatever of the same kind. But such magnitudes may always be *multiplied geometrically;* that is, a line may be produced till it becomes n times its original length—an area, or a solid, may be doubled, trebled, &c.—*geometrically.* Hence the strictness of Geometry requires such a definition as that which is the foundation of Euclid's 5th Book; and which may easily be shewn to follow from the Algebraic Definition.

For suppose a, b, c, d, to represent four quantities in proportion, according to the Algebraical definition; then

$$\frac{a}{b}=\frac{c}{d},$$

$$\text{and } \frac{m}{n}\cdot\frac{a}{b}=\frac{m}{n}\cdot\frac{c}{d},$$

$$\text{or } \frac{ma}{nb}=\frac{mc}{nd};$$

from which it follows, by the nature of fractions, that if $ma > nb$, then $mc > nd$; if $ma = nb$, $mc = nd$; if $ma < nb$, $mc < nd$: and ma, mc, are any equimultiples *whatever* of the 1st and 3rd quantities, nb, nd any equimultiples *whatever* of the 2nd and 4th. Therefore a, b, c, d are proportional also according to the Geometrical Definition.

[*Exercises Za.*]

258. *If two numbers* a *and* b, *be prime to each other, they are the least in that proportion.*

If possible, let $\frac{a}{b}=\frac{c}{d}$, where a and b are prime to each other, and respectively greater than c and d. If the latter numbers be not prime to each other, divide them by their greatest common measure. Then divide a by b, and c by d, as in Art. 103, thus

$$\begin{array}{lcl} b)\,\underline{a}\,(m & \qquad & d)\,\underline{c}\,(m \\ \overline{x)\,\underline{b}\,(n} & & \overline{r)\,\underline{d}\,(n} \\ \quad y & & \quad s \end{array}$$

and because $\frac{a}{b} = \frac{c}{d}$, the first quotients m, m, are equal; again, since $\frac{a}{b} = m + \frac{x}{b}$, and $\frac{c}{d} = m + \frac{r}{d}$, we have $\frac{x}{b} = \frac{r}{d}$, or $\frac{b}{x} = \frac{d}{r}$; also, because b is greater than d, x is greater than r. In the same manner, $\frac{x}{y} = \frac{r}{s}$, and y is greater than s; and so on, if there be more remainders. Thus the remainder in the latter division will become 1 sooner than the remainder in the former. Let $s = 1$; then $\frac{x}{y} = r$; and y, which is greater than 1, will be a common measure of a and b (Art. 105), which is contrary to the supposition.

COR. 1. Hence, if $\frac{a}{b} = \frac{c}{d}$, and a and b be prime to each other, c and d are equimultiples of a and b.

COR. 2. Hence, also, *a number cannot be made up of prime factors in more ways than one.* For, if possible,

1st. Let the number be made up of *two* prime factors, either a and d, or b and c; so that $ad = bc$, where b is not the same number as either a or d. Then, since $\frac{a}{b} = \frac{c}{d}$, and a and b are prime to each other, c and d are equimultiples of a and b, by Cor. 1; that is, d is *not* a prime number, which is contrary to the supposition.

2nd. Let the number be made up of more than *two* prime factors, in two different ways; and let a be *one* of the factors, and p the product of the rest, in one set—b *one* of the factors, and q the product of the rest, in the other set, where b is not the same as a, or as any of the factors in p. Then $ap = bq$, $\therefore \frac{a}{b} = \frac{q}{p}$, and, by Cor. 1, p is a multiple of b, which is contrary to the supposition.

Hence there are not two ways of resolving a number into its prime factors.

259. *If* a *and* b *be each of them prime to* c, ab *is prime to* c.

If not, let $ab = mr$, and $c = ms$; then since a and b are prime to c, they are respectively *prime* to ms, and therefore to m; and because $ab = mr$, we have $\frac{a}{m} = \frac{r}{b}$; therefore b is a *multiple* of m (Art. 258, Cor.), which is absurd, since it was before shewn to be *prime* to m.

COR. 1. If b be equal to a, then a^2 and c have no common measure; or $\frac{a^2}{c}$ is a fraction in its lowest terms.

COR. 2. In the same manner, $\frac{a^3}{c}$, $\frac{a^4}{c}$, &c. are fractions in their lowest terms.

COR. 3. If a, b, and c, be *each* of them prime to d, e, and f, abc is prime to def.

For, if a be prime to d and e, it is prime to de, and if it be prime to de and f, it is prime to def. In the same manner, b and c are prime to def; consequently, abc is prime to def.

COR. 4. If a be prime to b, a^2 is prime to b^2, and a^3 to b^3, &c.

SCHOLIUM.

260. In the definition of Proportion it is supposed that one quantity is some determinate multiple, part, or parts, of another; or that the fraction arising from the division of one by the other, (which expresses the multiple, part, or parts, that the former is of the latter), is a determinate fraction. This will be the case, whenever the two quantities have any Common Measure whatever.

Let x be a common measure of a and b, and let $a = mx$, $b = nx$; then $\frac{a}{b} = \frac{mx}{nx} = \frac{m}{n}$, where m and n are whole numbers.

But it sometimes happens that the quantities are *incommensurable*, that is, admit of no common measure whatever, as when one represents the circumference of a circle and the other its diameter; in such cases the value of $\frac{a}{b}$ cannot be exactly expressed by any fraction, $\frac{m}{n}$, whose numerator and denominator are whole numbers; yet a fraction of this kind may be found, which will express its value *to any required degree of accuracy.*

Suppose x to be a measure of b, and let $b = nx$; also let a be greater than mx but less than $(m+1)x$; then $\frac{a}{b}$ is greater than $\frac{m}{n}$ but less than $\frac{m+1}{n}$, or the difference between $\frac{m}{n}$ and $\frac{a}{b}$ is less than

$\frac{1}{n}$; and as x is diminished, since $nx=b$, n is increased, and $\frac{1}{n}$ diminished; therefore, by diminishing x, the difference between $\frac{m}{n}$ and $\frac{a}{b}$ may be made less than any that can be assigned.

If a and b, as well as c and d, be incommensurable, and if, when $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, $\frac{c}{d}$ lie also between $\frac{m}{n}$ and $\frac{m+1}{n}$, however the magnitudes m and n are increased, then $\frac{a}{b}$ is equal to $\frac{c}{d}$. For, if they are not equal, they must have some assignable difference; and because each of them lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, this difference is less than $\frac{1}{n}$; but since n may, by the supposition, be increased without limit, $\frac{1}{n}$ may be diminished without limit, that is, it may become less than any assignable magnitude; therefore $\frac{a}{b}$ and $\frac{c}{d}$ have no assignable difference; that is, $\frac{a}{b}$ is equal to $\frac{c}{d}$.

260*. *If four magnitudes* a, b, c, d *be proportionals according to the geometrical definition, they will be proportionals also according to the algebraical definition.*

It is required to prove, that if m and n be any integral multipliers *whatever*, and $ma > nb$ when $mc > nd$, and $ma = nb$ when $mc = nd$, and $ma < nb$ when $mc < nd$, then $\frac{a}{b} = \frac{c}{d}$.

If a, b, c, d be commensurable, m and n can always be assumed so as to make $ma = nb$: then also $mc = nd$, and $\frac{a}{b} = \frac{n}{m} = \frac{c}{d}$.

If, however, a, b, c, d be incommensurable, the above equalities cannot be obtained; but we can always make ma approach as near as we please to nb, by giving proper values to m and n: *i.e.* we can make ma differ from nb by a quantity less than b, or make ma lie between nb and $(n+1)b$. Then also will mc lie between nd and $(n+1)d$; *i.e.* both $\frac{a}{b}$ and $\frac{c}{d}$ lie between $\frac{n}{m}$ and $\frac{n+1}{m}$.

Also m and n may be increased without limit; therefore $\frac{a}{b}=\frac{c}{d}$ or a, b, c, d are proportionals according to the algebraical definition.

All the preceding propositions, therefore, respecting proportionals, will be equally true for incommensurable quantities as for commensurable.

VARIATION.

261. In the investigation of the relation which varying and dependent quantities bear to each other the conclusions are more readily obtained by expressing only *two* terms in each proportion, than by retaining the *four*.

But though, in considering the variation of such quantities, *two* terms only are expressed, it will be necessary for the Learner to keep constantly in mind that *four* are supposed; and that the operations, by which our conclusions are in this case obtained, are in reality the operations of *proportionals*.

262. DEF. 1. One quantity is said to *vary directly* as another, when the two quantities depend wholly upon each other, and in such a manner, that if one be changed, the other is changed *in the same proportion**.

Let A and B be mutually dependent upon each other, in such a way, that if A be changed to any other value a, B must be changed to another value b, such that $A : a :: B : b$; then A is said to *vary directly* as B.

N.B. When it is simply stated that one quantity 'varies' as another it is always meant that the one 'varies *directly*' as the other.

Ex. 1. If a man agrees to work for a certain sum per hour, the amount of his wages *varies as* the number of hours during which he works; for as the number of hours increases or decreases, his wages will increase or decrease, *and in the same proportion.*

Ex. 2. If the altitude of a triangle be invariable, the area varies as the base. For, if the base be increased, or diminished, the area is increased or diminished *in the same proportion*, (the area of a triangle being the half of the rectangle under the base and perpendicular. See Euclid, Book I. Props. 36 and 41).

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* That Variation is merely an abridgment of Proportion is a point to be carefully borne in mind; for one quantity is said to "vary" as another, not because the two increase and decrease together, but because as one increases or decreases, the other increases or decreases *in the same proportion.* Thus, if $y=\sqrt{ax}$, in which x and y are varying quantities and a is invariable, y increases as x increases, and diminishes as x diminishes, but y does *not "vary"* as x, because, as x increases, y does not increase *in the same proportion;* for instance, if x be doubled, y is not doubled.—ED.

The sign $\propto$ placed between two quantities signifies that they *vary* as each other.

263. DEF. 2. One quantity is said to *vary inversely* as another, when the former cannot be changed in any manner, but the reciprocal* of the latter is changed *in the same proportion.*

A varies inversely as B, $\left(A \propto \frac{1}{B}\right)$, if, when A is changed to a, B be changed to b, in such a manner that $A : a :: \frac{1}{B} : \frac{1}{b}$; or $A : a :: b : B$.

Ex. 1. If a letter-carrier has a fixed route, the *time* in which he will perform it *varies inversely* as his *speed.* If he *double* his speed, he will go in *half* the time; and so on.

Ex. 2. If the area of a triangle be given, the base *varies inversely* as the perpendicular altitude.

Let A and a represent the altitudes, B and b the bases, of two equal triangles; then $\frac{A \times B}{2} = \frac{a \times b}{2}$, or $A \times B = a \times b$; therefore (Art. 240), $A : a :: b : B :: \frac{1}{B} : \frac{1}{b}$.

264. DEF. 3. One quantity is said to *vary as two others jointly,* if, when the former is changed in any manner, the *product* of the other two be changed *in the same proportion.*

Thus A varies as B and C jointly, ($A \propto BC$), when A cannot be changed to a, but the *product* BC must be changed to bc, such that $A : a :: BC : bc$.

Ex. 1. The wages to be received by a workman will *vary* as the number of days he has worked and the wages per day *jointly;* for if either the number of days, or the wages per day, be doubled, trebled, &c., so as to double or treble, &c., their *product,* the whole wages will be doubled, trebled, &c., that is, altered *in the same proportion.*

Ex. 2. The area of a triangle varies as its base and perpendicular altitude jointly. Let A, B, P, represent the area, base, and perpendicular altitude of one triangle; a, b, p, those of another; then $A = \frac{1}{2}BP$, and $a = \frac{1}{2}bp$; therefore $\frac{A}{a} = \frac{BP}{bp}$, or $A : a :: BP : bp$.

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* DEF. The *reciprocal* of a quantity is $\frac{1}{\text{the quantity}}$; thus the *reciprocal* of a is $\frac{1}{a}$.—ED.

265. **Def.** 4. One quantity is said to vary *directly* as a second and *inversely* as a third, when the first cannot be changed in any manner, but the second multiplied by the reciprocal of the third is changed in the same proportion.

A varies directly as B, and inversely as C, $\left(A \propto \frac{B}{C}\right)$, when $A : a :: \frac{B}{C} : \frac{b}{c}$; A, B, C, and a, b, c, being corresponding values of the three quantities.

Ex. The base of a triangle varies as the area directly and the perpendicular altitude inversely. The notation in the last Article being retained, $\frac{BP}{bp} = \frac{A}{a}$; and multiplying both sides by $\frac{p}{P}$, we have $\frac{B}{b} = \frac{Ap}{aP} = \frac{A}{P} \div \frac{a}{p}$; therefore $B : b :: \frac{A}{P} : \frac{a}{p}$.

In the following articles, A, B, C, &c. represent corresponding values of any quantities, and a, b, c, &c. any other corresponding values of the same quantities.

266. *If one quantity vary as a second, and that second as a third, the first varies as the third.*

Let $A \propto B$, and $B \propto C$, then shall $A \propto C$. For $A : a :: B : b$, and $B : b :: C : c$, therefore (Art. 241), $A : a :: C : c$; that is, $A \propto C$.

In the same manner, if $A \propto B$, and $B \propto \frac{1}{C}$, then $A \propto \frac{1}{C}$.

267. *If each of two quantities vary as a third, their sum, or difference, or the square root of their product, will vary as the third.*

Let $A \propto C$, and $B \propto C$, then $\overline{A \pm B} \propto C$; also $\sqrt{AB} \propto C$.

By the supposition, $A : a :: C : c :: B : b$;

$\therefore A : a :: B : b$;

alternately $A : B :: a : b$ (Art. 243),

by composition or division, $A \pm B : B :: a \pm b : b$;

alternately $A \pm B : a \pm b :: B : b :: C : c$;

that is, $A \pm B \propto C$.

Again, $A : a :: C : c$,

and $B : b :: C : c$;

$\therefore AB : ab :: C^2 : c^2$ (Art. 252),

and $\sqrt{AB} : \sqrt{ab} :: C : c$ (Art. 253);

that is, $\sqrt{AB} \propto C$.

268. *If one quantity vary as another, it will also vary as any multiple, or part, of the other.*

Let $A \propto B$, and m be any constant quantity, then because $A : a :: B : b$, $A : a :: mB : mb$, or $A : a :: \frac{B}{m} : \frac{b}{m}$ (Art. 249); that is, $A \propto mB$, or $\propto \frac{B}{m}$.

269. Cor. 1. *If* A *vary as* B, A *is equal to* B *multiplied by some invariable quantity*[*].

For $A : a :: mB : mb$; altern. $A : mB :: a : mb$; if therefore m be so assumed that $A = mB$, then in all cases $a = mb$.

Conversely, if $A = mB$, and m is invariable, then $A \propto B$.

270. Cor. 2. If we know any corresponding values of A and B, the constant quantity m may be found.

Let a and b be the two values known, then $m = \frac{a}{b}$, and is therefore known; and, in general, $A = \frac{a}{b} \times B$.

Ex. 1. Let $s \propto t^2$, and when $t = 1$ suppose $s = 16$, then, since $s = mt^2$, $16 = m$, and $s = 16t^2$.

Ex. 2. Let $y^2 \propto a^2 - x^2$, and when $x = 0$, suppose $y = b$, then, since $y^2 = m(a^2 - x^2)$, $b^2 = ma^2$, and $m = \frac{b^2}{a^2}$; $\therefore$ $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$.

271. *If one quantity vary as another, any power or root of the former will vary as the same power or root of the latter.*

Let A vary as B, then $A : a :: B : b$, and by Art. 253, $A^n : a^n :: B^n : b^n$; that is, $A^n \propto B^n$, where n may be either whole or fractional.

272. *If one quantity vary as another, and each of them be multiplied or divided by any quantity, variable or invariable, the products, or quotients, will vary as each other.*

Let A vary as B, and let T be any other quantity. Then, by the supposition, $A : a :: B : b$;

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[*] By the application of this rule almost every question in Variation is readily solved, since the variation is convertible into an equation, to which the usual rules may be applied.—Ed.

$$\therefore AT : at :: BT : bt;\ \text{that is, } AT \propto BT.$$

$$\text{Also } \frac{A}{T} : \frac{a}{t} :: \frac{B}{T} : \frac{b}{t};\ \text{that is, } \frac{A}{T} \propto \frac{B}{T}.$$

273. Cor. If $A \propto B$, dividing both by B, $\frac{A}{B} \propto \frac{B}{B} \propto 1$; that is, $\frac{A}{B}$ is constant.

274. *If one quantity vary as two others jointly, either of the latter varies as the first directly and the other inversely.*

Let $V \propto FT$, then by Art. 272, $F \propto \frac{V}{T}$, or $T \propto \frac{V}{F}$.

275. Cor. If the product of two quantities be invariable, those quantities vary inversely as each other.

Let $B \times P$ be constant, or $B \times P \propto 1$; by division, $B \propto \frac{1}{P}$.

276. *If four quantities be always proportionals, and one or two of them be invariable, we may find how the others vary.*

Ex. Let p, q, r, s, be always proportionals, and let p be invariable, then $s \propto qr$. Because $ps = qr$ (Art. 237), $ps \propto qr$; and since p is constant, $s \propto qr$ (Art. 268). If both p and q be invariable, $s \propto r$.

277. *If one quantity vary as a second, and a third as a fourth, the product of the first and third will vary as the product of the second and fourth.*

Let $A \propto B$, and $C \propto D$, then $AC \propto BD$.

$$\begin{aligned} \text{Because } A &: a :: B : b, \\ \text{and } C &: c :: D : d, \\ AC &: ac :: BD : bd \text{ (Art. 252)}; \\ \text{that is, } AC &\propto BD. \end{aligned}$$

278. *When the increase or decrease of one quantity depends upon the increase or decrease of* two *others, and it appears that, if either of these latter be invariable, the first varies as the other; when they both vary, the first varies as their product.*

Let $S \propto V$ when T is given,

and $S \propto T$ when V is given;

when neither T nor V is given, $S \propto TV$.

For the variation of S depends upon the variations of the two quantities T and V; let the changes take place separately, and while T is changed to t, let S be changed to S'; then, by the supposition,

$$S : S' :: T : t;$$

but this value S' will again be changed to s, by the variation of V, and in the same proportion that V is changed; that is,

$$S' : s :: V : v;$$

and by compounding this with the last proportion,

$$S'S : S's :: TV : tv;$$

$$\text{or} \quad S : s :: TV : tv \text{ (Art. 249)},$$

$$\text{that is, } S \propto TV.$$

Ex. If 6 horses can plough 17 acres in 2 days, how many acres will 93 horses plough in $4\frac{1}{2}$ days?

The number of acres ploughed $\propto$ number of horses,

if the days are the same.

.... $\propto$ number of days,

if the horses are the same.

When both horses and days are different,

number of acres $\propto$ number of days × number of horses;

$\therefore$ number of acres required : 17 acres :: $93 \times 4\frac{1}{2} : 6 \times 2$;

$$\therefore \frac{\text{number of acres required}}{17} = \frac{93 \times 4\frac{1}{2}}{6 \times 2},$$

$$\therefore \text{number of acres required} = \frac{31 \times 2\frac{1}{4} \times 17}{2} = 592\frac{7}{8}.$$

279. *If there be any number of magnitudes,* P, Q, R, S, &c., *each of which varies as another,* V, *when the rest are constant; when they are all changed,* V *varies as their product.*

Let $V \propto P$ when Q, R, and S, are given;

$V \propto Q$ when P, R, and S, are given;

$V \propto R$ when P, Q, and S, are given;

$V \propto S$ when P, Q, and R, are given;

when P, Q, R, S, are all variable, $V \propto PQRS$.

Let the changes of V, dependent upon the changes of
place separately, and whilst P is changed to p, let V be ch
when Q is changed to q, let V_1 be changed to V_2; when R is cha
let V_2 be changed to V_3; and when S is changed to s, let V_3 be
to v. Then, by the supposition, these changes are such, that we ha
following proportions :— the

$$V : V_1 :: P : p,$$
$$V_1 : V_2 :: Q : q,$$
$$V_2 : V_3 :: R : r,$$
$$V_3 : v :: S : s;$$
$$\therefore V : v :: PQRS : pqrs; \text{ (Arts 252, 249)},$$
$$\text{that is, } V \propto PQRS.$$

Otherwise, $V \propto P$, when Q, R, &c. are invariable,

$\therefore V = mP$, where m does not contain P. (Art. 269.)

But if P be made constant, and Q be made to vary, $V \propto Q$, and $\therefore mP \propto Q$; $\therefore$ also $m \propto Q$, and consequently $m = nQ$, where n does not contain Q; and n cannot contain P, by what has been said before, $\therefore V = nPQ$. Now if R be considered alone to vary, by the same reasoning we have $n \propto R$, and $\therefore = pR$; $\therefore V = p.PQR$, where p does not contain P, Q, or R:

$$\therefore V \propto PQR.$$

This may be extended to any number of magnitudes P, Q, R, S, &c.

[*Exercises Zb.*]

ARITHMETICAL PROGRESSION.

280. **Def.** Quantities are said to be in *Arithmetical Progression* when they increase or decrease by a *Common Difference*. Thus each of the following series of quantities

$$\begin{array}{llllll} 1, & 3, & 5, & 7, & 9, & \&c. \\ 20, & 19, & 18, & 17, & 16, & \&c. \\ a, & a+b, & a+2b, & a+3b, & \&c. & \\ a, & a-b, & a-2b, & a-3b, & \&c. & \end{array}$$

is in Arithmetical Progression.

Hence it is manifest, that, if a be the first term and $a+b$ the second, $a+2b$ is the third, $a+3b$ the fourth, &c. and $a+(n-1)b$ the n^{th} term.

By giving the proper value to n any one term may be found independently of the rest. Thus, if the 50th term of the first series be required, we have

$$a = 1, \quad b = 2, \quad \text{and } n = 50,$$

$$\therefore \text{ the } 50^{\text{th}} \text{ term} = 1 + (50-1) \times 2 = 1 + 98 = 99.$$

e two extremes, and the number of terms, in an Arithmetical e given, the means, that is, the intervening terms, may be found.

and l be the extremes, that is, the first and last terms, of an etical Progression, n the number of terms, and b the unknown on difference; then the progression is

$$a,\ a+b,\ a+2b,\ \dots\ a+(n-1)b;$$

and since l is the last term, $\quad a+(n-1)b=l$;

$$\therefore\ b=\frac{l-a}{n-1};$$

and b being thus found, all the means $a+b$, $a+2b$, &c. are known.

Ex. There are four means, or intervening terms, in Arithmetical Progression between 1 and 36; find them.

Here $a=1$, $l=36$, $n=6$,

$$\therefore\ b=\frac{36-1}{5}=7;$$

and the means are 8, 15, 22, 29.

COR. A single Arithmetical mean between any two quantities a and b, which is called *the Arithmetical mean,* will be $a+\frac{b-a}{3-1}$, or $\frac{a+b}{2}$; but this may be shewn more simply thus,

Let a, x, b be in Arithmetical Progression,

then by Def. $\quad x-a=b-x$; $\therefore\ x=\frac{1}{2}(a+b)$.

282. *The sum of a series of quantities in Arithmetical Progression may be found by multiplying the sum of the first and last terms by half the number of terms**.

Let a be the first term, b the common difference†, n the number of terms, l the last term, and s the sum of the series:

$$\text{Then,}\quad a+(a+b)+(a+2b)+\text{\&c.}\dots\dots\dots+l=s;$$

also, reversing the order of the terms,

$$l+(l-b)+(l-2b)+\text{\&c.}\dots\dots\dots+a=s;$$

$$\text{adding,}\quad (a+l)+(a+l)+(a+l)+\text{\&c. to } n \text{ terms}=2s,$$

$$\text{that is,}\quad n \text{ times } a+l, \text{ or } (a+l)n=2s;$$

$$\therefore\ s=(a+l)\frac{n}{2}.$$

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* By this rule we are enabled to find the sum of *any number* of terms in Arithmetical Progression, without the trouble of adding them all together.—ED.

† The *Common Difference* in any proposed case is obviously found by subtracting *any* term from the term next following.—ED.

COR. 1. Since $l = a + (n-1)b$, we have also

$$s = \{2a + (n-1)b\}\frac{n}{2}.$$

COR. 2. Any three of the quantities s, a, n, b, being given, the fourth may be found from the equation

$$s = \{2a + (n-1)b\}\frac{n}{2}.$$

Ex. 1. To find the sum of 14 terms of the series 1, 3, 5, 7, &c.

$$\text{Here } a = 1, \quad b = 2, \quad n = 14;$$
$$\therefore s = (2 + 26) \times 7 = 196.$$

Ex. 2. Required the sum of 9 terms of the series 11, 9, 7, 5, &c.

In this case $a = 11$, $b = -2$, $n = 9$;

$$\therefore s = (22 - 16) \times \frac{9}{2} = 6 \times \frac{9}{2} = 27.$$

Ex. 3. If the first term of an arithmetical progression be 14, and the sum of 8 terms be 28, what is the common difference, and the series?

$$\text{Since } \{2a + (n-1)b\} \times \frac{n}{2} = s,$$
$$2a + (n-1)b = \frac{2s}{n},$$
$$(n-1)b = \frac{2s}{n} - 2a = \frac{2s - 2an}{n};$$
$$\therefore b = \frac{2s - 2an}{n(n-1)}.$$

In the case proposed $s = 28$, $a = 14$, $n = 8$;

$$\therefore b = \frac{56 - 224}{8 \times 7} = \frac{7 - 28}{7} = -3.$$

Hence the series is 14, 11, 8, 5, 2, -1, -4, -7, &c.

Ex. 4. The 1st term of a series in A. P. is 22, and the common difference is -4; how many terms will make 70?

Generally, $2s = (2a - b)n + bn^2$,

$\therefore$ here $140 = 48n - 4n^2$,

or $n^2 - 12n + 35 = 0$,

or $\underline{(n-5)(n-7) = 0}$; $\therefore n = 5$, or 7.

Both values of n satisfy the question; in the one case the series is 22, 18, 14, 10, 6, and in the other, 22, 18, 14, 10, 6, 2, −2.

With the same *data* as before, find the number of terms of which the sum is −90.

$$\text{Here } -180 = 48n - 4n^2,$$
$$\text{or} \quad n^2 - 12n - 45 = 0,$$
$$\text{or} \quad (n-15)(n+3) = 0; \ \therefore n = 15, \text{ or } -3.$$

Hence the number of terms is 15, and the series is

22, 18, 14, 10, 6, 2, −2, −6, −10, −14, −18, −22, −26, −30, −34, of which the sum is −90.

But in this and similar examples the *negative* value of n is excluded, as an answer to the question, by the supposition under which n *originally* enters the equation, since *a number of terms* can only be expressed by a positive integer.

A meaning can, however, be obtained for the *negative* value of n. For in the equation $s = \{2a + (n-1)b\}\frac{n}{2}$, write $-n$ for n, and it becomes

$$s = -\{2a - (n+1)b\}\frac{n}{2},$$

$$\text{or } s = \{(n+1)b - 2a\}\frac{n}{2}, \quad \text{or } \{2(b-a) + (n-1)b\}\frac{n}{2};$$

and if we compare this with the original equation, we see that the right-hand member is the sum of n terms of an arithmetical progression whose first term is $b-a$, and common difference b; this series therefore is indicated by the negative result.

Thus in the last Ex., where $n = -3$, $b - a = -26$, and the series will be −26, −30, −34, of which the sum is −90.

Conversely, it may be shewn, that, if n, and $-m$, be the roots of the equation $2s = (2a-b)n + bn^2$, the sum of m terms, beginning with $b-a$, is the same as the sum of n terms, beginning with a.

For $n - m = -\dfrac{2a-b}{b}$, (Art. 205), and therefore

$$\begin{aligned}\{2(b-a) + (m-1)b\} &= (b-2a)m + bm^2,\\ &= (b-2a)\left(n - \frac{b-2a}{b}\right) + b\left(n - \frac{b-2a}{b}\right)^2,\\ &= (b-2a)n + bn^2 - 2(b-2a)n,\\ &= (2a-b)n + bn^2 = \{2a + (n-1)b\}n.\end{aligned}$$

Another, and perhaps the best, way of interpreting a *negative* value for n, is derived thus :—putting $-m$ for n in the general equation, we get

$$-s = \{2(a-b) + (m-1).(-b)\}\frac{m}{2} = \text{sum of } m \text{ terms beginning with}$$

$a-b$, and with com. diff. $-b$; so that both values of n may be used *for the same series*, viz.

$$\&c., a-2b, a-b, \mid a, a+b, a+2b, \&c.$$

and reckoned from *the same starting point*, marked |, the one to the *right*, and the other to the *left*, the sum in either case being the same, but with different signs.

283. *The sum of any two terms taken equidistant from the beginning and end of an arithmetical progression is equal to the sum of the first and last terms.*

This is proved in the demonstration of Art. 282; or it may be shewn independently, thus:—Let a, b, c, d, e, f, g be in arithmetical progression, then, by Definition,

$$b-a=g-f, \text{ or } b+f=a+g.$$

Again, $c-b=f-e$; $\therefore c+e=b+f=a+g$. And so on, whatever the number of terms in the series may be.

284. *If the number of terms of an arithmetical progression be odd, twice the middle term is equal to the sum of the first and last terms.*

Let q represent the middle term, p the preceding, and r the succeeding term, then $q-p=r-q$, by Definition;

$$\therefore 2q=p+r,$$

and p and r are equidistant terms from the beginning and end;

$$\therefore p+r=a+l, \text{ by last Art.}$$
$$\text{and } 2q=a+l.$$

285. By the last two articles the summation of series in arithmetical progression is sometimes capable of being facilitated. Thus, if the number of terms whose sum is required be even, and more than half that number of terms are given, it is sometimes easier to add together two terms equidistant from the beginning and end of the series, than to find either the "common difference" or the last term.

If the number of terms be odd, and the middle term is one of those which are given, then, from what has been proved, *the middle term multiplied by the whole number of terms will be the sum required.*

Ex. 1. Required the sum of $\frac{2}{3}+\frac{7}{15}+\frac{4}{15}+\frac{1}{15}+$&c. to 7 terms.

Here $\frac{1}{15}$ is the middle term; $\therefore$ sum required $=\frac{1}{15}\times 7=\frac{7}{15}$.

Ex. 2. Required the sum of $1+\frac{3}{2}+2+\frac{5}{2}+$&c. to 6 terms.

Here the 3rd and 4th terms are equidistant from the beginning and end,

$$\therefore \text{sum}=\left(2+\frac{5}{2}\right)\times 3=\frac{27}{2}=13\tfrac{1}{2}.$$

[*Exercises Zc.*]

GEOMETRICAL PROGRESSION.

286. **Def.** Quantities are said to be in *Geometrical Progression*, or continual proportion, when the first is to the second as the second to the third, and as the third to the fourth, &c., that is, when every succeeding term is a certain multiple, or part, of the preceding term.

If a be the first term, and ar the second, the series will be a, ar, ar^2, ar^3, ar^4, &c., the n^{th} term being ar^{n-1}.

$$\text{For } a : ar :: ar : ar^2 :: ar^2 : ar^3, \text{ \&c.}$$

$$\text{or, } \frac{ar}{a} = \frac{ar^2}{ar} = \frac{ar^3}{ar^2} = \text{\&c.}$$

287. The constant multiplier, by which any term is derived from the preceding, is called the *Common Ratio*, and it may be found by dividing the second term by the first, or *any other term by that which precedes it.*

Ex. 1. 1, 3, 9, 27, &c. are in Geometrical Progression, find the *Common Ratio.*

$$\text{Here Common Ratio} = \frac{3}{1} = 3.$$

Ex. 2. $2\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{10}$, &c. are in Geometrical Progression, find the Common Ratio.

$$\text{Here Common Ratio} = \frac{1}{10} \div \frac{1}{2} = \frac{1}{5}.$$

288. *If any terms be taken at equal intervals in a geometrical progression, they will be in geometrical progression.*

Let a, $ar \ldots ar^n \ldots\ldots ar^{2n} \ldots ar^{3n} \ldots$&c. be the progression, then a, ar^n, ar^{2n}, ar^{3n}, &c. are at the interval of n terms, and form a geometrical progression, whose common ratio is r^n.

289. *If the two extremes, and the number of terms, in a geometrical progression be given, the* means, *that is, the intervening terms, may be found.*

Let a and l be the extremes, n the number of terms, and r the common ratio; then the progression is a, ar, ar^2, $ar^3 \ldots ar^{n-1}$; and since l is the last term,

$$ar^{n-1} = l, \text{ and } r^{n-1} = \frac{l}{a};$$

$$\therefore r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}};$$

and r being thus known, all the *means*, or intervening terms, ar, ar^2, ar^3, &c. are known.

Cor. A single Geometrical *mean* between a and b, called *the Geometrical mean*, will be $a\left(\frac{b}{a}\right)^{\frac{1}{2}}$, or $\sqrt{ab}$; but this may be shewn more simply thus,

Let a, x, b be in Geometrical Progression, then, by Def., $\frac{x}{a}=\frac{b}{x}$;

$$\therefore\ x^2=ab, \text{ and } x=\sqrt{ab}.$$

Ex. There are three means, or intervening terms, in a Geometrical Progression between 2 and 32; find them.

$$\text{Here } n=5,\ a=2, \text{ and } l=32;$$

$$\therefore\ r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}=16^{\frac{1}{4}}=2;$$

and the means are 4, 8, 16.

290. *To find the sum of a series of quantities in Geometrical Progression.*

Let a be the first term, r the common ratio, n the number of terms, and s the sum of the series:

$$\text{Then}\quad s=a+ar+ar^2+\ldots+ar^{n-2}+ar^{n-1},$$

$$\text{and}\quad rs=\quad ar+ar^2+\ldots+ar^{n-2}+ar^{n-1}+ar^n.$$

$$\text{Subtracting,}\quad rs-s=ar^n-a;$$

$$\therefore s=\frac{ar^n-a}{r-1}=a\cdot\frac{r^n-1}{r-1}.$$

Cor. 1. If l be the last term, $l=ar^{n-1}$,

$$\therefore s=\frac{rl-a}{r-1};$$ [*]

from which equation, any three of the quantities s, r, l, a, being given, the fourth may be found.

Cor. 2. When r is a proper fraction, as n increases, the value of r^n, or of ar^n, decreases, and when n is increased without limit, ar^n becomes less, with respect to a, than any magnitude that can be assigned[†]. Hence the sum of the series, which in general is equal

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[*] The following is another method of arriving at the same result equally simple:—

Let a, b, c, d, &c., h, k, l, be the series, s the sum, and r the common ratio; then, by definition, $b=ar,\ c=br,\ d=cr,\ldots k=hr,\ l=kr,$

$$\therefore\ b+c+d+\ldots+k+l=(a+b+c+\ldots+h+k)r,$$

$$\text{or } s-a=(s-l)r=rs-rl;$$

$$\therefore\ s=\frac{rl-a}{r-1}.$$

Ed.

[†] Thus, if $r=\frac{3}{10}$ or 0·3, $r^2=0{\cdot}09$, $r^3=0{\cdot}0027$; and so on, shewing that as n increases r^n decreases, and that such a power of r may be taken as to produce a quantity less than any number which shall be named, however small.—Ed.

to $\frac{ar^n - a}{r-1}$, or $\frac{ar^n}{r-1} - \frac{a}{r-1}$, is reduced in this case, to $\frac{-a}{r-1}$, that is, $\frac{a}{1-r}$.

This quantity, $\frac{a}{1-r}$, which we call the *sum* of the series, is the *limit* to which the sum of the terms approaches, but never actually attains*; it is however the true *representative* of the series continued *sine fine*, for this series arises from the division of a by $\overline{1-r}$; and therefore $\frac{a}{1-r}$ may without error be substituted for it.

Ex. 1. To find the sum of 20 terms of the series, 1, 2, 4, 8, &c.

$$\text{Here } a = 1,\ r = 2,\ n = 20;$$

$$\therefore s = \frac{1 \times 2^{20} - 1}{2-1} = 2^{20} - 1.$$

Ex. 2. Required the sum of 12 terms of the series, 64, 16, 4, &c.

$$\text{Here } a = 64,\ r = \frac{1}{4},\ n = 12;$$

$$\therefore s = 64 . \frac{\frac{1}{4^{12}} - 1}{\frac{1}{4} - 1} = \frac{4^3}{4^{11}} . \frac{4^{12} - 1}{4 - 1} = \frac{1}{4^8} . \frac{4^{12} - 1}{3}.$$

Ex. 3. Required the sum of 12 terms of the series 1, -3, 9, -27, &c.

$$\text{Here } a = 1,\ r = -3,\ n = 12;$$

$$\therefore s = \frac{(-3)^{12} - 1}{-3-1} = -\frac{3^{12} - 1}{4}.$$

Ex. 4. To find the sum of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} +$ &c. *in infinitum.*

$$\text{Here } a = 1,\ r = -\frac{1}{2};\ \therefore s = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}.$$

[*Exercises Zd.*]

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* That is, although no *definite* number of terms will amount to $\frac{a}{1-r}$, yet, by taking a sufficient number, the sum will reach as near as we please to it; and, whatever number be taken, their sum will not exceed it.—ED.

291. Recurring decimals are made up of quantities in geometrical progression, where $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. is the common ratio, according as one, two, three, &c. figures recur; and the vulgar fraction corresponding to such a decimal is found by summing the series.

Ex. 1. Required the vulgar fraction which is equivalent to the decimal ·123123123, &c.

Here the series is $\frac{123}{10^3}+\frac{123}{10^6}+\frac{123}{10^9}+\ldots\ldots$ *in inf.*; $a=\frac{123}{10^3}$, and $r=\frac{1}{10^3}$;

$$\therefore\ s=\frac{\frac{123}{10^3}}{1-\frac{1}{10^3}}=\frac{123}{999}=\frac{41}{333}.$$

Or, as follows:—

Let ·123123123 &c. $= s$; then, as in Art. 290, multiply both sides by 1000; and 123·123123123 &c. $= 1000s$, and by subtracting the former equation from the latter, $123 = 999s$;

$$\therefore\ s=\frac{123}{999}=\frac{41}{333}.$$

Ex. 2. Required the vulgar fraction which is equivalent to $\cdot PPP\ldots$ where P contains p digits recurring *in inf.*

$$\begin{aligned}\text{Let}\quad & s=\cdot PPP\ldots\\ \text{then}\quad & 10^p s=P\cdot PPP\ldots\\ \therefore\ & (10^p-1)s=P,\\ \text{and}\quad & s=\frac{P}{10^p-1}.\end{aligned}$$

Ex. 3. Required the vulgar fraction equivalent to $\cdot PQQQ\ldots$, where P contains p digits, and Q contains q digits recurring *in inf.*

$$\begin{aligned}\text{Let}\quad & s=\cdot PQQQ\ \&\text{c.},\\ \text{then}\quad & 10^{p+q}s=PQ\cdot QQ\ \&\text{c.},\\ \text{and}\quad & 10^p s=\ P\cdot QQ\ \&\text{c.};\\ \therefore\ & (10^{p+q}-10^p)s=PQ-P,\\ \text{and}\quad & s=\frac{PQ-P^*}{10^p(10^q-1)}.\end{aligned}$$

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[*] Observe, the notation here employed is liable to be misunderstood, being arithmetical, rather than algebraical. For when two of the letters, P and Q, stand together, or Q and Q, without sign between them, it is not multiplication which is signified, but addition. Thus in the final result PQ does not mean $P\times Q$, but $P\times10^q+Q$, just as 345 means $3\times10^2+45$.

Both the results in Exs. 2 and 3 may be easily verified by expressing the proposed quantities in geometrical progressions—

$$\text{the former being } \frac{P}{10^p}+\frac{P}{10^{2p}}+\frac{P}{10^{3p}}+\ldots\ldots \textit{in inf.};$$

$$\text{the latter being } \frac{P}{10^p}+\frac{Q}{10^{p+q}}+\frac{Q}{10^{p+2q}}+\frac{Q}{10^{p+3q}}+\ldots\ldots \textit{in inf.};$$

which may be summed by the rule in Art. 290, Cor. 2.

292. *In a Geometrical series continued* in inf. *any term* $>$, $=$, *or* $<$, *the sum of all that follow, according as the common ratio* $<$, $=$, *or* $>$, $\frac{1}{2}$.

Let $a+ar+ar^2+\ldots ar^{n-1}+ar^n+\ldots\ldots$ be the series. Then the sum of the series after n terms is the sum of

$$ar^n+ar^{n+1}+\ldots\ldots \text{which} = ar^n.\frac{1}{1-r} \quad \text{(Art. 290, Cor. 2)}$$

$$\text{and } ar^{n-1} >,\ =,\ \text{or} <,\ ar^n.\frac{1}{1-r},$$

$$\text{according as} \quad 1 >,\ =,\ \text{or} <,\ \frac{r}{1-r},\ \text{(Art. 219)},$$

$$\ldots\ldots\ldots \quad 1-r >,\ =,\ \text{or} <,\ r,\ \text{(Art. 219)},$$

$$\ldots\ldots\ldots \quad 1 >,\ =,\ \text{or} <,\ 2r,\ \text{(Art. 217)},$$

$$\ldots\ldots\ldots \quad r <,\ =,\ \text{or} >,\ \frac{1}{2}. \quad \text{(Art. 219)}.$$

HARMONICAL PROGRESSION.

Def. Any magnitudes a, b, c, d, e, &c. are said to be in *Harmonical Progression*, if $a : c :: a-b : b-c$; $b : d :: b-c : c-d$; $c : e :: c-d : d-e$; &c.

293. *The reciprocals of quantities in Harmonical Progression are in Arithmetical Progression.*

Let a, b, c, &c. be in Harmonical Progression;

then by Def. $a : c :: a-b : b-c$;

$$\therefore ab-ac=ac-bc, \text{ (Art. 237)},$$

and dividing both sides by abc,

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}.$$

Again, $b : d :: b-c : c-d$;

$$\therefore bc-bd=bd-cd,$$

and dividing by bcd,

$$\frac{1}{d}-\frac{1}{c}=\frac{1}{c}-\frac{1}{b};$$

and $\frac{1}{c}-\frac{1}{b}$ has been proved equal to $\frac{1}{b}-\frac{1}{a}$; therefore the quantities $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, have a common difference; that is, they are in Arithmetical Progression. And the same proof may be extended to any number of terms.

Hence every series of quantities in Harmonical Progression may be easily converted into an Arithmetical Progression, and then the rules of Arithmetical Progression may be applied to it. Thus,

Ex. *Given* a *and* b *the first two terms of an Harmonic Series, to find the* n^{th} *term.*

Since $\frac{1}{a}, \frac{1}{b}$, are contiguous terms of an Arithmetic Series,

$$\therefore \text{ the Common Difference of this Series} = \frac{1}{b}-\frac{1}{a},$$

$$\text{and its } n^{\text{th}} \text{ term} = \frac{1}{a}+(n-1)\left(\frac{1}{b}-\frac{1}{a}\right), \text{ (Art. 280)},$$

$$=\frac{1}{a}+(n-1)\frac{a-b}{ab}=\frac{(n-1)a-(n-2)b}{ab};$$

$$\therefore \text{ the } n^{\text{th}} \text{ term required} = \frac{ab}{(n-1)a-(n-2)b}.$$

294. *The two extremes, and the number of terms, in an Harmonical Progression being given, the means, or intervening terms, may be found.*

Let a and l be the extremes, and n the number of terms; then since the reciprocals of the terms are in Arithmetic Progression, let b be their common difference, and $\frac{1}{l}$ being the n^{th} term of the Arithmetic Progression, we have

$$\frac{1}{l}=\frac{1}{a}+(n-1)b, \text{ (Art. 280)},$$

$$b=\left(\frac{1}{l}-\frac{1}{a}\right)\div(n-1),$$

$$=\frac{a-l}{(n-1)al};$$

hence the Arithmetic Means are

$$\frac{1}{a}+\frac{a-l}{(n-1)al},\ \frac{1}{a}+\frac{2(a-l)}{(n-1)al},\ \frac{1}{a}+\frac{3(a-l)}{(n-1)al},\ \&c.$$

$$\text{or } \frac{a+(n-2)l}{(n-1)al},\ \frac{2a+(n-3)l}{(n-1)al},\ \frac{3a+(n-4)l}{(n-1)al},\ \&c.$$

∴ the Harmonic Means are the reciprocals of these quantities, viz.

$$\frac{(n-1)al}{a+(n-2)l},\ \frac{(n-1)al}{2a+(n-3)l},\ \frac{(n-1)al}{3a+(n-4)l},\ \&c.$$

COR. A single Harmonic Mean between a and b, called *the Harmonic Mean*, will be $\frac{(3-1)ab}{a+(3-2)b}$, or $\frac{2ab}{a+b}$; but this may be shewn more simply thus,

Let a, x, b be in Harmonic Progression,

then by Def. $a : b :: a-x : x-b$;

$$\therefore ax-ab=ab-bx, \text{ (Art. 237)},$$

$$(a+b)x=2ab,$$

$$\therefore x=\frac{2ab}{a+b}.$$

OBS. There is no general expression for the *Sum* of an Harmonic Series, since the sum of any number of quantities is not deducible from the sum of their reciprocals.

[*Exercises Ze.*]

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295. Series are sometimes proposed for summation which are not actually composed of terms in Arithmetical or Geometrical Progression, but which may be made to depend upon the rules of one or both by arrangement or artifice. Thus,

Ex. 1. Let the sum of n terms of the following series be required,

$$1+5+13+29+61+\&c.$$

The series is equivalent to

$$\overline{4-3}+\overline{8-3}+\overline{16-3}+\overline{32-3}+\&c.$$

$$=4+8+16+\&c. \text{ to } n \text{ terms} - 3\times n,$$

$$=4.\frac{2^n-1}{2-1}-3n=4(2^n-1)-3n.$$

Ex. 2. To find the sum of n terms of the series

$$1+2x+3x^2+4x^3+\&c.$$

Let S be the sum required; then

$$S=1+2x+3x^2+\dots\dots\dots+nx^{n-1},$$

$$\therefore xS=\quad x+2x^2+\dots\dots+(n-1)x^{n-1}+nx^n,$$

subtracting, $$S-xS=1+x+x^2+\dots\dots+x^{n-1}-nx^n,$$

$$\text{or } S(1-x)=\frac{1-x^n}{1-x}-nx^n, \text{ (Art. 290)},$$

$$\therefore S=\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x},$$

$$=\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}.$$

Ex. 3. To find the sum of n terms of the series

$$\frac{n-1}{n}.x+\frac{n-2}{n}.x^2+\frac{n-3}{n}.x^3+\&c.$$

$$\text{Sum required}=x+x^2+x^3+\dots\dots+x^n-\frac{x}{n}\{1+2x+3x^2+\dots+nx^{n-1}\},$$

$$=x.\frac{1-x^n}{1-x}-\frac{x}{n}.\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}, \text{ (Ex. 2)},$$

$$=\frac{(n-1)x-nx^2+x^{n+1}}{n(1-x)^2}.$$

Ex. 4. To find the sum of n terms of the series

$$a^2+(a+b)^2+(a+2b)^2+\&c.$$

Let $A_1, A_2, A_3, \dots A_n$ represent the several terms in order of the series a, $a+b$, $a+2b$, &c. and S the sum required; then

$$A_2^3-A_1^3=(A_1+b)^3-A_1^3=3A_1^2b+3A_1b^2+b^3,$$

$$A_3^3-A_2^3=(A_2+b)^3-A_2^3=3A_2^2b+3A_2b^2+b^3,$$

$$A_4^3-A_3^3=(A_3+b)^3-A_3^3=3A_3^2b+3A_3b^2+b^3,$$

$$\dots\dots\dots=\dots\dots\dots\dots\dots=\dots\dots\dots\dots\dots\dots$$

$$A_{n+1}^3-A_n^3=(A_n+b)^3-A_n^3=3A_n^2b+3A_nb^2+b^3;$$

$\therefore$ by addition,

$$A_{n+1}^3-A_1^3=3(A_1^2+A_2^2+\dots A_n^2)b+3(A_1+A_2+\dots+A_n)b^2+nb^3,$$

$$\text{or } (a+nb)^3-a^3=3bS+3\{2a+(n-1)b\}\frac{nb^2}{2}+nb^3,$$

$$\therefore S=\frac{1}{3b}\left\{3na^2b+3n^2ab^2+n^3b^3-3nab^2-\frac{3n^2b^3}{2}+\frac{nb^3}{2}\right\},$$

$$=na^2+n^2ab+\frac{n^3b^2}{3}-nab-\frac{n^2b^2}{2}+\frac{nb^2}{6},$$

$$=na(a+\overline{n-1}.b)+\frac{nb^2}{6}(2n^2-3n+1).$$

COR. If $a=b=1$, then

$$1^2+2^2+3^2+\ldots+n^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6},$$
$$=\frac{2n^3+3n^2+n}{6},$$
$$=\frac{(2n+1)n^2+(2n+1)n}{6},$$
$$=\frac{1}{6}.n(n+1)(2n+1).$$

If $a=1$, and $b=2$, then

$$1^2+3^2+5^2+\&c. \text{ to } n \text{ terms}=2n^2-n+\frac{4n^3}{3}-2n^2+\frac{2n}{3},$$
$$=\frac{n}{3}(4n^2-1)=\frac{(2n-1)2n(2n+1)}{1\times2\times3};$$

and so on, when any other values are given to a and b.

PERMUTATIONS AND COMBINATIONS.

296. DEF. The different orders in which any quantities can be arranged are called their *Permutations.*

Thus the *permutations* of *a, b, c,* taken two and two together, are *ab, ba, ac, ca, bc, cb*; taken three and three together are *abc, acb, bac, bca, cab, cba.*

Some writers on Algebra call them *Permutations* only when the quantities are taken *all* together; and in all other cases *Variations.*

297. DEF. The *Combinations* of quantities are the different collections that can be formed out of them, without regarding the order in which the quantities are placed.

Thus *ab, ac, bc,* are the *combinations* of the quantities, *a, b, c,* taken two and two; *ab* and *ba,* though different *permutations,* forming the same *combination.*

298. *The number of permutations that can be formed out of* n *quantities, taken two and two together, is* n(n – 1); *taken three and three together, is* n(n – 1)(n – 2).

In n things, *a, b, c, d,* &c. *a* may be placed before each of the rest, and thus form $\overline{n-1}$ *permutations;* in the same manner, there are $\overline{n-1}$ *permutations* in which *b* stands first; and so of the rest;

therefore there are, upon the whole, $n(n-1)$ *permutations* of this kind, *ab*, *ba*, *ac*, *ca*, &c.

Again, of $\overline{n-1}$ things b, c, d, &c. taken two and two together, there are $(n-1)(n-2)$ *permutations*, by the former part of the article, and by prefixing a to each of these, there are $(n-1)(n-2)$ *permutations*, taken three and three, in which a stands first; the same may be said of b, c, d, &c., therefore there are, upon the whole, $n(n-1)(n-2)$ such *permutations*.

Ex. The number of *permutations* of 7 things taken three together $=7\times6\times5=210$.

299. *To find the number of Permutations of* n *things taken* r *together.*

By Art. 298, the number taken two together $=n(n-1)$,

.... three .... $=n(n-1)(n-2)$.

Similarly, ... four .... $=n(n-1)(n-2)(n-3)$.

Now, suppose the law, which is here perceived, to hold generally, that is, let the number of *permutations* of n things a, b, c, d, &c. taken $\overline{r-1}$ together be

$$n(n-1)(n-2)......(n-r+2).$$

Then omitting a, it is equally true that the number of *permutations* of $\overline{n-1}$ things b, c, d, &c. taken $\overline{r-1}$ together is, (putting $\overline{n-1}$ for n),

$$(n-1)(n-2)......(n-r+1).$$

Prefix a to each of these last *permutations*, and there will be a set of *permutations* of n things taken r together in which a stands first in every *permutation*, the number of them being

$$(n-1)(n-2)......(n-r+1).$$

The same number may be made of similar *permutations* in which b stands first; and so for each of the n quantities a, b, c, d, &c.

Hence the whole number of *permutations* which can be made of n things taken r together is

$$n(n-1)(n-2)......(n-r+1),$$

if it be true that the number of *permutations* of n things taken $\overline{r-1}$ together is

$$n(n-1)(n-2)......(n-r+2);$$

that is, if the *assumed* law be true for any value of r it is *proved* true for the next higher value. But it has been shewn to hold (Art. 298) when $r=2$, and 3; therefore, it is true when $r=4$; and, if for 4, for 5; if for 5, for 6; and so on generally for any number.

COR. The number of *permutations* of n things taken all together is

$$n(n-1)(n-2)\ldots\ldots(n-n+1),$$
$$=n(n-1)(n-2)\ldots\ldots1,$$
$$\text{or } =1.2.3\ldots\ldots n, \text{ or } =\underline{|n}.$$

Ex. 1. Required the number of different ways in which 6 persons can be arranged at a dinner table.

Number required = number of *permutations* of 6 things taken all together,
$$=6\times5\times4\times3\times2\times1=720.$$

Ex. 2. Required the number of changes which can be rung upon 12 bells.

$$\text{Number required}=12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1,$$
$$=479001600.$$

300. *The number of Combinations that can be formed out of* n *things, taken two and two together, is* $n.\frac{n-1}{2}$; *taken three and three together, the number is* $n.\frac{n-1}{2}.\frac{n-2}{3}$.

The number of *permutations* in the first case is $n(n-1)$, (Art. 298), but each *combination*, *ab*, admits of two *permutations*, *ab*, *ba*; therefore there are twice as many *permutations* as *combinations*, or the number of *combinations* is $n.\frac{n-1}{2}$.

Again, there are $n(n-1)(n-2)$ *permutations* in n things, taken three and three together; and each *combination* of three things admits of 3.2.1 *permutations* (Art. 299, Cor.); therefore there are 3.2.1 times as many *permutations* as *combinations*, and consequently the number of *combinations* is

$$\frac{n(n-1)(n-2)}{1.2.3}.$$

301. *To find the number of combinations of* n *things taken* r *together.*

Number of *permutations* of n things taken r together
$$=n(n-1)(n-2)\ldots\ldots(n-r+1). \quad \text{(Art. 299).}$$

But every *combination* of r things will make $1.2.3\ldots r$ *permutations* r together (Art. 299, Cor.); and no two of these can be the same; therefore there are $1.2.3\ldots r$ times as many *permutations* as *combinations*; and consequently the number of *combinations* is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\,r}.$$

This may be expressed in a very convenient form: for by multiplying the numerator and denominator of the above fraction by $1.2.3\ldots(n-r)$, it becomes

$$\frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots3.2.1}{1\,.\,2\,\,.\,\,3\ldots\ldots\ldots\ldots r\times1.2.3\ldots\ldots\ldots\ldots(n-r)},$$

$$\text{or}\quad\frac{|\underline{n}}{|\underline{r}\,.\,|\underline{n-r}}.$$

Ex. Required the number of combinations of 24 different letters taken four and four together.

$$\text{Here } n=24,\ r=4,$$

$$\therefore\ \text{number required}=\frac{24\times23\times22\times21}{1\times2\times3\times4},$$

$$=23\times22\times21=10626.$$

302. *To find when the number of combinations of a given number of things is the greatest.*

Since the number of combinations of n things taken r together is equal to the number taken $\overline{r-1}$ together multiplied by $\frac{n-r+1}{r}$, the number of combinations will go on increasing as r increases, so long as $\frac{n-r+1}{r}>1$, that is, so long as $r<\frac{1}{2}(n+1)$, but not longer.

If n be odd, $\frac{1}{2}(n+1)$ is a whole number, and the next less number is $\frac{1}{2}(n+1)-1$, or $\frac{1}{2}(n-1)$. Thus the last increase to the number of combinations made by increasing r is when r is made $\frac{1}{2}(n-1)$. But r may be further increased by 1, that is, made $\frac{1}{2}(n+1)$, and the number of combinations remain the same; for this value for r in $\frac{n-r+1}{r}$ gives $\frac{\frac{1}{2}(n+1)}{\frac{1}{2}(n+1)}$, or 1. Therefore for n odd, the greatest number of combinations is, when $r=\frac{1}{2}(n\mp1)$.

If n be even, the highest number for r, consistent with $r<\frac{1}{2}(n+1)$, is clearly $\frac{1}{2}n$. But $\frac{1}{2}n+1$ for r would make $\frac{n-r+1}{r}$ become $\frac{\frac{1}{2}n}{\frac{1}{2}n+1}$, which <1; so that for the greatest number of combinations, when n is even, $r=\frac{1}{2}n$.

Ex. Of six things how many must be taken together that the number of combinations may be the greatest possible?

Here $n=6$, an even number; $\therefore$ the number (r) to be taken together $=\frac{1}{2}n=3$; which will give $\frac{6\times5\times4}{1\times2\times3}$, or 20, combinations.

303. *The number of combinations of* n *things taken* r *together is the same as the number of combinations of* n *things taken* $\overline{n-r}$ *together.*

Number of combinations of n things taken r together is

$$\frac{|\underline{n}}{|\underline{r}\,.\,|\underline{n-r}}:$$

and writing $n-r$ for r, which may be done, because it is true for all values of r less than n, we have

Number of combinations of n things taken $\overline{n-r}$ together

$$=\frac{\underline{|n}}{\underline{|n-r}\,.\,\underline{|n-(n-r)}},$$

$$=\frac{\underline{|n}}{\underline{|n-r}\,.\,\underline{|r}},$$

the same as the number taken r together.

The truth of this proposition will also appear from a very simple consideration, viz. that of n things if r be taken, $\overline{n-r}$ things will always be left; and for every different parcel containing r things, there will be a different one left containing $\overline{n-r}$; therefore the *number* of the former parcels must be equal to that of the latter.

Hence in finding the number of combinations taken r together in certain cases, that is, when $r>\frac{1}{2}n$, it will be a shorter operation to find the number taken $\overline{n-r}$ together.

Ex. Required the number of combinations of 20 things taken 18 together.

Number required = number taken 2 together,

$$=\frac{20\times19}{1\times2}=10\times19=190.$$

304. *To find the number of permutations of* n *things taken all together, when the quantities recur.*

Let a recur p times, b recur q times, c recur r times, &c. And let P represent the number of permutations required. Then if all the a's be changed into different letters, these alone will form $1.2.3\ldots p$ permutations instead of one, and out of *each* of the P permutations we should form $1.2.3\ldots p$ permutations; therefore the whole number would be $P\times1.2.3\ldots p$. If again, all the b's be changed to different letters, in the same manner, the b's would of themselves form $1.2.3\ldots q$ permutations, and the whole number of permutations would be increased to

$$P\times1.2.3\ldots p.1.2.3\ldots q.$$

And so on, till all the quantities are different. But, when all are different, the number of permutations is $1.2.3\ldots n$, (Art. 299. Cor.);

$$\therefore\ P\times1.2.3\ldots p.1.2.3\ldots q.1.2.3\ldots r.\&c.=1.2.3\ldots n;$$

$$\text{and } P=\frac{1.2.3\ldots n}{1.2.3\ldots p.1.2.3\ldots q.1.2.3\ldots r.\&c.};$$

$$\text{or } P=\frac{\underline{|n}}{\underline{|p}\,.\,\underline{|q}\,.\,\underline{|r}\,.\,\&c.}.$$

Ex. Required the number of permutations that can be formed out of the letters of the word "Mississippi."

Here the whole number of letters is 11.

i recurs 4 times

s ...... 4 ......

p ...... 2 ......

$$\therefore P = \frac{1.2.3.4.5.6.7.8.9.10.11}{1.2.3.4.1.2.3.4.1.2},$$

$$= 5.7.9.10.11 = 34650.$$

305. *To find the number of combinations of* n *sets of things, containing respectively* p, q, r, *&c. things, one being taken out of each set for each combination.*

1. First, suppose there are two sets of things, containing p and q things respectively; then the number of combinations made by taking one out of each set is clearly p taken q times, or pq.

2. Next, let there be another set introduced containing r things; then each one of these being combined with pq combinations, there will be pqr combinations of three sets of things, one being taken out of each set.

And so on: the number of combinations required being the continued product of the numbers which express the number of things in each set.

COR. If there be the same number in each set, or $p = q = r = \&c.$, then the number of combinations is p^n.

Ex. There are 7 men, 5 women, and 3 boys; required the number of ways in which they can be taken, so as always to have one and no more out of each set.

The number required $= 7 \times 5 \times 3 = 105$.

306. *To find the number of combinations of two sets of things, containing respectively* p *and* q *things,* m *being taken out of one set and* n *out of the other for each combination.*

The number of combinations of the first set taken m together is $\frac{p(p-1)\ldots(p-m+1)}{1\,.\,2\ldots\ldots\ldots\ldots m}$; and the number of combinations of the second set taken n together is $\frac{q(q-1)\ldots(q-n+1)}{1\,.\,2\ldots\ldots\ldots\ldots n}$.

And, to form the combinations required, each of the latter must be combined with each of the former; therefore the number of them will be the product of these two quantities, or

$$\frac{p(p-1)(p-2)\ldots(p-m+1)}{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots m} \times \frac{q(q-1)\ldots(q-n+1)}{1\,.\,2\ldots\ldots\ldots\ldots n}.$$

And similarly if there be more than two sets of things, always taking the continued product of the respective numbers of combinations in each set.

Ex. Out of 10 consonants and 3 vowels how many different collections of letters may be made with 4 consonants and 2 vowels in each?

Here $p=10$, $q=3$, $m=4$, $n=2$;

$$\therefore \text{ the number required} = \frac{10\times9\times8\times7}{1\times2\times3\times4}\times\frac{3\times2}{1\times2} = 10\times9\times7 = 630.$$

[*Exercises Zf.*]

THE BINOMIAL THEOREM.

307. The method of raising a binomial to any power by repeated multiplication has been before laid down in Art. 140. The same thing may be done much more expeditiously by a general rule, which is called the *Binomial Theorem.*

This *Theorem* was first published by Sir Isaac Newton. It may have been discovered as follows :—By actual multiplication,

$$(x+a)^2 = x^2+2ax+a^2,$$
$$(x+a)^3 = x^3+3ax^2+3a^2x+a^3,$$
$$(x+a)^4 = x^4+4ax^3+6a^2x^2+4a^3x+a^4; \text{ and so on.}$$

Or
$$\frac{(x+a)^2}{1.2} = \frac{x^2}{1.2}+\frac{a}{1}.\frac{x}{1}+\frac{a^2}{1.2},$$
$$\frac{(x+a)^3}{1.2.3} = \frac{x^3}{1.2.3}+\frac{a}{1}.\frac{x^2}{1.2}+\frac{a^2}{1.2}.\frac{x}{1}+\frac{a^3}{1.2.3},$$
$$\frac{(x+a)^4}{1.2.3.4} = \frac{x^4}{1.2.3.4}+\frac{a}{1}.\frac{x^3}{1.2.3}+\frac{a^2}{1.2}.\frac{x^2}{1.2}+\frac{a^3}{1.2.3}.\frac{x}{1}+\frac{a^4}{1.2.3.4};$$

and so on; in which a *law* of formation of the terms in each expansion is distinctly perceptible in relation to the *index* of the power of the binomial. And the same *law* is found to hold for $(x+a)^5$, $(x+a)^6$, &c. Hence it is reasonable to *assume, as a thing to be proved,* that the same law may hold *generally,* viz. that, if n be *any* positive integer,

$$\frac{(x+a)^n}{\lfloor\underline{n}} = \frac{x^n}{\lfloor\underline{n}}+\frac{a}{\lfloor\underline{1}}.\frac{x^{n-1}}{\lfloor\underline{n-1}}+\frac{a^2}{\lfloor\underline{2}}.\frac{x^{n-2}}{\lfloor\underline{n-2}}+\frac{a^3}{\lfloor\underline{3}}.\frac{x^{n-3}}{\lfloor\underline{n-3}}+\ldots\ldots+\frac{a^n}{\lfloor\underline{n}};$$

$$\text{or } (x+a)^n = x^n+nax^{n-1}+\frac{n(n-1)}{1.2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3}+\ldots\ldots+a^n,$$

which is called the *Binomial Theorem.*

308. *To prove the Binomial Theorem for a positive integral index.*

By actual multiplication it appears, that

$$(x+a)(x+b) = x^2+(a+b)x+ab,$$
$$(x+a)(x+b)(x+c) = x^3+(a+b+c)x^2+(ab+ac+bc)x+abc,$$
$$\begin{aligned}(x+a)(x+b)(x+c)(x+d) = x^4 &+ (a+b+c+d)x^3\\ &+(ab+ac+bc+ad+bd+cd)x^2\\ &+(abc+acd+bcd+abd)x+abcd;\end{aligned}$$

and the same law of formation of the continued product is observed to hold whatever be the number of binomial factors, $x+a$, $x+b$, $x+c$, &c., actually multiplied together, viz. that it is composed of a descending series of powers of x, the index of the highest being the number of factors, and the other indices decreasing by 1 in each succeeding term. Also the coefficient of the first term is 1; of the second the sum of the quantities a, b, c, &c.; of the third the sum of the products of every two; of the fourth the sum of the products of every three; and so on; of the last the product of all the n quantities a, b, c, &c.

Suppose, then, this law to hold for n binomial factors, $x+a$, $x+b$, $x+c\dots\dots$, $x+k$; so that

$$(x+a)(x+b)(x+c)\dots(x+k)=x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\dots+K,$$

$$\begin{aligned}\text{where } A&=a+b+c+\dots+k,\\ B&=ab+ac+bc+\dots\\ C&=abc+acd+\dots\\ \&\text{c.}&=\&\text{c.}\\ K&=abcd\dots k;\end{aligned}$$

introducing a new factor, $x+l$, we have

$$(x+a)(x+b)(x+c)\dots(x+k)(x+l)=x^{n+1}+(A+l)x^n+(B+lA)x^{n-1}+\dots+Kl.$$

$$\begin{aligned}\text{Hence } A+l&=a+b+c+\dots+k+l,\\ B+lA&=ab+ac+bc+\dots+al+bl+\dots+kl,\\ \&\text{c.}&=\&\text{c.}\\ Kl&=abcd\dots kl;\end{aligned}$$

so that, if the law above described holds when n binomial factors are multiplied together, the same law is *proved* to hold for $\overline{n+1}$ factors. But it has been shewn to hold up to 4 factors, *therefore* it is true for 5; and, if for 5, then also for 6; and so on, generally, for any number whatever*.

Now, let $a=b=c=\&\text{c.}$,

then $A=a+a+a+\&\text{c.}$ to n terms $=na$.

$$\left.\begin{array}{l}B=a^2+a^2+\&\text{c. to as many}\\ \quad\text{terms as is equal to the No.}\\ \quad\text{of combinations of } n \text{ things}\\ \quad\text{taken } \textit{two} \text{ together,}\end{array}\right\}=n.\frac{n-1}{2}a^2,$$

$$\left.\begin{array}{l}C=a^3+a^3+\&\text{c. to as many}\\ \quad\text{terms as is equal to the No.}\\ \quad\text{of combinations of } n \text{ things}\\ \quad\text{taken } \textit{three} \text{ together.}\end{array}\right\}=n.\frac{(n-1)(n-2)}{2\ .\ 3}a^3,$$

$\&\text{c.}=\&\text{c.}$

$K=a.a.a\dots$to n factors $=a^n$.

Also $(x+a)(x+b)(x+c)\dots(x+k)$ becomes $(x+a)^n$;

$$\therefore\ (x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1\times2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1\times2\times3}a^3x^{n-3}+\dots+a^n.$$

Cor. $(1+x)^n=1+nx+\dfrac{n(n-1)}{1\times2}x^2+\dfrac{n(n-1)(n-2)}{1\times2\times3}x^3+\dots+x^n.$

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* **This method of proof is called proof *by induction*.**

309. *Having given the Binomial Theorem for a positive integral index, to prove it, when the index is either fractional or negative.* [EULER'S PROOF.]

Let the series $1+mx+\frac{m(m-1)}{1\times2}x^2+\&c.$ be represented for all values of m, whether positive, or negative, integral, or fractional, by the *symbol* $f(m)$; then it has been shewn that, when m is a positive integer,

$$f(m)=(1+x)^m.$$

It remains to prove that this equation is also true when m is either fractional or negative.

By the notation assumed

$$1+nx+\frac{n(n-1)}{1\times2}x^2+\&c.=f(n);$$

therefore, by multiplication, $f(m).f(n)=$ the product of the two series, which will evidently be a series of the form

$$1+ax+bx^2+cx^3+\&c.,$$

ascending regularly by the integral powers of x, the coefficients a, b, c, &c. being different combinations of m and n.

Now, although by changing the values of m and n the *values* of a, b, c, &c. are altered, yet their *forms*, that is, the manner in which m and n enter the series will remain the same. Whatever, therefore, be the *forms* of a, b, c, &c. when m and n are positive integers, the same will they be when m and n are fractional or negative. But in the former case

$$f(m)=(1+x)^m, \text{ and } f(n)=(1+x)^n;$$

$$\therefore f(m).f(n)=(1+x)^{m+n}=1+(m+n)x+\frac{(m+n)(m+n-1)}{1\times2}x^2+\&c.$$

$\because$ $m+n$ is a positive integer.

These then are the forms in which m and n appear in the product when they are positive integers; and therefore they appear in these same forms, whatever be their values: *i.e.* whether m and n be positive or negative, integral or fractional, the multiplication of $f(m)$ by $f(n)$ *always* produces the series

$$1+(m+n)x+\frac{(m+n)(m+n-1)}{1\times2}x^2+\&c.$$

which by the notation is represented by $f(m+n)$;

$$\therefore \textit{ universally, } f(m).f(n)=f(m+n).$$

Since, then, this last equation is true whatever be the values of m and n, for n write $n+p$, and we have

$$f(m+n+p)=f(m).f(n+p)=f(m).f(n).f(p);$$

and proceeding similarly, we have generally

$$f(m+n+p+\&c.)=f(m).f(n).f(p).\&c. \text{ to any number of terms.}$$

Now, let $m=n=p=\&c.=\frac{h}{k}$, h and k being positive integers; then

$$f\left(\frac{h}{k}+\frac{h}{k}+\&c.\right)=f\left(\frac{h}{k}\right).f\left(\frac{h}{k}\right).f\left(\frac{h}{k}\right).\ \&c.$$

and, if the number of terms be k,

$$f\left(\frac{h}{k}+\frac{h}{k}+\&c.\text{ to }k\text{ terms}\right)=f\left(\frac{h}{k}\right).f\left(\frac{h}{k}\right).\ \&c.\text{ to }k\text{ factors,}$$

$$\text{or } f(h)=\left\{f\left(\frac{h}{k}\right)\right\}^k;$$

but $f(h)=(1+x)^h$, because h is a positive integer:

$$\therefore\ (1+x)^h=\left\{f\left(\frac{h}{k}\right)\right\}^k;$$

$$\therefore\ (1+x)^{\frac{h}{k}}=f\left(\frac{h}{k}\right)=1+\frac{h}{k}x+\frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1\times2}x^2+\dots\text{ by the notation,}$$

which proves the Theorem for a fractional positive index.

Again, $\because f(m).f(n)=f(m+n)$ for all values of m and n, let $n=-m$, then

$$f(m).f(-m)=f(m-m)=f(0),$$

$$=1,\ \because\text{ the assumed series becomes 1 when } m=0;$$

$$\therefore\ f(-m)=\frac{1}{f(m)}=\frac{1}{(1+x)^m},$$

$$\text{or } (1+x)^{-m}=f(-m)=1+(-m)x+\frac{-m(-m-1)}{1\times2}x^2+\dots$$

which proves the Theorem for a negative index, integral or fractional.

310. *To prove the Binomial Theorem for any index whatever.* [GRIFFITH'S PROOF.]

Let α_2 denote $\frac{\alpha(\alpha-1)}{1\ .\ 2}$; $\quad\therefore\ 2\alpha_2=(\alpha-1)\alpha,$

α_3 .... $\frac{\alpha(\alpha-1)(\alpha-2)}{1\ .\ 2\ .\ 3}$; $\quad\therefore\ 3\alpha_3=(\alpha-2)\alpha_2$; and so on:

α_r .... $\frac{\alpha(\alpha-1)\dots(\alpha-r+1)}{1\ .\ 2\dots\dots\dots r}$; $\quad\therefore\ r\alpha_r=(\alpha-r+1)\alpha_{r-1}.$

β_2 .... $\frac{\beta(\beta-1)}{1\ .\ 2}$; $\quad\therefore\ 2\beta_2=(\beta-1)\beta,$

β_3 .... $\frac{\beta(\beta-1)(\beta-2)}{1\ .\ 2\ .\ 3}$; $\quad\therefore\ 3\beta_3=(\beta-2)\beta_2$; and so on:

β_r .... $\frac{\beta(\beta-1)\dots(\beta-r+1)}{1\ .\ 2\dots\dots\dots r}$; $\quad\therefore\ r\beta_r=(\beta-r+1)\beta_{r-1}.$

Now the product of the two series

$$1+\alpha x+\alpha_2x^2+\alpha_3x^3+\dots+\alpha_rx^r+\dots\dots\dots\dots\dots (1)$$

$$1+\beta x+\beta_2x^2+\beta_3x^3+\dots+\beta_rx^r+\dots\dots\dots\dots\dots (2)$$

by actual multiplication, is $1+(a+\beta)x+$ terms of x^2, x^3, $\ldots x^r$, $\ldots$

the coeff

$$\text{the coeff}^{\text{t}}.\text{ of } x^r \text{ being } \quad a_r+a_{r-1}\beta+a_{r-2}\beta_2+a_{r-3}\beta_3+\ldots+a\beta_{r-1}+\beta_r,$$

$$\text{and } \ldots\ldots\ldots\ldots\ldots\, x^{r-1} \;\ldots\; a_{r-1}+a_{r-2}\beta+a_{r-3}\beta_2+a_{r-4}\beta_3+\ldots+a\beta_{r-2}+\beta_{r-1}.$$

But

$$\begin{aligned}
ra_r. &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = (a-r+1)a_{r-1},\\
ra_{r-1}\beta &= (r-1)a_{r-1}\beta + a_{r-1}\beta = (a-r+2)a_{r-2}\beta+\beta a_{r-1},\\
ra_{r-2}\beta_2 &= (r-2)a_{r-2}\beta_2+2a_{r-2}\beta_2 = (a-r+3)a_{r-3}\beta_2+(\beta-1)\beta a_{r-2},\\
ra_{r-3}\beta_3 &= (r-3)a_{r-3}\beta_3+3a_{r-3}\beta_3 = (a-r+4)a_{r-4}\beta_3+(\beta-2)\beta_2 a_{r-3},\\
\ldots\ldots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
ra\beta_{r-1} &= a\beta_{r-1}+(r-1)a\beta_{r-1} = a\beta_{r-1}+(\beta-r+2)\beta_{r-2}a,\\
r\beta_r &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = (\beta-r+1)\beta_{r-1};
\end{aligned}$$

∴ adding, and observing that there is a pair of *similar* terms in every two lines, we have

$$r\times\text{coeff}^{\text{t}}.\text{ of } x^r = (a+\beta-\overline{r-1})(a_{r-1}+a_{r-2}\beta+a_{r-3}\beta_2+\ldots+\beta_{r-1});$$

$$\begin{aligned}
\therefore\ \text{coeff}^{\text{t}}.\text{ of } x^r &= \frac{a+\beta-(r-1)}{r}\times\text{coeff}^{\text{t}}.\text{ of } x^{r-1},\\
&= \frac{a+\beta-(r-1)}{r}\cdot\frac{a+\beta-(r-2)}{r-1}\times\text{coeff}^{\text{t}}.\text{ of } x^{r-2},\\
&= \frac{a+\beta-(r-1)}{r}\cdot\frac{a+\beta-(r-2)}{r-1}\cdots\frac{a+\beta-1}{2}\times\text{coeff}^{\text{t}}.\text{ of } x,\\
&= \frac{a+\beta-(r-1)}{r}\cdot\frac{a+\beta-(r-2)}{r-1}\cdots\frac{a+\beta-1}{2}\cdot\frac{a+\beta}{1},\\
&= \frac{(a+\beta)(a+\beta-1)\ldots(a+\beta-r+1)}{1\ .\ 2\ \ldots\ r};
\end{aligned}$$

and writing 2, 3, 4, ... successively for r, the product of the series (1) and (2) becomes

$$1+(a+\beta)x+\frac{(a+\beta)(a+\beta-1)}{1\ .\ 2}x^2+\ldots+\frac{(a+\beta)(a+\beta-1)\ldots(a+\beta-r+1)}{1\ .\ 2\ \ldots\ r}x^r+\ldots;$$

therefore calling a the '*characteristic*' of series (1), and β of (2), it is proved that the *product* of two such series is a similar series with the ***sum*** of their 'characteristics' for its 'characteristic.'

Hence also this product multiplied by another such series, that is, the product of three such series, will be an exactly similar series with the sum of their 'characteristics' for its 'characteristic': and so on for any number whatever of such series.

1. Let the number of series be n, and the 'characteristic' of each 1; then all the series being alike, and the sum of their 'characteristics' n, we have

$$\left(1+1\times x+\frac{1\times(1-1)}{2}x^2+\ldots\right)^n = 1+nx+\frac{n(n-1)}{1\ .\ 2}x^2+\&\text{c.}\ldots$$

$$\text{or}\quad (1+x)^n = 1+nx+\frac{n(n-1)}{1\ .\ 2}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{1\ .\ 2\ \ldots\ r}x^r+\ldots,$$

which proves the Binomial Theorem, when the index is a positive integer.

2. Let the number of series be n, and the 'characteristic' of each $\frac{m}{n}$; then, since $\frac{m}{n}+\frac{m}{n}+$ &c. to n terms $=m$,

$$\left(1+\frac{m}{n}x+\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{1\,.\,2}x^2+\dots\right)^n=1+mx+\frac{m(m-1)}{1\,.\,2}x^2+\dots=(1+x)^m, \text{ by } 1^{\text{st}} \text{ case;}$$

$$\therefore\ (1+x)^{\frac{m}{n}}=1+\frac{m}{n}x+\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{1\,.\,2}x^2+\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{1\,.\,2\,.\,3}x^3+\dots$$

which proves the Theorem, when the index is a positive fraction.

3. Let there be two series, of which the 'characteristics' are m and $-m$, then

$$\left(1+mx+\frac{m(m-1)}{1\,.\,2}x^2+\dots\right)\left(1-mx+\frac{-m(-m-1)}{1\,.\,2}x^2+\dots\right)=1+(m-m)x+\&\text{c}.$$
$$=1;$$

$$\therefore\ \frac{1}{(1+x)^m},\ \text{or}\ (1+x)^{-m}=1-mx+\frac{-m(-m-1)}{1\,.\,2}x^2+\dots,$$

which proves the Theorem for a negative index, whole or fractional.

311. It was observed, in Art. 307, that the Binomial Theorem may be written in the following form:—

$$\frac{(x+a)^n}{\underline{|n}}=\frac{x^n}{\underline{|n}}+\frac{a^1}{\underline{|1}}\cdot\frac{x^{n-1}}{\underline{|n-1}}+\frac{a^2}{\underline{|2}}\cdot\frac{x^{n-2}}{\underline{|n-2}}+\dots+\frac{a^{n-1}}{\underline{|n-1}}\cdot\frac{x^1}{\underline{|1}}+\frac{a^n}{\underline{|n}},$$

a form which possesses peculiar advantages. Thus, if we want any term, independently of the rest, as, for example, that which contains x^r, we have it at once, $\frac{a^{n-r}}{\underline{|n-r}}\cdot\frac{x^r}{\underline{|r}}$; and the corresponding term in $(x+a)^n$ is

$$\underline{|n}\cdot\frac{a^{n-r}}{\underline{|n-r}}\cdot\frac{x^r}{\underline{|r}},\ \text{or}\ \frac{n(n-1)\dots(n-r+1)}{1\,.\,2\dots r}a^{n-r}x^r.$$

Again, it is worthy of notice, that the index of a in any term gives the number of terms preceding, and the index of x the number of terms following, if n be a positive integer*. And since the *sum* of these indices is always n, it follows at once that the whole *number of terms* is $n+1$.

Thus, also, since the numerical *coefficient* of the term, which has r terms preceding it, is $\frac{\underline{|n}}{\underline{|r}\,.\,\underline{|n-r}}$, and the coefficient of the term, which has r terms succeeding it, is $\frac{\underline{|n}}{\underline{|n-r}\,.\,\underline{|r}}$, it follows that the numerical coefficients of any two terms equidistant from the beginning and end are the same.

312. In applying the Binomial Theorem to any proposed case it is well to observe, that, if each term of the given Binomial be of *one* dimension, every term of the expansion will be of n dimensions, n being the index of the power to which the binomial is raised.

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* This is a most useful fact to remember.

Also, if each term of the proposed binomial be of *two* dimensions, every term in the expansion will be of $2n$ dimensions; and so on.

And it will be found convenient in practice to reduce the proposed binomial to such a form that it may have 1 for its first term. Thus we reduce $(a+b)^n$ to $a^n\left(1+\frac{b}{a}\right)^n$, then expand $\left(1+\frac{b}{a}\right)^n$, and multiply every term of the expansion by a^n.

Ex. 1. $$(a^2+x^2)^n=\left\{a^2\left(1+\frac{x^2}{a^2}\right)\right\}^n=a^{2n}\times\left(1+\frac{x^2}{a^2}\right)^n,$$
$$=a^{2n}.\left(1+n.\frac{x^2}{a^2}+n.\frac{n-1}{2}.\frac{x^4}{a^4}+\frac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}.\frac{x^6}{a^6}+\&c.\right),$$
$$=a^{2n}+na^{2n-2}x^2+n.\frac{n-1}{2}.a^{2n-4}x^4+\frac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}.a^{2n-6}x^6+\&c.$$

Ex. 2. $$(1+x)^{\frac{1}{n}}=1+\frac{1}{n}x+\frac{1}{n}.\frac{\frac{1}{n}-1}{2}x^2+\frac{1}{n}.\frac{\frac{1}{n}-1}{2}.\frac{\frac{1}{n}-2}{3}x^3+\&c.$$
$$=1+\frac{1}{n}x-\frac{n-1}{2n^2}x^2+\frac{(n-1)(2n-1)}{2\ .\ 3\ .\ n^3}x^3-\&c.$$

Ex. 3. $$(1+x)^{-\frac{1}{n}}=1-\frac{1}{n}x+\frac{n+1}{2n^2}x^2-\frac{(n+1)(2n+1)}{2\ .\ 3\ .\ n^3}x^3+\&c.$$

Ex. 4. $$(a^{\frac{1}{2}}+b^{\frac{1}{2}})^4=(a^{\frac{1}{2}})^4\left\{1+\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right\}^4,$$
$$=a^2\left\{1+4\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}+\frac{4\times3}{1\times2}\left(\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)^2+\frac{4\times3\times2}{1\times2\times3}\left(\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)^3+\frac{4\times3\times2\times1}{1\times2\times3\times4}\left(\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)^4\right\},$$
$$=a^2+4a^{\frac{3}{2}}b^{\frac{1}{2}}+6ab+4a^{\frac{1}{2}}b^{\frac{3}{2}}+b^2.$$

Ex. 5. $$\frac{1}{(1+x^2)^3}=(1+x^2)^{-3},$$
$$=1-3x^2+\frac{-3\times-4}{1\times2}x^4+\frac{-3\times-4\times-5}{1\times2\times3}x^6+\dots$$
$$=1-3x^2+6x^4-10x^6+\dots\dots$$

Ex. 6. $$(ax+by)^{\frac{1}{2}}=(ax)^{\frac{1}{2}}.\left\{1+\frac{by}{ax}\right\}^{\frac{1}{2}},$$
$$=(ax)^{\frac{1}{2}}\left\{1+\frac{1}{2}.\frac{by}{ax}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\ .\ 2}\left(\frac{by}{ax}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\ .\ 2\ .\ 3}\left(\frac{by}{ax}\right)^3+\dots\right\},$$
$$=(ax)^{\frac{1}{2}}+\frac{1}{2}.\frac{by}{(ax)^{\frac{1}{2}}}-\frac{1}{8}.\frac{(by)^2}{(ax)^{\frac{3}{2}}}+\frac{1}{16}.\frac{(by)^3}{(ax)^{\frac{5}{2}}}-\dots\dots$$

313. If either term of the binomial be negative, every *odd* power of that term will be negative; and consequently the signs of the terms, in which those *odd* powers are found, will be changed.

Ex. 1. $(1-x)^n = 1 - nx + n\frac{n-1}{2}x^2 - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3 + \&c.$

Ex. 2. $(a^2-x^2)^n = a^{2n} - na^{2n-2}x^2 + n\,.\,\frac{n-1}{2}a^{2n-4}x^4 - \&c.$

314. If the index of the power, to which a binomial is to be raised, be *negative*, and the second term of the binomial be negative, then *every* sign in the series is positive.

Ex. 1. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1\,.\,2}x^2 + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}x^3 + \ldots\ldots$

The particular form of this result is worth remembering.

Ex. 2. $(1-x)^{-3} = 1 + 3x + \frac{3\times4}{1\times2}x^2 + \frac{3\times4\times5}{1\times2\times3}x^3 + \ldots\ldots$

$$= 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots\ldots\ldots$$

Ex. 3. $\frac{1}{\sqrt{a^2-x^2}} = (a^2-x^2)^{-\frac{1}{2}} = a^{-1}\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}},$

$$= a^{-1}\left\{1 + \frac{1}{2}\,.\,\frac{x^2}{a^2} + \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{1\times2}\,.\,\frac{x^4}{a^4} + \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)}{1\,.\,2\,.\,3}\,.\,\frac{x^6}{a^6} + \ldots\ldots\right\},$$

$$= \frac{1}{a} + \frac{1}{2}\,.\,\frac{x^2}{a^3} + \frac{3}{8}\,.\,\frac{x^4}{a^5} + \frac{5}{16}\,.\,\frac{x^6}{a^7} + \ldots\ldots\ldots$$

315. *To find the* general term *of the expansion of* $(x+a)^n$.

The 1st term is x^n,

2nd .... na^1x^{n-1},

3rd .... $n\,.\,\frac{n-1}{2}\,a^2x^{n-2}$,

4th .... $n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,a^3x^{n-3}$, and so on,

in which we observe that the coefficient of any term is formed of the product of the factors $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, &c. in number one less than the number which expresses the position of the term; therefore the coefficient of the r^{th} term will be

$$\frac{n}{1}\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\ldots\ldots\frac{\overline{n-r-2}}{r-1}\,{}^{*},$$

$$\text{or } \frac{n(n-1)(n-2)\ldots\ldots(\overline{n-r-2})}{1\,.\,2\,.\,3\ldots\ldots\ldots(r-1)}.$$

Also the index of a is always the number of terms preceding, that is,

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[*] The learner often finds a difficulty in putting down correctly the terminal fraction in a proposed case. Let him write the *denominator* first, and after that the numerator, bearing in mind that their *sum* is always $n+1$.

$r-1$; and the index of x is the difference between n and the index of a*; therefore the whole r^{th} term is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1\,.\,2\,.\,3\ldots\ldots\ldots(r-1)}\,a^{r-1}x^{n-r+1}.$$

Or thus, more briefly, by Art. 311. The r^{th} term, that is, the term which has $r-1$ terms preceding it, is

$$\underline{|n}\,.\,\frac{a^{r-1}}{\underline{|r-1}}\,.\,\frac{x^{n-(r-1)}}{\underline{|n-(r-1)}},$$

which is easily shewn to be the same as that before found.

By substituting in this expression any proposed number for r any single term of the expansion may be found independently of the rest.

Ex. Required the fifth term of $(a^2-b^2)^{12}$.

$$\text{Here } r=5, \text{ and } n=12;$$

$$\therefore \text{ term required} = \frac{12.11.10.9}{1\,.\,2\,.\,3\,.4}\,.\,(-b^2)^4\,.\,(a^2)^8,$$

$$=495a^{16}b^8.$$

316. *If the index of the binomial be a positive integer, every coefficient in the expansion, formed from the index, is a positive integer.*

For the coefficient of the $(r+1)^{\text{th}}$ term, where r may have any integral value from 1 to n inclusive, is

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots r};$$

and this, by Art. 301, is the same as the number of combinations of n things taken r together. Now this latter number, by the nature of the thing, must be a whole number, if n and r be positive whole numbers; therefore also

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots r} \text{ is a whole number.}$$

Another Method. If all the coefficients of the expansion of $(1+x)^n$ are positive integers for *any one* value of n, it is obvious that they will be also for the *next higher* value, since no *fractional* quantities can be introduced by merely multiplying by $1+x$. Now we know that the coefficients are whole numbers in $(1+x)^2$, and therefore *it follows* that they must be so in $(1+x)^3$; then again in $(1+x)^4$; and so on generally in $(1+x)^n$.

317. *To find the* number of terms *in the expansion of a binomial.*

The $(r+1)^{\text{th}}$ term of $(x+a)^n$ is derived from the r^{th} by multiplying the latter by $\dfrac{n-r+1}{r}\,.\,\dfrac{a}{x}$; hence the r^{th} term will be the last, when $n-r+1$ is equal to 0, that is, if n be a positive whole number and $r=n+1$, there is no term after the r^{th}, or the number of terms is $n+1$, that is, greater by 1 than the index.

The same result was obtained more simply in Art. 311.

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* Observe the *sum* of the indices of a and x is always n.

If n be negative or fractional, since r must necessarily be a positive integer, no value of r can make $n-r+1$ equal to 0, and therefore in these cases the number of terms is unlimited.

Thus the number of terms in the expansions of $(x+a)^3$, $(x+a)^7$, is 4 and 8 respectively; but the number for $(x+a)^{-2}$, or $(x+a)^{\frac{1}{2}}$ is unlimited, or indefinitely great.

318. *To prove that, in an expanded binomial, when the index is a positive integer, the coefficients, formed from the index, of any two terms taken equidistant from the beginning and end, are the same.*

Since the number of terms is $n+1$, the $(r+1)^{\text{th}}$ term from the end, having r terms after it, is the $(\overline{n+1-r})^{\text{th}}$ or $(n-r+1)^{\text{th}}$ term from the beginning; and its coefficient (Art. 315, putting $n-r+1$ for r,) is

$$\frac{n(n-1)(n-2)\ldots(n-\overline{n-r}+1)}{1\;.\;2\;.\;3\;\ldots\ldots\;(n-r)},$$

$$=\frac{n(n-1)(n-2)\ldots(r+1)}{1\;.\;2\;.\;3\;\ldots\;(n-r)}\times\frac{r(r-1)\ldots3.2.1}{1\;.\;2\;.\ldots\ldots\;r},$$ and div$^{\text{g}}$. num$^{\text{r}}$. and denom$^{\text{r}}$. by $1.2.3\ldots(n-r)$,

$$=\frac{n(n-1)(n-2)\ldots(n-r+1)}{1\;.\;2\;.\;3\;\ldots\ldots\ldots\;r},$$

= coefficient of the $(r+1)^{\text{th}}$ term from the beginning.

This result is shewn more simply by Art. 311. Or it may be obtained by writing a for x, and x for a, in the expansion of $(x+a)^n$, so as to deduce that of $(a+x)^n$, and then equating the coefficients of similar terms in the two expansions, which are clearly equal to each other.

COR. Hence in expanding a binomial, with the index a positive integer, the latter half of the expansion may be taken from the first half.

Ex. 1. Required to expand $(a+b)^7$.

Here the number of terms is 8; and it will be necessary to calculate the coefficient up to the 4^{th} term only;

$$\therefore\;(a+b)^7=a^7+7a^6b+\frac{7.6}{1.2}a^5b^2+\frac{7.6.5}{1.2.3}a^4b^3+\&\text{c.}$$

$$=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$$

Ex. 2. Required the 11^{th} term of $(a+x)^{13}$.

Since the number of terms is 14, the 11^{th} is the 4^{th} from the end, and its coefficient the same as that of the 4^{th} from the beginning,

$$\therefore\;\text{term required}=\frac{13.12.11}{1\;.\;2\;.\;3}a^3x^{10}=286a^3x^{10}.$$

319. *To find the greatest term in the expansion of* $(\mathrm{a}+\mathrm{b})^{\mathrm{n}}$.

The $(r+1)^{\text{th}}$ term of the expansion is $\dfrac{n(n-1)\ldots(n-r+1)}{1\;.\;2\;\ldots\ldots\;r}a^{n-r}b^r$;

and the r^{th} term is $\dfrac{n(n-1)\ldots(n-r+2)}{1\;.\;2\;\ldots\ldots(r-1)}.a^{n-r+1}b^{r-1}$.

Therefore the $(r+1)^{\text{th}}$ term is obtained from the r^{th} by multiplying the latter by $\frac{n-r+1}{r}\cdot\frac{b}{a}$; and as r takes its successive values, the numerator of this fraction continually diminishes, and the denominator increases.

Hence the r^{th} term will be the greatest when $\frac{n-r+1}{r}\cdot\frac{b}{a}$ *first* < 1,

$$\text{or } (n-r+1)b < ar, \text{ (Art. 219)},$$

$$\text{or } r(a+b) > (n+1)b, \text{ (Art. 217)},$$

$$\text{or } r > (n+1)\frac{b}{a+b}. \text{ (Art. 219.)}$$

Take r, therefore, the *first* whole number greater than $(n+1)\frac{b}{a+b}$, and the r^{th} term will be the greatest.

If $(n+1)\frac{b}{a+b}$ be a whole number, when r is equal to it we shall have $\frac{n-r+1}{r}\cdot\frac{b}{a}=1$, and then *two* terms, the r^{th} and the $(r+1)^{\text{th}}$, will be equal, and each of them greater than any of the other terms.

If the index be negative, $-n$, the r^{th} term will be the greatest when $\frac{-n-r+1}{r}\cdot\frac{b}{a}$ first <1, irrespectively of sign; *i.e.* when $\frac{n+r-1}{r}\cdot\frac{b}{a}$ first <1, or r first $>(n-1)\frac{b}{a-b}$. And if this is a whole number, two terms will be equal, and each greater than any of the others.

Cor. By thus ascertaining the greatest term we determine the point from which the terms of a series become less and less, or, as it is usually stated, the point at which the series begins to *converge*.

Ex. Required to find which is the greatest term in the expansion of $(3+5x)^8$, when $x=\frac{1}{2}$.

$$\text{Here } (n+1)\frac{b}{a+b}=(8+1)\cdot\frac{\frac{5}{2}}{3+\frac{5}{2}}=9\times\frac{5}{11}=\frac{45}{11}=4\tfrac{1}{11}.$$

The *first* whole number greater than $4\frac{1}{11}$ is 5; therefore the term required is the 5^{th}.

320. *To find the sum of all the coefficients of an expanded binomial.*

Since $(1+x)^n=1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\&c.$, for any value whatever of x, let $x=1$, then

$$(1+1)^n=1+n+\frac{n(n-1)}{1\,.\,2}+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}+\&c.$$

$$\text{or } 2^n=\text{the sum of the coefficients.}$$

Ex. $(x+a)^5=x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5$, and the sum of the coefficients $=1+5+10+10+5+1=32=2^5$.

Also by putting $x=-1$, we have

$$(1-1)^n=1-n+\frac{n(n-1)}{1\,.\,2}-\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}+\&c.$$

or $0=$ sum of the odd coefficients $-$ the sum of the even ones;

$\therefore$ the sums of the odd and even coefficients are equal, and consequently each $=\frac{1}{2}.2^n=2^{n-1}$.

321. *To find the approximate roots of numbers by the Binomial Theorem.*

The theorem being proved for a fractional index, we have

$$\sqrt[n]{1\pm x}=(1\pm x)^{\frac{1}{n}}=1\pm\frac{1}{n}x-\frac{1}{n}\,.\,\frac{n-1}{2n}x^2\pm\ldots\ldots$$

Now, if N represent a proposed number whose n^{th} root is required, take p such that p^n is the nearest perfect n^{th} power to N, so that $N=p^n\pm q$, q being small compared with p^n, and + or – according as $N>$ or $<p^n$;

then $\sqrt[n]{N}=p\left(1\pm\frac{q}{p^n}\right)^{\frac{1}{n}}$; and writing $\frac{q}{p^n}$ for x,

$$=p\left\{1\pm\frac{1}{n}.\frac{q}{p^n}-\frac{1}{n}.\frac{n-1}{2n}\left(\frac{q}{p^n}\right)^2\pm\ldots\right\},$$

of which series a few terms only will give the required root to a considerable degree of accuracy.

Ex. Required the approximate cube root of 128.

Here $\sqrt[3]{128}=\sqrt[3]{5^3+3}=5\sqrt[3]{1+\frac{3}{125}}$,

$$=5\left\{1+\frac{1}{3}.\frac{3}{125}-\frac{1}{3}.\frac{1}{3}\left(\frac{3}{125}\right)^2+\frac{1}{3^2}.\frac{5}{9}\left(\frac{3}{125}\right)^3-\ldots\right\},$$

$$=5+\frac{1}{5^2}-\frac{1}{5^5}+\frac{1}{3}.\frac{1}{5^7}-\ldots=5+\frac{2^2}{10^2}-\frac{2^5}{10^5}+\frac{1}{3}.\frac{2^7}{10^7}-\ldots\ldots$$

$$=5+0{\cdot}04-0{\cdot}00032+0{\cdot}0000042-\ldots\ldots=5{\cdot}0396842.$$

322. A trinomial, $a+b+c$, may be raised to any power by considering two terms as one, and making use of the Binomial Theorem. Thus,

$$(a+b+c)^n=(\overline{a+b}+c)^n,$$

$$=(a+b)^n+n(a+b)^{n-1}c+\frac{n(n-1)}{1\,.\,2}(a+b)^{n-2}c^2+\&c.$$

in which the several powers of $\overline{a+b}$ may be replaced by their expansions found by the Binomial Theorem.

Ex. Required the cube of $1+x+x^2$.

$$(1+x+x^2)^3=(\overline{1+x}+x^2)^3,$$

$$=(1+x)^3+3(1+x)^2x^2+3(1+x)x^4+x^6,$$

$$=1+3x+3x^2+x^3+3x^2+6x^3+3x^4+3x^4+3x^5+x^6,$$

$$=1+3x+6x^2+7x^3+6x^4+3x^5+x^6.$$

Similarly $(a+b+c+d)^n$ may be expanded by considering $a+b$ as one term, and $c+d$ as another; and any multinomial may be expanded in a similar manner by dividing the whole into two terms and considering it as a binomial.

Also any *particular term* of an expanded trinomial may be easily found :—thus

To find the term involving x^4 in the expansion of $(1+x+x^2)^3$.

$$(\overline{1+x}+x^2)^3=(1+x)^3+3(1+x)^2x^2+3(1+x)x^4+x^6,$$

and without further expansion it is seen that

$$\text{the term required}=3x^2\times x^2+3x^4=6x^4.$$

Again, the *number of terms* in the expansion of $(a+b+c)^n$ may be found; for it will obviously be the aggregate number of the terms in the expansions of the several powers of $\overline{a+b}$, from $(a+b)^n$ down to $(a+b)^0$, that is, (Art. 317),

$$=(n+1)+n+(n-1)+\&c. \text{ to } \overline{n+1} \text{ terms},$$

$$=\{2n+2-n\}\frac{n+1}{2}\ (\text{Art. } 282)=\frac{(n+1)(n+2)}{1\ .\ 2}.$$

[*Exercises Zg.*]

323. *Required to find the Remainder after taking* r *terms of the expansion of* $(1-x)^{-2}$.

By the Binomial Theorem,

$$(1-x)^{-2}=1+2x+3x^2+\ldots+rx^{r-1}+R,$$

R representing the remainder after r terms;

$$\therefore\ \frac{1}{1-2x+x^2}=1+2x+3x^2+\ldots+rx^{r-1}+R,$$

$$\therefore\ 1=\left\{\begin{array}{l}1+2x+3x^2+\ldots\ldots\ldots\quad+rx^{r-1}+R\\ \quad-2x-4x^2+\ldots-2(r-1)x^{r-1}-2rx^r-2Rx\\ \qquad+x^2+\ldots+\ (r-2)x^{r-1}+(r-1)x^r+rx^{r+1}+Rx^2,\end{array}\right.$$

$$\text{or}\quad 1=1-(r+1)x^r+rx^{r+1}+R(1-x)^2;$$

$$\therefore\ R=\frac{(r+1)x^r-rx^{r+1}}{(1-x)^2}.$$

In the same manner may be found the remainder after taking r terms of the expansions of $(1-x)^{-3}$, $(1-x)^{-4}$, &c.

324. *To find the number of homogeneous products of* r *dimensions which can be made of* n *things* a, b, c, d, *&c. and their powers.*

By common division, or the Binomial Theorem,

$$\frac{1}{1-ax}=1+ax+a^2x^2+a^3x^3+\ldots\ldots$$

$$\frac{1}{1-bx}=1+bx+b^2x^2+b^3x^3+\ldots\ldots$$

$$\frac{1}{1-cx}=1+cx+c^2x^2+c^3x^3+\ldots\ldots$$

$$\&c.\ =\qquad\qquad \&c.$$

$$\therefore\ \frac{1}{1-ax}\cdot\frac{1}{1-bx}\cdot\frac{1}{1-cx}\cdot\text{\&c.}=1+(a+b+c+\text{\&c.})x$$
$$+(a^2+ab+b^2+ac+bc+c^2+\text{\&c.})x^2$$
$$+(a^3+a^2b+ab^2+b^3+a^2c+b^2c+ac^2+c^3+\text{\&c.})x^3$$
$$+\quad\text{\&c.}\qquad\qquad\text{\&c.}$$

the coefficient of x^r being the sum of the homogeneous products of n things a, b, c, d, &c. of r dimensions.

Now to obtain the *number* of these products, let $a=b=c=d=\text{\&c.}=1$, then the coefficient of x^r will give the number required. But on this supposition, the left-hand side of the equation becomes $(1-x)^{-n}$, which, by the Binomial Theorem, is

$$1+nx+n.\frac{n+1}{2}x^2+n.\frac{n+1}{2}.\frac{n+2}{3}x^3+\text{\&c.}\ldots+\frac{n(n+1)(n+2)\ldots(n+r-1)}{1\ .\ 2\ .\ 3\ \ldots\ r}x^r+\ldots;$$

$$\therefore\ \text{number required}=\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ r}.$$

COR. Hence also the number of terms in the expansion of any multinomial, as $(a_1+a_2+a_3+\ldots a_r)^n$, is known; for it is obviously the same as the number of homogeneous products of r things taken n together, that is,

$$\frac{r(r+1)(r+2)\ldots\ldots(r+n-1)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ n}.$$

If $r=2$, that is for a binomial, $(a+b)^n$, the expression becomes

$$\frac{2.3.4\ldots\ldots(n+1)}{1.2.3\ldots\ldots\ n},\ \text{or}\ n+1.$$

If $r=3$, the number is, for $(a+b+c)^n$,

$$\frac{3.4.5\ldots\ldots(n+2)}{1.2.3\ldots\ldots\ n},\ \text{or}\ \frac{(n+1)(n+2)}{1\ .\ 2}.$$

If $r=4$, or the quantity to be expanded be $(a+b+c+d)^n$, the number of terms is

$$\frac{4.5.6\ldots\ldots(n+3)}{1.2.3\ldots\ldots\ n},\ \text{or}\ \frac{(n+1)(n+2)(n+3)}{1\ .\ 2\ .\ 3}.$$

And so on for any value of r.

THE EXPONENTIAL THEOREM.

325. *To expand* a^x *in a series of powers of* x.

$a^x=\{(1+\overline{a-1})^n\}^{\frac{x}{n}}$; and expanding by the Binomial Theorem,

$$=\{1+n(a-1)+n.\frac{n-1}{2}(a-1)^2+n.\frac{n-1}{2}.\frac{n-2}{3}(a-1)^3+\text{\&c.}\}^{\frac{x}{n}},$$

$$=\{1+[(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\text{\&c.}]n+Bn^2+Cn^3+\ldots\}^{\frac{x}{n}},\quad B,\ C,\ \text{\&c.}$$

containing powers of $\overline{a-1}$ only;

$$=\{1+An+Bn^2+Cn^3+\&c.\}^{\frac{x}{n}},\text{ if } a-1-\frac{1}{2}(a-1)^2+\&c.=A,$$

$$=1+\frac{x}{n}(An+Bn^2+\ldots)+\frac{x}{n}.\frac{\frac{x}{n}-1}{2}(An+Bn^2+\ldots)^2+\ldots\ldots$$

$$=1+x(A+Bn+\ldots)+x.\frac{x-n}{2}(A+Bn+\ldots)^2+\ldots\ldots$$

Now, since n is arbitrary, and a^x is clearly independent of n, so as to have the same value whatever be the value of n, it follows, that the terms in the above series, which contain n, must mutually destroy each other, and the reduced series will therefore consist only of those terms in which n is not found;

$$\therefore\ a^x=1+Ax+\frac{A^2x^2}{1.2}+\frac{A^3x^3}{1.2.3}+\&c.$$

COR. If ϵ be the value of a which makes A equal to 1, then

$$\epsilon^x=1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots\ldots$$

Hence, making $x=1$,

$$\epsilon=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots=2{\cdot}71828,\ \text{(Art. 57, Ex. 1).}$$

Another method of expanding a^x will be found in p. 215, Ex. 9.

THE MULTINOMIAL THEOREM.

326. The Multinomial Theorem is a rule or formula for expanding any power of a quantity which consists of more than two terms.

The expansion of a multinomial may frequently be effected by the Binomial Theorem, as is done for a trinomial in Art. 322; for $(a+b+c+d+\&c.)^m$ may be expanded as a binomial by considering any number of terms as one term, and the remainder as another term. But a more general method is to find *the general term*, and to deduce the whole expansion from that term as follows:—

327. *To find the* general term *of the expansion of* $(\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}+\&c.)^m$.

Let $b+c+d+\&c.=z$, then $(a+b+c+d+\&c.)^m=(a+z)^m$, of which the general term, expressed by the $\overline{\alpha+1}^{\text{th}}$, is

$$\frac{m(m-1)\ldots\ldots(m-\alpha+1)}{1\ .\ 2\ \ldots\ldots\ \alpha}a^pz^\alpha,\text{ where } p+\alpha=m,$$

$$\text{or } m(m-1)\ldots\ldots(p+1).a^p.\frac{z^\alpha}{\underline{|\alpha}},\ \textit{where } \alpha \textit{ is a positive integer.}$$

Again, if $c+d+\&c.=y$, then $z^\alpha=(y+b)^\alpha$, of which the general term, expressed by the $\overline{q+1}^{\text{th}}$, is

$$\frac{\alpha(\alpha-1)\ldots\ldots(\alpha-q+1)}{1\,.\,2\ \ldots\ldots\ q}b^q y^\beta, \text{ (where } q+\beta=\alpha, \text{ or } p+q+\beta=m),$$

or $$\frac{\alpha(\alpha-1)\ldots(\beta+1)}{1\,.\,2\ \ldots\ q}.\frac{\beta(\beta-1)\ldots3.2.1}{1\,.\,2\ \ldots\ \beta}b^q y^\beta, \text{ (}\beta \text{ being a pos. integer)},$$

or $\underline{|\alpha}.\frac{b^q}{\underline{|q}}.\frac{y^\beta}{\underline{|\beta}}$; so that the general term of the multinomial becomes

$$m(m-1)\ldots\ldots(p+1).a^p.\frac{b^q}{\underline{|q}}.\frac{y^\beta}{\underline{|\beta}}.$$

Again, if $\overline{d+\&c.}=x$, $y^\beta=(x+c)^\beta$, of which the general term, expressed by the $\overline{r+1}^{\text{th}}$, is

$$\frac{\beta(\beta-1)\ldots\ldots(\beta-r+1)}{1\,.\,2\ \ldots\ldots\ r}c^r x^\gamma, \text{ where } (r+\gamma=\beta, \text{ or } p+q+r+\gamma=m),$$

or $$\frac{\beta(\beta-1)\ldots(\gamma+1)}{1\,.\,2\ \ldots\ r}.\frac{\gamma(\gamma-1)\ldots3.2.1}{1\,.\,2\ \ldots\ \gamma}c^r x^\gamma, \text{ (}\gamma \text{ being a pos. integer)},$$

or $\underline{|\beta}.\frac{c^r}{\underline{|r}}.\frac{x^\gamma}{\underline{|\gamma}}$; so that the general term of the multinomial becomes

$$m(m-1)\ldots\ldots(p+1).a^p.\frac{b^q}{\underline{|q}}.\frac{c^r}{\underline{|r}}.\frac{x^\gamma}{\underline{|\gamma}};$$

and so on, until the terms of the multinomial are exhausted.

Hence the general term required is

$$m(m-1)\ldots\ldots(p+1).a^p.\frac{b^q}{\underline{|q}}.\frac{c^r}{\underline{|r}}.\frac{d^s}{\underline{|s}}.\ \&\text{c., where } p+q+r+s+\&\text{c.}=m,$$

p being fractional or negative when m is fractional or negative, but q, r, s, &c. always positive integers.

Cor. If m be a positive integer, then, since p is a positive integer, the expression for the general term may be written

$$m(m-1)\ldots(p+1)p(p-1)\ldots3.2.1.\frac{a^p}{\underline{|p}}.\frac{b^q}{\underline{|q}}.\frac{c^r}{\underline{|r}}.\&\text{c.}$$

$$\text{or } \underline{|m}.\frac{a^p}{\underline{|p}}.\frac{b^q}{\underline{|q}}.\frac{c^r}{\underline{|r}}.\&\text{c.}$$

The last result may also be arrived at by the following method, assuming *the index a positive integer*:—

328. *To expand* $(\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}+\&\mathrm{c.})^{\mathrm{m}}$, *when* m *is a positive integer.*

$$\epsilon^{(a+b+c+d+\&c.)x}=\epsilon^{ax}.\epsilon^{bx}.\epsilon^{cx}.\epsilon^{dx}.\&\text{c.}$$

and if $\epsilon=2{\cdot}7182818$, expanding by the Exponential Theorem, (Art. 325),

$$1+(a+b+c+d+\&\text{c.})\frac{x}{1}+(a+b+c+d+\&\text{c.})^2\frac{x^2}{\underline{|2}}+\ldots$$
$$+(a+b+c+d+\&\text{c.})^m\frac{x^m}{\underline{|m}}+\ldots$$

$$\begin{aligned}
&= \left(1+ax+\frac{a^2x^2}{\lfloor\underline{2}}+\frac{a^3x^3}{\lfloor\underline{3}}+\ldots+\frac{a^mx^m}{\lfloor\underline{m}}+\ldots\right)\\
&\times\left(1+bx+\frac{b^2x^2}{\lfloor\underline{2}}+\frac{b^3x^3}{\lfloor\underline{3}}+\ldots+\frac{b^mx^m}{\lfloor\underline{m}}+\ldots\right)\\
&\times\left(1+cx+\frac{c^2x^2}{\lfloor\underline{2}}+\frac{c^3x^3}{\lfloor\underline{3}}+\ldots+\frac{c^mx^m}{\lfloor\underline{m}}+\ldots\right)\\
&\times \&c.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Now, as this operation merely exhibits the same quantity expanded in two different ways by the same theorem, the corresponding terms, that is, the terms involving the same powers of x will be equal to each other; therefore equating the coefficient of x^m on the one side with the coefficient of x^m on the other, and observing that each separate term on this side of the equation which involves x^m will be the product of as many terms as there are series to be multiplied, one of which is taken out of each series, and will therefore be of the form

$$\frac{a^px^p}{\lfloor\underline{p}}\cdot\frac{b^qx^q}{\lfloor\underline{q}}\cdot\frac{c^rx^r}{\lfloor\underline{r}}.\&c. \text{ or } \frac{a^pb^qc^r.\&c.}{\lfloor\underline{p}\cdot\lfloor\underline{q}\cdot\lfloor\underline{r}.\&c.}\,x^{p+q+r+\&c.},$$

where $p+q+r+\&c.=m$, we have

$$\frac{(a+b+c+d+\&c.)^m}{\lfloor\underline{m}} = \Sigma\frac{a^pb^qc^r.\&c.}{\lfloor\underline{p}\cdot\lfloor\underline{q}\cdot\lfloor\underline{r}.\&c.}\,{}^{*},$$

$$\therefore\ (a+b+c+d+\&c.)^m = \Sigma\,\frac{\lfloor\underline{m}}{\lfloor\underline{p}\cdot\lfloor\underline{q}\cdot\lfloor\underline{r}.\&c.}\,a^pb^qc^r.\&c.^{\dagger}$$

Cor. 1. If $q+r+s+\&c.=\pi$, then $p=m-\pi$, and the general term becomes

$$\frac{m(m-1)\ldots(m-\pi+1)}{1.2\ldots q.1.2\ldots r.\&c.}\,a^{m-\pi}b^qc^r.\&c.$$

which form is sometimes found more convenient.

Cor. 2. If it be required to expand $(a_0+a_1x+a_2x^2+a_3x^3+\&c.)^m$, the general term may be obtained from that of $(a+b+c+d+\&c.)^m$ by writing a_0, a_1x, a_2x^2, a_3x^3, &c. in place of a, b, c, d, &c. respectively, by which it becomes

$$\frac{\lfloor\underline{m}}{\lfloor\underline{p}\cdot\lfloor\underline{q}\cdot\lfloor\underline{r}.\&c.}\,.a_0^{\,p}a_1^{\,q}a_2^{\,r}.\&c..\,x^{q+2r+3s+\&c.};$$

and all the terms of the expansion may be found as before, by giving p, q, r, s, &c. all possible values which the condition $p+q+r+s+\&c.=m$ admits of.

Also any particular term involving a proposed power of x, as x^n, will be found by taking the sum of the values of this general term, when p, q, r, s, &c. are made to assume *all* the values, which satisfy the two equations $p+q+r+s+\&c.=m$, and $q+2r+3s+\&c.=n$.

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* Σ stands for the expression "the sum of all the quantities of the form of."

† The proof here given of the Multinomial Theorem extends only to the case of a positive integral index, for by the Exponential Theorem m cannot be any thing but a positive integer. But if the Multinomial be deduced from the Binomial Theorem (as in Art. 327), then since the latter is proved for fractional and negative indices, the former is also proved to hold equally for such indices.

COR. 3. Assuming the theorem for a positive integral index, it may be proved for a fractional or negative index thus :—

Let $b+c+d+\&c.=x$, then $(a+b+c+d+\&c.)^m=(a+x)^m$, of which the general term, expressed by the $\overline{\alpha+1}^{\text{th}}$, is

$$\frac{m(m-1)\dots(p+1)}{1\,.\,2\;\dots\;\alpha}\,a^p x^\alpha, \text{ where } p+\alpha=m;$$

and since α is a positive integer, by what has been proved the general term of x^α, or $(b+c+d+\&c.)^\alpha$, is $\underline{|\alpha}\,.\,\dfrac{b^q}{\underline{|q}}\,.\,\dfrac{c^r}{\underline{|r}}\,.\,\&c.$

$$\therefore \text{ term required}=m(m-1)\dots(p+1).a^p.\frac{b^q}{\underline{|q}}.\frac{c^r}{\underline{|r}}.\,\&c.$$

Ex. 1. Required the term in the expansion of $(a-b-c)^7$ which involves $a^2b^3c^2$.

Here $m=7$, $m-\pi=2$, $q=3$, $r=2$;

$$\therefore \text{ the term required}=\frac{7.6.5.4.3}{1.2.3.1.2}\,.\,a^2.(-b)^3.(-c)^2,$$

$$=-210a^2b^3c^2.$$

Ex. 2. Required the term in the expansion of $(a+bx+cx^2+dx^3)^4$ which involves x^8.

$$\text{The required term}=\Sigma\,\frac{\underline{|4}}{\underline{|p}\,.\,\underline{|q}\,.\,\underline{|r}\,.\,\underline{|s}}\,.\,a^pb^qc^rd^s.\,x^8,$$

where $p+q+r+s=4$, and $q+2r+3s=8$; and it remains to find *all* the values of p, q, r, s which satisfy these equations. To do this, it is most convenient to take in order, beginning with the highest or the lowest, the several values of q, r, and s, which satisfy the latter, and reject those which are inconsistent with the former, equation. And it is most advantageous to begin by assigning values to that quantity which has the greatest coefficient, *i. e.* in this case s, and to take them in descending order of magnitude. Thus we see from the second equation that s cannot be greater than 2 : *i. e.* it may have the values 2, 1, 0 : let $s=2$; then $q+2r=2$; whence $r=1$, or 0, and $q=0$, or 2. Next, let $s=1$; then $q+2r=5$; and $r=2$, 1, or 0 ; $q=1$, 3, or 5. Lastly, let $s=0$; then $q+2r=8$: and $r=4$, 3, 2, 1, or 0 ; $q=0$, 2, 4, 6, 8. Of these values, only so many must be taken as will satisfy the first equation, when we shall have to reject all except the underwritten :

s	r	q	p
2	1	0	1
2	0	2	0
1	2	1	0
0	4	0	0

where the simultaneous values of the quantities are written from left to right.

Hence the required term is

$$|\underline{4}\left(\frac{acd^2}{|\underline{2}}+\frac{b^2d^2}{|\underline{2}\,.\,|\underline{2}}+\frac{bc^2d}{|\underline{2}}+\frac{c^4}{|\underline{4}}\right)x^8,$$

or $(12acd^2+6b^2d^2+12bc^2d+c^4)x^8$.

Ex. 3. Find the coefficient of x^8 in the expansion of $(a-bx+cx^2)^{12}$.

Here $\left.\begin{aligned} p+q+r&=12\\ q+2r&=\ 8\end{aligned}\right\}$ are the equations of condition:

p	q	r
4	8	0
5	6	1
6	4	2
7	2	3
8	0	4

$$\therefore \text{ coeff}^{\text{t}}.\text{ required}=|\underline{12}\,.\left\{\frac{a^4b^8}{|\underline{4}\,.\,|\underline{8}}+\frac{a^5b^6c}{|\underline{5}\,.\,|\underline{6}}+\frac{a^6b^4c^2}{|\underline{6}\,.\,|\underline{4}\,.\,|\underline{2}}+\frac{a^7b^2c^3}{|\underline{7}\,.\,|\underline{2}\,.\,|\underline{3}}+\frac{a^8c^4}{|\underline{8}\,.\,|\underline{4}}\right\},$$

$$=495a^4b^8+5544a^5b^6c+13860a^6b^4c^2+7920a^7b^2c^3+495a^8c^4.$$

Ex. 4. Required the term involving x^{14} in the expansion of

$$(ax-bx^3+cx^5-\&c.)^{10}.$$

Here $(ax-bx^3+cx^5-\&c.)^{10}=x^{10}(a-bx^2+cx^4-\&c.)^{10}$; therefore it will only be necessary to find the term involving x^4 in $(a-bx^2+cx^4-\&c.)^{10}$. Hence the equations of condition are

$$\left.\begin{aligned} p+q+r+\&c.&=10,\\ 2q+4r+\&c.&=\ 4,\end{aligned}\right\}$$

p	q	r
8	2	0
9	0	1

and by virtue of the second equation all the quantities after r must separately $=0$.

$$\therefore \text{ the term required}=|\underline{10}\,.\left\{\frac{a^8b^2}{|\underline{8}\,.\,|\underline{2}}+\frac{a^9c}{|\underline{9}}\right\}x^{14},$$

$$=(45a^8b^2+10a^9c)x^{14}.$$

Ex. 5. Required the coefficient of x^3 in $(a+bx+cx^2+dx^3)^{\frac{1}{2}}$.

$$\text{Here}\quad \left.\begin{aligned} p+q+r+s&=\frac{1}{2}\\ q+2r+3s&=3 \end{aligned}\right\}\text{ are the equations of condition :}$$

p	q	r	s
$-\frac{5}{2}$	3	0	0
$-\frac{3}{2}$	1	1	0
$-\frac{1}{2}$	0	0	1

$$\therefore \text{ coefficient required} = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{a^{-\frac{5}{2}}b^3}{\underline{\lvert 3}} + \frac{1}{2}\left(\frac{1}{2}-1\right)a^{-\frac{3}{2}}bc + \frac{1}{2}a^{-\frac{1}{2}}d,$$

$$= \frac{b^3}{16a^{\frac{5}{2}}} - \frac{bc}{4a^{\frac{3}{2}}} + \frac{d}{2a^{\frac{1}{2}}}.$$

EVOLUTION OF SURDS.

A practical method of finding the square root of a binomial surd was given in Art. 182; the following is the one more usually adopted :—

329. *To extract the square root of a quantity which is under the form* a$+\sqrt{\text{b}}$.

$$\text{Assume}\quad \sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}},$$

$$\text{then squaring,}\quad x+y+2\sqrt{xy}=a+\sqrt{b};$$

$$\left.\begin{aligned} \therefore\ x+y&=a,\\ \text{and}\ 2\sqrt{xy}&=\sqrt{b}, \end{aligned}\right\}\text{(Art. 179):}$$

from these two equations we find x and y thus :—

$$\left.\begin{aligned} x^2+2xy+y^2&=a^2,\\ 4xy&=b, \end{aligned}\right\}$$

$$\therefore\ x^2-2xy+y^2=a^2-b,$$

$$x-y=\sqrt{a^2-b}.$$

$$\text{And}\ x+y=a;$$

$$\therefore\ 2x=a+\sqrt{a^2-b},$$

$$\text{and } 2y = a - \sqrt{a^2 - b},$$

$$\therefore\ x = \frac{a + \sqrt{a^2 - b}}{2},$$

$$\text{and } y = \frac{a - \sqrt{a^2 - b}}{2};$$

$$\therefore\ \sqrt{x} + \sqrt{y} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}.$$

From this conclusion it appears that the square root of $a + \sqrt{b}$ can only be expressed by a binomial, one or both of whose terms are *quadratic* surds, when $a^2 - b$ is a perfect square.

If the proposed surd be of the form $a - \sqrt{b}$, then we assume $\sqrt{x} - \sqrt{y} = \sqrt{a - \sqrt{b}}$, and proceed as before.

330. It must be observed that this method applies only to cases in which *one* of the terms of the binomial is a *quadratic* surd, and *the other rational.*

If, however, a binomial is proposed which can be put under the form $\sqrt{a^2 c} + \sqrt{bc}$, or $\sqrt{c}(a + \sqrt{b})$, its square root may be found, by finding the square root of $a + \sqrt{b}$, and multiplying the result by $\sqrt[4]{c}$.

Ex. 1. Required the square root of $\frac{3}{2} + \sqrt{2}$.

$$\text{Assume } \sqrt{x} + \sqrt{y} = \sqrt{\frac{3}{2} + \sqrt{2}};$$

$$\text{then } x + y + 2\sqrt{xy} = \frac{3}{2} + \sqrt{2};$$

$$\left.\begin{aligned} \therefore\ x + y &= \frac{3}{2}, \\ \text{and } 2\sqrt{xy} &= \sqrt{2}, \end{aligned}\right\}\ \text{(Art. 179)}$$

$$\left.\begin{aligned} x^2 + 2xy + y^2 &= \frac{9}{4}, \\ 4xy &= 2, \end{aligned}\right\}$$

$$x^2 - 2xy + y^2 = \frac{1}{4},$$

$$\left.\begin{aligned} x - y &= \frac{1}{2}, \\ x + y &= \frac{3}{2}, \end{aligned}\right\}$$

$$\therefore\ 2x = 2, \text{ or } x = 1,$$

$$\text{and } 2y=1, \text{ or } y=\frac{1}{2};$$

$$\therefore \sqrt{x}+\sqrt{y}=1+\frac{1}{2}\sqrt{2}, \text{ the root required.}$$

Ex. 2. Required the square root of $\sqrt{27}+\sqrt{24}$.

$$\text{Here } \sqrt{27}+\sqrt{24}=\sqrt{9\times3}+\sqrt{8\times3}=\sqrt{3}(3+\sqrt{8});$$

$$\therefore \sqrt{\sqrt{27}+\sqrt{24}}=\sqrt[4]{3}.\sqrt{3+\sqrt{8}};$$

and applying the method of Art. 329, $\sqrt{3+\sqrt{8}}$ is found to be $1+\sqrt{2}$; (or see Art. 182. Ex. 1): therefore

$$\text{the root required}=\sqrt[4]{3}(1+\sqrt{2}), \text{ or } \sqrt[4]{3}+\sqrt[4]{12}.$$

331. Lemma. *If* $\sqrt[3]{\mathrm{a}+\sqrt{\mathrm{b}}}=\mathrm{x}+\sqrt{\mathrm{y}}$, *then also* $\sqrt[3]{\mathrm{a}-\sqrt{\mathrm{b}}}=\mathrm{x}-\sqrt{\mathrm{y}}$.

$$\text{For if } \sqrt[3]{a+\sqrt{b}}=x+\sqrt{y},$$

$$a+\sqrt{b}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y},$$

$$\therefore \text{ (Art. 179) } a=x^3+3xy,$$

$$\text{and } \sqrt{b}=3x^2\sqrt{y}+y\sqrt{y},$$

$$\text{hence } a-\sqrt{b}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y},$$

$$\therefore \sqrt{a-\sqrt{b}}=x-\sqrt{y}.$$

332. *To find the cube root of a binomial Surd of the form* $\mathrm{a}+\sqrt{\mathrm{b}}$, *when it can be expressed by a binomial of the same description.*

$$\text{Assume } x+\sqrt{y}=\sqrt[3]{a+\sqrt{b}},$$

$$\text{then } x-\sqrt{y}=\sqrt[3]{a-\sqrt{b}}, \text{ (Art. 331)};$$

$$\therefore x^2-y=\sqrt[3]{a^2-b}.$$

Now, if a^2-b be a perfect cube, let it be equal to c^3;

$$\text{then } x^2-y=c,$$

$$\text{but } x^3+3xy=a, \text{ (Art. 331)};$$

$$\therefore x^3+3x(x^2-c)=a,$$

$$\text{or } 4x^3-3cx=a.$$

From this equation x must be found by trial, and then y is known from the equation $y=x^2-c$; thus $x+\sqrt{y}$ is known, which is the required root.

It appears from the operation that the cube root of $a+\sqrt{b}$ can only be expressed by a binomial of the same form when a^2-b is a perfect cube.

This test, therefore, ought to be applied to every proposed case in the first instance.

Ex. Required the cube root of $10+\sqrt{108}$.

Here $a^2-b=100-108=-8=(-2)^3$; therefore the method of Art. 332 may be applied.

$$\begin{aligned} \text{Let}\quad & x+\sqrt{y}=\sqrt[3]{10+\sqrt{108}}, \\ \text{then}\quad & x-\sqrt{y}=\sqrt[3]{10-\sqrt{108}}; \\ \therefore\quad & x^2-y=\sqrt[3]{100-108}=-2. \\ \text{Also}\quad & x^3+3xy=10; \\ \therefore\quad & x^3+3x(x^2+2)=10, \\ \text{or}\quad & 4x^3+6x=10, \end{aligned}$$

an equation which is satisfied by $x=1$; therefore $y=3$; and

$$x+\sqrt{y}=1+\sqrt{3}.$$

If therefore the cube root of $10+\sqrt{108}$ can be expressed in the proposed form, it is $1+\sqrt{3}$; which on trial is found to succeed.

[*Exercises Zh.*]

333. LEMMA. *If* n *be an odd number,* a *and* b *one or both quadratic surds, and* x *and* y *involve the same surds that* a *and* b *do respectively, and also* $(a+b)^{\frac{1}{n}}=x+y$, *then* $(a-b)^{\frac{1}{n}}=x-y$.

By involution $a+b=(x+y)^n$,

or $a+b=x^n+nx^{n-1}y+n.\frac{n-1}{2}x^{n-2}y^2+n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}y^3+\&c.$

where the odd terms involve the same surd that x does, because n is an odd number, and the even terms the same surd that y does; and since no part of a can consist of y, or its multiples or parts (Art. 181),

$$a=x^n+n.\frac{n-1}{2}x^{n-2}y^2+\&c.$$

$$\text{and}\quad b=nx^{n-1}y+n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}y^3+\&c.$$

hence,

$$\begin{aligned} a-b&=x^n-nx^{n-1}y+n.\frac{n-1}{2}x^{n-2}y^2-n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}y^3+\&c. \\ &=(x-y)^n; \end{aligned}$$

$$\therefore\ (a-b)^{\frac{1}{n}}=x-y.$$

334. *The* nth *root of a binomial, one or both of whose terms are possible quadratic surds, may sometimes be expressed by a binomial of that description.*

Let $A+B$ be the given binomial surd, in which both terms are possible; the quantities under the radical signs *whole numbers; A* greater than B; and n an odd number.

$$\text{Assume } \sqrt[n]{(A+B)\times\sqrt{Q}} = x+y,$$

$$\text{then } \sqrt[n]{(A-B)\times\sqrt{Q}} = x-y, \text{ (Art. 333.)}$$

$$\text{by mult. } \sqrt[n]{(A^2-B^2)\times Q} = x^2-y^2;$$

let Q be so assumed that $(A^2-B^2)\times Q$ may be a perfect n^{th} power, as p^n, then $x^2-y^2=p$.

Again, by squaring both sides of the first two equations, we have

$$\sqrt[n]{(A+B)^2\times Q} = x^2+2xy+y^2,$$

$$\sqrt[n]{(A-B)^2\times Q} = x^2-2xy+y^2,$$

$$\text{hence } \sqrt[n]{(A+B)^2\times Q} + \sqrt[n]{(A-B)^2\times Q} = 2x^2+2y^2,$$

which is always a whole number, when the root is a binomial of the supposed form. Take therefore s and t the *nearest* integer values of

$$\sqrt[n]{(A+B)^2\times Q}, \text{ and } \sqrt[n]{(A-B)^2\times Q},$$

one of which is greater, and the other less, than the true value of the corresponding quantity; then since the sum of these surds is an integer, the fractional parts must destroy each other, and $2x^2+2y^2=s+t$ exactly, when the root of the proposed quantity can be obtained. We have therefore these two equations

$$\left.\begin{aligned} x^2-y^2&=p, \\ x^2+y^2&=\frac{s+t}{2}, \end{aligned}\right\}$$

$$\therefore\ 2x^2 = p+\frac{s+t}{2} = \frac{s+t+2p}{2},$$

$$x^2 = \frac{s+t+2p}{4},$$

$$x = \frac{\sqrt{s+t+2p}}{2}.$$

$$\text{Also } 2y^2 = \frac{s+t-2p}{2},$$

$$\text{and } y = \frac{\sqrt{s+t-2p}}{2};$$

therefore, if the root of the binomial $\sqrt[n]{(A+B)\times\sqrt{Q}}$ be of the form $x+y$, it is $\frac{\sqrt{s+t+2p}+\sqrt{s+t-2p}}{2}$; and the n^{th} root of $A+B$ is $\frac{\sqrt{s+t+2p}+\sqrt{s+t-2p}}{2\sqrt[2n]{Q}}$.

335. In the same manner, the n^{th} root of $A-B$ is

$$\frac{\sqrt{s+t+2p}-\sqrt{s+t-2p}}{2\sqrt[2n]{Q}};$$

in which expression, when A is less than B, p is negative.

336. If the index of the root to be extracted be an *even* number, the square root of the proposed quantity may be found by Art. 329, when it can be expressed by a binomial of the same description; and if half the index be an even number, the square root may again be taken, and so on, until the root remaining to be extracted is expressed by an odd number, and then the method either of the preceding Arts., or of Art. 332, may be applied.

Ex. 1. Required the cube root of $11+5\sqrt{7}$.

Here $A=5\sqrt{7}$, $B=11$, $A^2-B^2=54$; therefore $Q=4$, and $p^3=216$, or $p=6$.

$$\text{Also } \sqrt[3]{(A+B)^2\times Q} = \sqrt[3]{(296+110\sqrt{7})\times 4},$$

$$= \sqrt[3]{2268\cdot 44},$$

$$= 13+f;$$

$$\text{Similarly } \sqrt[3]{(A-B)^2\times Q} = 3-f;$$

or $s=13$, and $t=3$; therefore, by substitution, $x=\sqrt{7}$, and $y=1$; hence $x+y=\sqrt{7}+1$; and the quantity to be tried for the root is $\frac{\sqrt{7}+1}{\sqrt[3]{2}}$, which is found to succeed.

Ex. 2. Required the cube root of $2\sqrt{7}+3\sqrt{3}$.

Here $A=2\sqrt{7}$, $B=3\sqrt{3}$, $A^2-B^2=1$; hence $Q=1$, and $p=1$.

$$\text{Also } \sqrt[3]{(A+B)^2.Q}=\sqrt[3]{(55+12\sqrt{21})\times 1},$$

$$=\sqrt[3]{109{\cdot}96},$$

$$=4+f.$$

$$\text{Similarly } \sqrt[3]{(A-B)^2.Q}=1-f;$$

or $s=4$, and $t=1$; therefore $x=\frac{\sqrt{7}}{2}$, and $y=\frac{\sqrt{3}}{2}$; hence $x+y=\frac{\sqrt{7}+\sqrt{3}}{2}$, the quantity to be tried for the root, which is found to succeed.

337. In the operation it is required to find a number Q, such, that $(A^2-B^2)\times Q$ may be a perfect n^{th} power; this will always be the case, if Q be taken equal to $(A^2-B^2)^{n-1}$; but to find a less number which will answer this condition, let A^2-B^2 be divisible by a, a, &c.…α times; b, b, &c.…β times; c, c, &c.…γ times, &c. in succession; that is, let $A^2-B^2=a^{\alpha}b^{\beta}c^{\gamma}$. &c. Also let $Q=a^xb^yc^z$. &c. then

$$(A^2-B^2).Q=a^{\alpha+x}\times b^{\beta+y}\times c^{\gamma+z}\times\&c.$$

which is a perfect n^{th} power, if x, y, z, &c. be so assumed that $\alpha+x$, $\beta+y$, $\gamma+z$, &c. are respectively equal to n, or some multiple of n.

Thus, to find a number which multiplied by 180 will produce a perfect cube, divide 180 as often as possible by 2, 3, 5, &c., and it appears that $2.2.3.3.5=180$; if therefore, it be multiplied by $2.3.5.5$, it becomes $2^3.3^3.5^3$, or $(2.3.5)^3$, which is a perfect cube.

338. If A and B be divided by their greatest common measure, either integer or quadratic surd, in all cases where the n^{th} root can be obtained by this method, Q will either be unity, or some power of 2, less than 2^n. See *Dr. Waring's Med. Alg.* Chap. v.*

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* The proof here omitted, and the reference to it in Dr. Waring's work, may both be safely neglected, as it is of no practical use whatever.—ED.

339. The square root of a *multinomial*, as $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$, of which one term is rational, and the rest quadratic surds, may sometimes be found by assuming

$$\sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}}=\sqrt{x}+\sqrt{y}+\sqrt{z},$$

and proceeding to find x, y, and z, as in Art. 329.

Ex. Required the square root of $21+6\sqrt{5}+6\sqrt{7}+2\sqrt{35}$.

Let $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{21+6\sqrt{5}+6\sqrt{7}+2\sqrt{35}}$,

then $x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}=21+6\sqrt{5}+6\sqrt{7}+2\sqrt{35}$;

$$\left.\begin{aligned}\therefore\ x+y+z&=21,\\ 2\sqrt{xy}&=6\sqrt{5},\\ 2\sqrt{xz}&=6\sqrt{7},\\ 2\sqrt{yz}&=2\sqrt{35},\end{aligned}\right\}\text{ to find } x,\ y, \text{ and } z.$$

Now $2\sqrt{xy}\times2\sqrt{xz}=4x\sqrt{yz}$, or $6\sqrt{5}\times6\sqrt{7}=4x\sqrt{35}$,

$$\therefore\ x=9,\ \text{and}\ \sqrt{x}=3.$$

Also $2\sqrt{xy}\times2\sqrt{yz}=4y\sqrt{xz}$, or $6\sqrt{5}\times2\sqrt{35}=12y\sqrt{7}$,

$$\therefore\ y=5,\ \text{and}\ \sqrt{y}=\sqrt{5}.$$

Again, $x+y+z=21$, or $9+5+z=21$,

$$\therefore\ z=7,\ \text{and}\ \sqrt{z}=\sqrt{7}.$$

Hence $\sqrt{x}+\sqrt{y}+\sqrt{z}=3+\sqrt{5}+\sqrt{7}$, the root required.

INDETERMINATE COEFFICIENTS.

340. *If* $\mathrm{A}+\mathrm{Bx}+\mathrm{Cx}^2+\&\mathrm{c}.=\mathrm{a}+\mathrm{bx}+\mathrm{cx}^2+\&\mathrm{c}.$ *be an identical equation, that is, if it hold for all values whatever of* x, *then the coefficients of like powers of* x *are equal to each other, that is,* A = a, B = b, C = c, &c.*

For if $A+Bx=a+bx$, then

$$A-a+(B-b)x=0,$$

an equation which admits of one value of x only (Art. 193), unless $B-b=0$, or $B=b$, and therefore also $A-a=0$, or $A=a$.

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* In the proof which is usually given x is assumed equal to 0, and afterwards the equal quantities are divided by x, whereas it is not proved that we may divide any quantity by x when x stands for 0, in the same manner as when it stands for a finite magnitude; and that such a proceeding will in certain cases lead to erroneous results is well known.

Again, if $A+Bx+Cx^2=a+bx+cx^2$, then

$$A-a+(B-b)x+(C-c)x^2=0,$$

a quadratic equation with respect to x which admits of no more than *two* distinct values of x (Art. 204), unless $C-c=0$, or $C=c$, and $B-b=0$, or $B=b$, and therefore also $A-a=0$, or $A=a$.

Similarly, if *any number of terms* be taken, or

$$(A-a)+(B-b)x+(C-c)x^2+\&c.=0,$$

there are certain values of x, and none other, which will satisfy the equation as long as it remains an *equation* with respect to x.

But, by the supposition, the equation must be true for *any value whatever* which we may please to give to x, and consequently for *any number* of values of x; and this, therefore, can only be attained by that which is apparently an *equation* with respect to x ceasing to be such, that is, by the *coefficients* of the powers of x being separately equal to 0; that is, we must have

$$A-a=0, \qquad B-b=0, \qquad C-c=0, \ \&c.$$

$$\text{or } A=a, \qquad B=b, \qquad C=c, \ \&c.$$

COR. If there be found any power of x on one side of the proposed equation, and no corresponding one on the other, then the whole coefficient of that power is of itself equal to 0. Thus, if $A+Bx+Cx^2+\&c.=0$, *for all values whatever of* x, then $A=0$, $B=0$, $C=0$, &c.

This may also be arrived at in the following manner.

It has been already proved (Arts. 193, 204) that if a simple or a quadratic equation be known to be satisfied by more different values of the unknown quantity than the dimensions of the equation, the coefficients of the several powers of the unknown, and the term independent of it, are separately equal to 0. This proposition will be now extended.

If $a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}+a_nx^n=0$ *be satisfied by* $n+1$ *different values of* x, *then* a_0, a_1, a_2,...a_n are separately equal to 0.

Let the $n+1$ values of x be x_1, x_2, x_3,...x_{n+1}.

Then we have

$$a_0+a_1x_{n+1}+a_2x_{n+1}^2+\ldots+a_{n-1}x_{n+1}^{n-1}+a_nx_{n+1}^n=0,$$

$$a_0+a_1x' \quad +a_2x'^2 \quad +\ldots+a_{n-1}x'^{n-1}+a_nx'^n \quad =0,$$

where x' may be equal to any of the n quantities

$$x_1,\ x_2,\ldots x_n.$$

By subtraction we have

$$a_1(x'-x_{n+1})+a_2(x'^2-x_{n+1}^2)+\ldots+a_{n-1}(x'^{n-1}-x_{n+1}^{n-1})+a_n(x'^n-x_{n+1}^n)=0,$$

and as $x'-x_{n+1}$ is not equal to 0, dividing by it we obtain

$$a_1+a_2(x'+x_{n+1})+\ldots+a_{n-1}(x'^{n-2}+\ldots+x_{n+1}^{n-2})+a_n(x'^{n-1}+\ldots+x_{n+1}^{n-1})=0;$$

or, arranging in powers of x',

$$(a_1+a_2x_{n+1}+\ldots)+(a_2+\ldots)x'+\ldots+a_nx'^{n-1}=0,$$

where x' may take any of the n values

$$x_1,\ x_2,\ldots x_n.$$

Comparing this with the original equation, we find that it is of one dimension less, and that the coefficient of the highest power of the unknown is the same as before, viz. a_n.

By repeating this process we shall have an equation of $n-2$ dimensions, the coefficient of the highest power of the unknown being still a_n, and we shall know that it is satisfied by $n-1$ values of the unknown, viz. $x_1,\ x_2,\ldots x_{n-1}$; and so on. At last we shall arrive at an equation of one dimension, where the coefficient of the unknown will be a_n, and this will be satisfied by two values, x_1, x_2.

$$\therefore \text{ by Art. 193, } a_n=0.$$

And our original equation will then become

$$a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}=0,$$

satisfied by more than $n-1$ values of x;

$$\therefore \text{ as before, } a_{n-1}=0,$$

$$\text{so } a_{n-2}=0, \text{ \&c.}$$

This, however, cannot satisfactorily be applied to the case of an infinite series.

341. *Otherwise.* Let $A+Bx+Cx^2+\ldots=a+bx+cx^2+\ldots$ be an identical equation, that is, such as will hold for any value whatever of x; then

$$A\sim a+(B\sim b)x+(C\sim c)x^2+\ldots=0,$$

and, if $A\sim a$ is not equal to 0, let it be equal to some quantity p; then we have

$$(B\sim b)x+(C\sim c)x^2+\text{\&c.}=-p.$$

And since A, a, are invariable quantities, their difference p, and $\therefore -p$, must be invariable; but $-p=(B\sim b)x+(C\sim c)x^2+\ldots$ a quantity which may have various values by the variation of x; that is, we have the same quantity (p) proved to be both fixed and variable, which is absurd. Therefore there is no quantity (p) which can express the difference $A\sim a$, or, in other words,

$$A\sim a=0, \text{ and } \therefore A=a.$$

$$\text{Also } (B\sim b)x+(C\sim c)x^2+\ldots=0,$$

$$\text{or } B\sim b+(C\sim c)x+\ldots=0. \quad \text{(Art. 82.)}$$

Therefore, by what has been proved, $B=b$. And so on, for the remaining coefficients of like powers of x.

This is open to the same objection as the first proof, in which it is assumed that *every* equation has only a *certain* number of values of x which will satisfy it: for as long as x takes the value of any of the roots of the equation here written, the equality will hold; and as the right-hand side may be an infinite series, there may be an infinite number of values of x that will satisfy it, considered as an *equation*. The conclusion that p is variable as well as fixed is therefore hardly correct.

The same result may also be deduced from the corollary to Art. 392: for by virtue of it we can make $(B \sim b)x+(C \sim c)x^2+$&c. less than $-p$, by putting for x any quantity less than $\frac{-p}{-p+k}$, where k is the greatest coefficient in the series involving x. But this cannot be, unless $p=0$;

$$\therefore\ A=a.$$

$$\therefore \text{ as before, } B=b, \quad C=c, \quad \text{\&c.}$$

This however supposes, *a priori*, that the coefficients do not increase without limit.

It will be seen that none of these proofs are altogether satisfactory: the reason of this is, that an *identity* is treated in them in all respects as an *equation*. The fact of the matter is, that we predicate of x properties quite distinct from those that belong to the symbols we have hitherto had to deal with. These symbols standing for abstract or concrete quantities considered with reference to number, are obtained by the repetition of certain units which may be as small as ever we please. By that repetition therefore, all numbers, and all quantities that can be represented under any circumstances by numbers generally, are essentially discontinuous. But in the identity we are dealing with we predicate of x *continuity*, making it thereby a symbol of quantity in the most general and unrestricted sense it can possibly be conceived in. Now an *equation* in which x appears in an infinite series, certainly has an infinite number of roots; but these roots are symbols of quantity not unrestricted: and the equation is satisfied only when x takes the value of one or another of these roots, and not under any other circumstances. However near these roots may lie to each other, as x passes from one to another of them, it changes discontinuously. But supposing x to possess continuity, which it does in the identity here discussed, it changes by *insensible degrees* from any one value to any other however near to the former, and consequently the infinitude of the number of values which x takes is of an infinitely higher order than the infinitude of the number of roots of an equation in the form of an infinite series. On this principle, therefore, and not otherwise, in the first proof, we can say that the infinite equality is satisfied by *more* values of x than the number of its dimensions; and in the second proof we can conclude that p is variable as well as fixed, and by this manner we overcome the objections to those proofs. Or we may proceed to reason as follows:

When we say $a+bx+cx^2+\text{\&c.}=0$ is an identity, for all values of x, we thereby assign to x the property of continuity, and make the series continuous likewise, and therefore capable of taking all manner of different values by the variation of x. These may, or may not, lie between certain limits, but that is quite another question: what we say is merely that the series does depend upon x for its value, and that therefore it can change in value, and moreover that if x is unrestricted, it *does* change, however much it may not when x is restricted.

But by its *continual* equality to zero, we see that it is incapable of taking different values, during the variation of x which is unrestricted. This cannot be unless it only involves x apparently and not really, *i. e.* unless x disappears. Now the only way in which x can disappear is by each of the coefficients vanishing separately, therefore we have

$$a=0, \quad b=0, \quad c=0, \quad \&c.$$

And by the same reasoning we shall have, if

$$A+Bx+Cx^2+\&c. = a+bx+cx^2+\&c.,$$

$$A=a, \quad B=b, \quad C=c, \quad \&c.$$

342. *If*
$$\left.\begin{array}{r} A+Bx+Cx^2+\ldots \\ +A'y+B'xy+\ldots \\ +A''y^2+\ldots \end{array}\right\} = \left\{\begin{array}{r} a+bx+cx^2+\ldots \\ +a'y+b'xy+\ldots \\ +a''y^2+\ldots \end{array}\right.$$
for all values whatever of x *and* y, *then the coefficients of like quantities are equal to each other, that is*

$$A=a, \quad B=b, \quad C=c, \quad A'=a', \quad B'=b', \quad A''=a'', \quad \&c.$$

Since x may receive any value whatever, suppose it to have some fixed value while y is variable, then the equation may be put under the form

$$A_{,}+B_{,}y+C_{,}y^2+\ldots = a_{,}+b_{,}y+c_{,}y^2+\ldots$$

where $A_{,}$, $B_{,}$, $C_{,}$, &c. $a_{,}$, $b_{,}$, $c_{,}$, &c. are invariable coefficients, and

$$\begin{aligned} A_{,} &= A+Bx+Cx^2+\ldots \\ B_{,} &= A'+B'x+C'x^2+\ldots \\ \&c. &= \&c. \\ a_{,} &= a+bx+cx^2+\ldots \\ b_{,} &= a'+b'x+c'x^2+\ldots \\ \&c. &= \&c. \end{aligned}$$

Now, by Art. 340, $A_{,}=a_{,}$, $B_{,}=b_{,}$, $C_{,}=c_{,}$, &c.; therefore

$$\begin{aligned} A+bx+Cx^2+\ldots &= a+bx+cx^2+\ldots \\ A'+B'x+C'x^2+\ldots &= a'+b'x+c'x^2+\ldots \\ A''+B''x+C''x^2+\ldots &= a''+b''x+c''x^2+\ldots \\ \&c. &= \&c. \end{aligned}$$

Then by Art. 340 again, since x may have any value whatever,

$$A=a, \quad B=b, \quad C=c, \quad A'=a', \quad B'=b', \quad A''=a'', \quad \&c.$$

The application of the preceding Theory will be shewn in the following Examples :—

Ex. 1. Expand $\dfrac{a-bx}{a+cx}$ to four terms; that is, divide $a-bx$ by $a+cx$.

Let $\frac{a-bx}{a+cx} = A + Bx + Cx^2 + Dx^3 + \dots$; in which the coefficients A, B, C, D, &c. remain to be determined,

$$\begin{aligned} a - bx &= Aa + Bax + Cax^2 + Dax^3 + \dots \\ &\qquad\quad + Acx + Bcx^2 + Ccx^3 + \dots \\ &= Aa + (Ba + Ac)x + (Ca + Bc)x^2 + (Da + Cc)x^3 + \dots; \end{aligned}$$

and equating coefficients of like powers of x,

$$Aa = a, \quad \text{or } A = 1.$$

$$Ba + Ac = -b, \qquad \therefore\ Ba = -(b+c), \qquad \text{or } B = -\frac{b+c}{a};$$

$$Ca + Bc = 0, \qquad \therefore\ Ca = \frac{b+c}{a}.c, \qquad \text{or } C = \frac{b+c}{a^2}.c;$$

$$Da + Cc = 0, \qquad \therefore\ Da = -\frac{b+c}{a^2}.c^2, \qquad \text{or } D = -\frac{b+c}{a^3}.c^2;$$

$$\therefore\ \frac{a-bx}{a+cx} = 1 - \frac{b+c}{a}x + \frac{b+c}{a^2}cx - \frac{b+c}{a^3}c^2x^2 + \dots\dots$$

Ex. 2. Extract the square root of $1 + x$.

$$\text{Let } \sqrt{1+x} = 1 + Ax + Bx^2 + Cx^3 + \dots\dots$$

assuming 1 for the first term, since that is the root when $x = 0$;

$$\text{Squaring, } \left.\begin{aligned} 1 + x = 1 + 2Ax + 2Bx^2 + 2Cx^3 + \dots\dots\dots \\ + A^2x^2 + 2ABx^3 + \dots\dots\dots \\ + B^2x^4 + \dots \end{aligned}\right\} \text{(Art. 142.)}$$

Hence, equating coefficients of like powers of x,

$$2A = 1, \qquad \therefore\ A = \frac{1}{2};$$

$$2B + A^2 = 0, \qquad \therefore\ B = -\frac{A^2}{2} = -\frac{1}{8};$$

$$2C + 2AB = 0, \qquad \therefore\ C = -AB = +\frac{1}{16}; \text{ and so on:}$$

$$\therefore\ \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots\dots$$

Ex. 3. Let $y^3 - 3y + x = 0$; required the value of y in a series of ascending powers of x.

Assume $y = Ax + Bx^3 + Cx^5 + Dx^7 + \&c.$*,

---

* The even powers of x are omitted because, from the given equation, it appears that the relation betwixt x and y is such that, if $-x$ be written for x, and $-y$ for y, the equation is not altered. If the even powers were retained, their coefficients would be found separately equal to 0.—Ed.

$$\left.\begin{array}{rl} \text{then } y^3 = & \quad A^3x^3 + 3A^2Bx^5 + 3A^2Cx^7 + \&c.\\ & \qquad\qquad\qquad\quad + 3AB^2x^7 + \&c.\\ -3y = & -3Ax - 3Bx^3 - 3Cx^5 - \quad 3Dx^7 - \&c.\\ +x = & +x \end{array}\right\} = 0,$$

and supposing the coefficient of each power of x to be equal to 0, (Art. 340. Cor.)

$$-3A + 1 = 0, \text{ or } A = \frac{1}{3}; \quad A^3 - 3B = 0, \text{ or } B = \frac{A^3}{3} = \frac{1}{3^4};$$

$$3A^2B - 3C = 0, \text{ or } C = A^2B = \frac{1}{3^6}; \ \&c.$$

$$\therefore\ y = \frac{x}{3} + \frac{x^3}{3^4} + \frac{x^5}{3^6} + \&c.$$

Ex. 4. Let $x = ay + by^2 + cy^3 + \&c.$ required the value of y in terms of x.

Assume $y = Ax + Bx^2 + Cx^3 + \&c.$

$$\left.\begin{array}{rl} \text{then } ay = & aAx + aBx^2 + \quad aCx^3 + \&c.\\ by^2 = & \qquad\quad bA^2x^2 + 2bABx^3 + \&c.\\ cy^3 = & \qquad\qquad\qquad\quad cA^3x^3 + \&c.\\ \&c. = & \qquad\qquad\qquad\quad \&c.\\ -x = & -x \end{array}\right\} = 0;$$

$$\text{hence } aA - 1 = 0, \text{ or } A = \frac{1}{a};$$

$$aB + bA^2 = 0, \text{ or } B = \frac{-bA^2}{a} = \frac{-b}{a^3};$$

$$aC + 2bAB + cA^3 = 0, \text{ or } C = \frac{-2bAB}{a} - \frac{cA^3}{a} = \frac{2b^2 - ac}{a^5}; \ \&c.$$

$$\therefore\ y = \frac{x}{a} - \frac{bx}{a^3} + \frac{(2b^2 - ac)x^3}{a^5} + \&c.$$

Ex. 5. Let $x = y - ay^3 + by^5 - \&c.$ required the value of y in terms of x.

Assume $y = Ax + Bx^3 + Cx^5 + \&c.$ (See Note, Ex. 3.)

$$\left.\begin{array}{rl} \text{then}\quad y = & Ax + Bx^3 + Cx^5 + \&c. \\ -ay^3 = & \qquad -aA^3x^3 - 3aA^2Bx^5 - \&c. \\ +by^5 = & \qquad\qquad + \quad bA^5x^5 + \&c. \\ \&c. = & \qquad\qquad\qquad \&c. \\ -x = & -x \end{array}\right\} = 0;$$

$$\text{hence } A-1=0, \quad \text{or } A=1; \quad B-aA^3=0, \quad \text{or } B=0;$$

$$C-3aA^2B+bA^5=0; \quad \text{or } C=3a^2-b; \ \&c.$$

$$\therefore\ y = x + ax^3 + (3a^2+b)x^5 + \&c.$$

[*Exercises Zi.*]

Ex. 6. *To find the sum of the series* $1^2+2^2+3^2+\ldots+n^2$ *by the method of Indeterminate Coefficients.*

Assume $1^2+2^2+3^2+\ldots+n^2 = A+Bn+Cn^2+Dn^3+\&c.$ A, B, C, &c. being unknown coefficients independent of n; then

$$1^2+2^2+\ldots+(n+1)^2 = A+B(n+1)+C(n+1)^2+D(n+1)^3+\&c.$$

$\therefore$ by subtraction, $\quad n^2+2n+1 = B+2Cn+C+3Dn^2+3Dn+D,$

the following terms being omitted, because their coefficients are separately equal to 0, being the coefficients in the last equation of n^3, n^4, &c. Hence, equating coefficients of like powers of n (Art. 340),

$$3D=1, \qquad \text{or } D=\frac{1}{3},$$

$$2C+3D=2, \qquad \text{or } C=\frac{1}{2},$$

$$B+C+D=1, \qquad \text{or } B=\frac{1}{6};$$

$$\therefore \text{ sum required} = \frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}+A,$$

$$= \frac{1}{6}n(n+1)(2n+1)+A.$$

To find A. It will be observed that all this investigation will equally hold good if we had begun at any other term of the proposed series, instead of 1^2; therefore the value of A depends upon the term of the series from which we start. Putting then $n=1$, which we can do, because we do not thereby alter the value of A, which is independent of n, our original equation becomes $1^2=A+B+C+D$; but we already have obtained $B+C+D=1$,

$$\therefore A=0, \text{ and the sum required}$$

$$= \frac{1}{6}n(n+1)(2n+1), \quad \text{or } \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{2\times 3},$$

which is the most convenient form for recollection.

Ex. 7. *To find the sum of the series* $1^3+2^3+3^3+\dots+n^3$.

Assume $1^3+2^3+3^3+\dots+n^3=A+Bn+Cn^2+Dn^3+En^4+$&c.

$\therefore\ 1^3+2^3+\dots+(n+1)^3=A+B(n+1)+C(n+1)^2+D(n+1)^3+E(n+1)^4+\dots$

and by subtraction,

$$n^3+3n^2+3n+1=B+2Cn+C+3Dn^2+3Dn^3+D+4En^3+6En^2+4En+E,$$

omitting the remaining terms for the reason assigned in the preceding Example. Hence

$$4E=1, \qquad \text{or } E=\frac{1}{4},$$

$$3D+6E=3, \qquad \text{or } D=\frac{1}{2},$$

$$2C+3D+4E=3, \qquad \text{or } C=\frac{1}{4},$$

$$B+C+D+E=1, \qquad \text{or } B=0;$$

$$\therefore \text{ sum required}=\frac{n^2}{4}+\frac{n^3}{2}+\frac{n^4}{4}+A=\frac{1}{4}n^2(n^2+2n+1)+A,$$

$$=\left\{\frac{n(n+1)}{1\ .\ 2}\right\}^2+A.$$

As in the last example, to find A we put $n=1$, when we obtain by a similar process $A=0$, and

$$\text{sum required}=\left\{\frac{n(n+1)}{1\ .\ 2}\right\}^2.$$

COR. Since $1+2+3+\dots+n=\dfrac{n(n+1)}{1\ .\ 2}$, therefore

$$1^3+2^3+3^3+\dots+n^3=(1+2+3+\dots+n)^2.$$

Ex. 8. *To solve the equation* $x^4+1=0$.

Assume $x^4+1=(x^2-mx+1)(x^2-nx+1)$, then

$$x^4+1=x^4-(m+n)x^3+(mn+2)x^2-(m+n)x+1;$$

and equating coefficients of like powers of x,

$$m+n=0, \text{ and } mn+2=0,$$

$$\therefore\ n=-m, \quad \text{and } m^2=2, \quad \text{or } m=\pm\sqrt{2};$$

$$\therefore\ x^4+1=(x^2+\sqrt{2}.x+1)(x^2-\sqrt{2}.x+1)=0,$$

$$\text{or } x^2+\sqrt{2}.x+1=0, \quad \text{and } x^2-\sqrt{2}.x+1=0,$$

from which two quadratics we obtain the four roots,

$$x=\frac{-1\pm\sqrt{-1}}{\sqrt{2}}, \text{ and } \frac{1\pm\sqrt{-1}}{\sqrt{2}}.$$

Ex. 9. *To expand* a^x *in a series of powers of* x.

Since $a^x=(1+\overline{a-1})^x=1+x(a-1)+x.\dfrac{x-1}{2}(a-1)^2+x.\dfrac{x-1}{2}.\dfrac{x-2}{3}(a-1)^3+\&c.$

$$=1+\{a-1-\tfrac{1}{2}(a-1)^2+\ldots\}x+Bx^2+Cx^3+\ldots^*,$$

assume $a^x=1+Ax+Bx^2+Cx^3+\ldots\ldots$

then $a^y=1+Ay+By^2+Cy^3+\ldots\ldots,$

and $a^{x+y}=1+A(x+y)+B(x+y)^2+C(x+y)^3+\ldots\ldots$

Multiply the first two series together, and equate coefficients of y with the last, and the result will be

$$A+2Bx+3Cx^2+4Dx^3+\ldots=A+A^2x+ABx^2+ACx^3+\ldots$$

from which, by equating coefficients of like powers of x, we have

$$A=A\;;\quad 2B=A^2,\ \therefore\ B=\frac{A^2}{2}\;;$$

$$3C=AB,\ \therefore\ C=\frac{A^3}{1.2.3}\;;\quad 4D=AC,\ \therefore\ D=\frac{A^4}{1.2.3.4}\;;\text{ and so on.}$$

Hence $a^x=1+\dfrac{Ax}{1}+\dfrac{A^2x^2}{1.2}+\dfrac{A^3x^3}{1.2.3}+\dfrac{A^4x^4}{1.2.3.4}+\ldots\ldots$ where A is equal to $a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\ldots\ldots$

Ex. 10. Resolve $\dfrac{1}{(x+a)(x+b)(x+c)}$ into its partial fractions.

Let $\dfrac{1}{(x+a)(x+b)(x+c)}=\dfrac{A}{x+a}+\dfrac{B}{x+b}+\dfrac{C}{x+c},$

then $1=A(x+b)(x+c)+B(x+a)(x+c)+C(x+a)(x+b).$

Now, since this equation (by the theory) is true for *any* value of x,

let $x=-a$, that is, $x+a=0$,

then $1=A(\overline{a-b}.\overline{a-c})$; or $A=\dfrac{1}{(a-b)(a-c)}$;

let $x=-b$, that is, $x+b=0$,

then $1=-B(\overline{a-b}.\overline{b-c})$; or $B=-\dfrac{1}{(a-b)(b-c)}$;

let $x=-c$, that is, $x+c=0$,

then $1=C(\overline{a-c}.\overline{b-c})$; or $C=\dfrac{1}{(a-c)(b-c)}$;

$$\therefore\ \frac{1}{(x+a)(x+b)(x+c)}=\frac{1}{(a-b)(a-c)(x+a)}-\frac{1}{(a-b)(b-c)(x+b)}+\frac{1}{(a-c)(b-c)(x+c)}.$$

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* This transformation is made to shew that the series will involve no other than positive integral powers of x.

Ex. 11. Resolve $\dfrac{A+Bx+Cx^2}{(1+ax)(1+bx)(1+cx)}$ into its partial fractions.

Let $\dfrac{A+Bx+Cx^2}{(1+ax)(1+bx)(1+cx)}=\dfrac{P}{1+ax}+\dfrac{Q}{1+bx}+\dfrac{R}{1+cx}$,

then $A+Bx+Cx^2=P(1+bx)(1+cx)+Q(1+ax)(1+cx)+R(1+ax)(1+bx)$,

let $x=-\dfrac{1}{a}$, that is, $1+ax=0$,

$$\text{then } A-\frac{B}{a}+\frac{C}{a^2}=P\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right);\quad\text{or } P=\frac{Aa^2-Ba+C}{(a-b)(a-c)};$$

let $x=-\dfrac{1}{b}$, that is, $1+bx=0$,

$$\text{then } A-\frac{B}{b}+\frac{C}{b^2}=Q\left(1-\frac{a}{b}\right)\left(1-\frac{c}{b}\right);\quad\text{or } Q=-\frac{Ab^2-Bb+C}{(a-b)(b-c)};$$

let $x=-\dfrac{1}{c}$, that is, $1+cx=0$,

$$\text{then } A-\frac{B}{c}+\frac{C}{c^2}=R\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right);\quad\text{or } R=\frac{Ac^2-Bc+C}{(a-c)(b-c)};$$

$$\therefore\ \frac{A+Bx+Cx^2}{(1+ax)(1+bx)(1+cx)}=\frac{Aa^2-Ba+C}{(a-b)(a-c)(1+ax)}-\frac{Ab^2-Bb+C}{(a-b)(b-c)(1+bx)}$$
$$+\frac{Ac^2-Bc+C}{(a-c)(b-c)(1+cx)}.$$

Ex. 12. Resolve $\dfrac{3x-1}{x^2(x+1)^2}$ into its partial fractions.

Let $\dfrac{3x-1}{x^2(x+1)^2}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+1}+\dfrac{D}{(x+1)^2}$, (it is necessary to assume so many fractions to include every possible case which can produce the proposed fraction, although either A, or C, or both, *may* possibly be equal to 0),

$$\text{then } 3x-1=Ax(x+1)^2+B(x+1)^2+Cx^2(x+1)+Dx^2;$$

let $x=0$, then $-1=B$,

let $x=-1$, that is, $x+1=0$, then $-4=D$,

let $x=1$, then $2=4A+4B+2C+D$, or $4A+2C=10$, ... (1)

let $x=2$, then $5=18A+9B+12C+4D$,

or $18A+12C=30$, or $3A+2C=5$, ... (2)

subtracting (2) from (1), $A=5$, and $C=5-2A=-5$,

$$\therefore\ \frac{3x-1}{x^2(x+1)^2}=\frac{5}{x}-\frac{1}{x^2}-\frac{5}{x+1}-\frac{4}{(x+1)^2}.$$

Ex. 13. Resolve $\dfrac{3x+2}{x(x+1)^3}$ into its partial fractions.

$$\text{Let } \frac{3x+2}{x(x+1)^3}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3},$$

$$\text{then } 3x+2=A(x+1)^3+Bx(x+1)^2+Cx(x+1)+Dx,$$

let $x=0$, then $2=A$,

let $x=-1$, then $-1=-D$, or $D=1$,

let $x=1$, then $5=8A+4B+2C+D$, or $4B+2C=-12$, ... (1)

let $x=2$, then $8=27A+18B+6C+2D$,

or $18B+6C=-48$, or $6B+2C=-16$, ... (2)

subtracting (1) from (2), $2B=-4$, or $B=-2$; and $C=-2B-6=-2$.

$$\therefore \frac{3x+2}{x(x+1)^3}=\frac{2}{x}-\frac{2}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}.$$

342*. The four preceding are examples of the resolution of a "rational fraction" into its partial fractions, of which we shall now proceed to describe the general method. By a rational fraction is meant one whose numerator and denominator, when arranged according to the powers of a variable quantity x, contain only positive integral powers of x; and furthermore it is supposed that the numerator is of lower dimensions in x than the denominator: if however this be not the case, we can perform the division as far as it will go, thus obtaining an integral quotient and a fraction with its numerator of lower dimensions than its denominator; to this therefore we can confine our attention.

Let $\dfrac{U}{V}$ be such a fraction; then V can always be resolved into factors of one or two dimensions, which may or may not be repeated, that is,

$$V=(x-a)\dots(x-b)^r\dots\dots(x^2+\alpha x+\beta)\dots(x^2+\gamma x+\delta)^s,$$

$x-a$ being one of the linear factors not repeated, $x-b$ one repeated r times, $x^2+\alpha x+\beta$ a quadratic factor that cannot be resolved any further, and $x^2+\gamma x+\delta$ a quadratic factor repeated s times. The following assumptions must then be made: for every factor such as $x-a$, assume a partial fraction $\dfrac{A}{x-a}$: for every factor repeated, as $(x-b)^r$, assume a series of partial fractions

$$\frac{B_1}{(x-b)^r}+\frac{B_2}{(x-b)^{r-1}}+\dots\frac{B_r}{x-b}:$$

for every quadratic factor, $x^2+\alpha x+\beta$, assume a partial fraction

$$\frac{Cx+D}{x^2+\alpha x+\beta};$$

and for the other, assume a series of fractions

$$\frac{E_1x+F_1}{(x^2+\gamma x+\delta)^s}+\ldots\ldots\frac{E_sx+F_s}{x^2+\gamma x+\delta};$$

A, B &c. being independent of x. It will be obvious on consideration that these assumptions must be made, because all that we know is that the numerator of the proposed fraction is of lower dimensions than the denominator.

(1) To find A. Multiply by $x-a$, the denominator corresponding to A, and suppose

$$V=Q.(x-a), \text{ then } \frac{U}{Q}=A+(\text{the other fractions})\times(x-a).$$

Now neither Q, nor any one of the denominators of the other fractions, contains $x-a$; and if we put $x=a$, the value of A is not changed, and neither $\frac{U}{Q}$ nor any of the other fractions becomes infinite, and the other fractions are all multiplied by zero;

$$\therefore\ A=\frac{U}{Q},\text{ when } x=a.$$

In this manner all the other corresponding partial fractions can be determined.

(2) To find B_1, B_2, &c. Multiply by $(x-b)^r$, and suppose

$$V=Q(x-b)^r,\text{ then } \frac{U}{Q}=B_1+B_2(x-b)+\ldots+(\text{the other fractions})\times(x-b)^r;$$

$$\therefore\ B_1=\frac{U}{Q},\text{ when } x=b.$$

By transposing this fraction we have

$$\frac{U}{Q(x-b)^r}-\frac{B_1}{(x-b)^r}=\frac{B_2}{(x-b)^{r-1}}+\ldots$$

And if the left-hand member be reduced to a single fraction, $x-b$ will be a factor of the numerator, which if cancelled will make the denominator involve $(x-b)^{r-1}$; B_2 can then be determined in the same way as B_1; and so on, if there be more.

(3) To find C and D. Multiply by $x^2+\alpha x+\beta$, and suppose

$$V=Q(x^2+\alpha x+\beta),\text{ then } \frac{U}{Q}=Cx+D+(\text{the other fractions})\times(x^2+\alpha x+\beta).$$

If then we put $x^2+\alpha x+\beta=0$, we have $\frac{U}{Q}=Cx+D$.

We then must substitute for x^2 continually $-(\alpha x+\beta)$, and after multiplying out, we shall have at last a simple equation in x, whose terms involve C and D. Now we know that this is satisfied whenever

$$x^2+\alpha x+\beta=0,$$

i.e. it is satisfied by two values of x, therefore the two terms are each $=0$, thus giving two equations to find C and D.

(4) To find E_1, F_1, &c. Multiply by $(x^2+\gamma x+\delta)^r$, and we can determine E_1 and F_1 as in (3): let this first fraction be transposed as in (2), then we shall have $x^2+\gamma x+\delta$ to be cancelled, and we can then find E_2, F_2; and so on, for any others.

Ex. Resolve $\dfrac{x-1}{x^2(x^2+x+1)}$ into its partial fractions.

$$\text{Let it} = \frac{A}{x^2}+\frac{B}{x}+\frac{Cx+D}{x^2+x+1}.$$

Then by (2), multiplying by x^2, and putting $x=0$,

we have $A=-1$.

Transposing this first fraction,

$$\frac{x-1}{x^2(x^2+x+1)}+\frac{1}{x^2}=\frac{B}{x}+\frac{Cx+D}{x^2+x+1};$$

$$\therefore\ \frac{x+2}{x(x^2+x+1)}=\frac{B}{x}+\frac{Cx+D}{x^2+x+1};$$

therefore by a similar process $B=2$.

Applying (3), we have

$$Cx+D=\frac{x-1}{x^2},\ \text{when } x^2+x+1=0,\ \text{or } x^2=-x-1;$$

$$\therefore\ Cx+D=\frac{x-1}{-x-1}=\frac{-x+1}{x+1};$$

$$\therefore\ -x+1=(x+1)(Cx+D)=Cx^2+(C+D)x+D,$$

$$=C(-x-1)+(C+D)x+D;$$

$$\therefore\ 0=(D+1)x-C+D-1,$$

an equation which is satisfied by the two roots of $x^2+x+1=0$;

$$\therefore\ D+1=0,\quad \text{and } -C+D-1=0;$$

$$\therefore\ D=-1,\qquad C=-2;$$

therefore the proposed fraction

$$=-\frac{1}{x^2}+\frac{2}{x}-\frac{2x+1}{x^2+x+1}.$$

CONTINUED FRACTIONS.

343. *To represent $\frac{a}{b}$ in a continued fraction*.*

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* Although a '*continued fraction*' (see Def. Art. 8) *may* be of the form $p+\cfrac{\alpha}{q+\cfrac{\beta}{r+\&c.}}$ the term is commonly restricted to those of the form $p+\cfrac{1}{q+\cfrac{1}{r+\&c.}}$, or $\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\&c.}}}$—Ed.

Let b be contained p times in a, with a remainder c; again, let c be contained q times in b, with a remainder d, and so on; then we have

$$\begin{array}{l} b)\,a\,(p \\ \overline{c)\,b\,(q} \\ \quad \overline{d)\,c\,(r} \\ \qquad \overline{e\ \&c.} \end{array}$$

$$a = pb + c, \quad b = qc + d, \quad c = rd + e, \text{ \&c.}$$

$$\text{or } \frac{a}{b} = p + \frac{c}{b} = p + \frac{c}{qc + d} = p + \cfrac{1}{q + \cfrac{d}{c}},$$

$$= p + \cfrac{1}{q + \cfrac{d}{rd + e}} = p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{e}{d}}}, \text{ \&c.}$$

$$\text{that is, } \frac{a}{b} = p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s + \&c.}}}$$

344. COR. 1. An approximation may thus be made to the value of a fraction whose numerator and denominator are in too high terms; and the further the division is continued, the nearer will the approximation be to the true value.

This is strictly true, but hardly *proved* here. It follows directly from Art. 350, Cor. 2.

345. COR. 2. This approximation is alternately less and greater than the true value. Thus p is less than $\frac{a}{b}$; and $p + \frac{1}{q}$ is greater, because a part of the denominator of the fraction is omitted; $q + \frac{1}{r}$ is too great for the denominator, therefore $p + \cfrac{1}{q + \cfrac{1}{r}}$ is less than $\frac{a}{b}$; and so on.

It is obvious that, when $\frac{a}{b}$ is a *proper* fraction, $p = 0$, and $c = a$, so that the operation of converting $\frac{a}{b}$ into a continued fraction in this case commences with dividing b by a.

DEF. The quantities p, q, r, &c., which are always positive integers, are called the *Partial Quotients;* and $\frac{p}{1}$, $p+\frac{1}{q}$, $p+\cfrac{1}{q+\cfrac{1}{r}}$, when reduced to simple fractions, are called the *Converging Fractions,* or *Convergents,* to $\frac{a}{b}$.

Ex. To find a fraction which shall be nearly equal to $\frac{314159}{100000}$, and in lower terms.

$$\begin{array}{rrl}
100000) & 314159 & (3 \\
 & \underline{300000} & \\
 & 14159) & 100000\ (7 \\
 & & \underline{99113} \\
 & & 887)\ 14159\ (15 \\
 & & \qquad\ \ \underline{887} \\
 & & \qquad\ \ 5289 \\
 & & \qquad\ \ \underline{4435} \\
 & & \qquad\ \ 854)\ 887\ (1 \\
 & & \qquad\qquad\quad \underline{854} \\
 & & \qquad\qquad\quad\ \ 33\ \&c.
\end{array}$$

Here $p=3$, $q=7$, $r=15$, $s=1$, &c. therefore

$$\frac{314159}{100000}=3+\cfrac{1}{7+\cfrac{1}{15+\&c.}}$$

The first approximation is 3, which is too little, the next is $3+\frac{1}{7}=\frac{22}{7}$, too great; the next is $3+\cfrac{1}{7+\cfrac{1}{15}}=3+\frac{15}{106}=\frac{333}{106}$, too little; and so on.

The proposed fraction expresses nearly the circumference of a circle whose diameter is 1; therefore the circumference is greater than 3 diameters, less than $\frac{22}{7}$ diameters, and greater than $\frac{333}{106}$ diameters, &c.

346. *To convert any continued fraction into a series of converging fractions.*

Let the continued fraction be (Art. 343)

$$p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s+\&c.,}}}$$

then the quotients are p, q, r, s, &c.
and the converging fractions are respectively equal to

$$\frac{p}{1},\quad p+\frac{1}{q},\quad p+\cfrac{1}{q+\cfrac{1}{r}},\quad p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}}},\ \&c.$$

$$\text{or } \frac{p}{1},\quad \frac{pq+1}{q},\quad \frac{pqr+p+r}{qr+1},\quad \frac{pqrs+ps+rs+pq+1}{qrs+q+s},\ \&c.$$

$$\text{or } \frac{p}{1},\quad \frac{pq+1}{q},\quad \frac{(pq+1)r+p}{qr+1},\quad \frac{(\overline{pq+1}.r+p)s+pq+1}{(qr+1)s+q},\ \&c.$$

in which the law of formation is observed to be as follows:—

Write down in one line the quotients p, q, r, s, &c., and the first and second converging fractions at sight; then the other fractions may be obtained thus:—

For the 3rd,

$$\begin{cases}\text{num}^{\text{r}}. = \text{3rd quot.} \times \text{num}^{\text{r}}. \text{ of 2nd fract.} + \text{num}^{\text{r}}. \text{ of 1st fract.}\\ \text{denom}^{\text{r}}. = \text{3rd quot.} \times \text{denom}^{\text{r}}. \text{ of 2nd fract.} + \text{denom}^{\text{r}}. \text{ of 1st fract.}\end{cases}$$

For the 4th,

$$\begin{cases}\text{num}^{\text{r}}. = \text{4th quot.} \times \text{num}^{\text{r}}. \text{ of 3rd fract.} + \text{num}^{\text{r}}. \text{ of 2nd fract.}\\ \text{denom}^{\text{r}}. = \text{4th quot.} \times \text{denom}^{\text{r}}. \text{ of 3rd fract.} + \text{denom}^{\text{r}}. \text{ of 2nd fract.}\end{cases}$$

And generally, for the n^{th} fraction in the series,

Multiply the n^{th} quotient by the numerator of the $\overline{n-1}|^{\text{th}}$ fraction, and add the product to the numerator of the $\overline{n-2}|^{\text{th}}$ fraction. This will give the numerator.

Multiply the n^{th} quotient by the denominator of the $\overline{n-1}|^{\text{th}}$ fraction, and add the product to the denominator of the $\overline{n-2}|^{\text{th}}$ fraction. This will give the denominator.

To prove this generally:—

Let q_1, q_2, q_3, q_4, be *any* 4 successive quotients, and $\frac{N_1}{D_1}$, $\frac{N_2}{D_2}$, $\frac{N_3}{D_3}$, $\frac{N_4}{D_4}$[*], the corresponding convergents; and suppose the law above stated to hold for the 3rd fraction, so that

---

[*] This notation, although at first sight somewhat complex, is very expressive, and greatly relieves the memory throughout the operation, the *first letter* of each word being used, q for quotient, N for *Numerator*, and D for *Denominator*, whilst the small figures indicate the *position* of each with respect to the others here employed. Thus, for instance, the final result in this proof may be as easily read and understood, as if it were written at full length in ordinary language.

$$N_3 = q_3 N_2 + N_1, \text{ and } D_3 = q_3 D_2 + D_1;$$

then, since any convergent is obtained from the preceding one by merely bringing to account another quotient, it is clear that $\frac{N_4}{D_4}$ may be found from $\frac{N_3}{D_3}$ by writing $q_3 + \frac{1}{q_4}$ instead of q_3, that is,

$$\frac{N_4}{D_4} = \frac{\left(q_3 + \frac{1}{q_4}\right)N_2 + N_1}{\left(q_3 + \frac{1}{q_4}\right)D_2 + D_1} = \frac{q_4(q_3 N_2 + N_1) + N_2}{q_4(q_3 D_2 + D_1) + D_2} = \frac{q_4 N_3 + N_2}{q_4 D_3 + D_2}.$$

Now it may be proved that these fractions are both in their lowest terms by the application of the following:—

Lemma. If an integer be added to a fraction in its lowest terms, the result, when reduced to an improper fraction, will also be in its lowest terms.

Let a be the integer, and $\frac{b}{c}$ the fraction; then $a + \frac{b}{c} = \frac{ac+b}{c}$; and if this is not in its lowest terms, its numerator and denominator have a common measure, which will therefore also measure $(ac+b) - ac$ or b, thus making the fraction $\frac{b}{c}$ not in its lowest terms.

It is important here to observe that this reduction to an improper fraction is to be effected by multiplying by the denominator.

Now in the given continued fraction, $\frac{1}{q_4}$ is clearly in its lowest terms: $\therefore$ $q_3 + \frac{1}{q_4}$ is so when reduced, $\therefore$ its reciprocal $\cfrac{1}{q_3 + \cfrac{1}{q_4}}$ is in its lowest terms, and $\therefore$ $q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_4}}$ is so likewise; and so on to any extent. $\therefore$ $\frac{N_4}{D_4}$, $\frac{N_3}{D_3}$, &c. are all in their lowest terms, considered as arising from the continued fraction, and not as derived from each other.

And when we derive $\frac{N_4}{D_4}$ from $\frac{N_3}{D_3}$, as we have written $q_3 + \frac{1}{q_4}$ for q_3, and multiplied by the denominator q_4, we clearly must get the same result, whether we retain the form $q_3 + \frac{1}{q_4}$ until the end of the operation, or perform the multiplication by q_4 first. But under this latter mode of reducing the fraction the result we obtain is in its lowest terms, therefore it is also in its lowest terms in the other case; that is, $\frac{q_4 N_3 + N_2}{q_4 D_3 + D_2}$ is in its lowest terms. Also $\frac{N_4}{D_4}$ has been proved to be in its lowest terms, $\therefore$ the numerators and denominators are separately equal, that is,

$$N_4 = q_4 N_3 + N_2, \quad \text{and} \quad D_4 = q_4 D_3 + D_2,$$

which *proves* that, if the law holds for *any* one convergent, it also holds for the next; and as we know it does hold for each of those near the beginning of the series, it follows that the law holds generally.

Ex. To find a series of converging fractions to $\frac{84}{227}$.

$$\frac{84}{227} = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\frac{1}{2}}}}}}};$$

$\therefore$ the quotients are 0, 2, 1, 2, 2, 1, 3, 2,

and the fractions are $\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{7}{19}, \frac{10}{27}, \frac{37}{100}, \frac{84}{227}$.

347. *To express* $\sqrt{a^2+1}$ *in the form of a Continued Fraction.*

$$\sqrt{a^2+1} = a + \sqrt{a^2+1} - a = a + \frac{1}{\sqrt{a^2+1}+a}, \text{......}(1)$$

$$\therefore \sqrt{a^2+1}+a = 2a + \frac{1}{\sqrt{a^2+1}+a}, \text{............}(2)$$

Substituting from (2) in the denominator of the fraction in (1)

$$\sqrt{a^2+1} = a + \cfrac{1}{2a+\cfrac{1}{\sqrt{a^2+1}+a}}.$$

Substituting again from (2),

$$\sqrt{a^2+1} = a + \cfrac{1}{2a+\cfrac{1}{2a+\cfrac{1}{\sqrt{a^2+1}+a}\text{ ; and so on.}}}\text{ ; and so on.}$$

$$= a + \cfrac{1}{2a+\cfrac{1}{2a+\cfrac{1}{2a+\dots\dots}}}$$

a result which is easily remembered and applied to any proposed case.

Ex. $\sqrt{17}=\sqrt{4^2+1}=4+\cfrac{1}{8+\cfrac{1}{8+\frac{1}{8}+\ldots\ldots}}$

The converging fractions will be equal to $\frac{4}{1}$, $4+\frac{1}{8}$, $4+\cfrac{1}{8+\frac{1}{8}}$, &c.

or $\frac{4}{1}$, $\frac{33}{8}$, $\frac{268}{65}$, &c., each of which is nearer to the true value of $\sqrt{17}$ than the one preceding.

348. *To express* $\sqrt{n}$ *in a continued fraction; and to find the converging fractions.*

Let a be the greatest whole number less than $\sqrt{n}$; then

$$\sqrt{n}=a+\sqrt{n}-a=a+\frac{n-a^2}{\sqrt{n}+a}=a+\cfrac{1}{\cfrac{\sqrt{n}+a}{r}},\ \text{where } r=n-a^2.$$

Let b be the greatest whole number less than $\frac{\sqrt{n}+a}{r}$; then

$$\frac{\sqrt{n}+a}{r}=b+\frac{\sqrt{n}+a-rb}{r}=b+\frac{\sqrt{n}-a'}{r},\ \text{if } a'=rb-a,$$

$$=b+\cfrac{1}{\cfrac{\sqrt{n}+a'}{r'}},\ \text{if } r'=\frac{n-a'^2}{r}.$$

Similarly $\frac{\sqrt{n}+a'}{r'}=b'+\cfrac{1}{\cfrac{\sqrt{n}+a''}{r''}}$, if $a''=r'b'-a'$, and $r''=\frac{n-a''^2}{r'}$;

and so on, until the quantity corresponding to r'' is equal to 1, after which the quotients will *recur*; for then the fraction corresponding to $\frac{\sqrt{n}+a''}{r''}$ becomes $a+a''+\sqrt{n}-a$, giving the same expression to be transformed as the original one. The *quotients* of the continued fraction being thus found, the *converging fractions* will be found by the Rule in Art. 346.

Ex. $\sqrt{11}=3+\sqrt{11}-3=3+\frac{2}{\sqrt{11}+3}=3+\cfrac{1}{\cfrac{\sqrt{11}+3}{2}}$,

$$\frac{\sqrt{11}+3}{2}=3+\frac{\sqrt{11}-3}{2}=3+\frac{2}{2(\sqrt{11}+3)}=3+\frac{1}{3+\sqrt{11}};$$

$$\therefore \sqrt{11} = 3 + \cfrac{1}{3 + \cfrac{1}{3+\sqrt{11}}} = 3 + \cfrac{1}{3 + \cfrac{1}{3+3+\cfrac{1}{3+\cfrac{1}{3+3+\cfrac{1}{3+\&c.}}}}}$$

$$= 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6+\&c.}}}}}}$$

the quotients are 3, 3, 6, 3, 6, &c.

the fractions are $\frac{3}{1}$, $\frac{10}{3}$, $\frac{63}{19}$, $\frac{199}{60}$, $\frac{1257}{379}$, &c.

349. As a test of the correctness of the converging fractions, it may be observed, that *the difference between any two consecutive fractions is always a fraction having* +1 *or* −1 *for its numerator*, according as that which is subtracted from the other is in an odd or even place. For

$$\frac{p}{1} - \frac{pq+1}{q} = \frac{-1}{q}, \quad \frac{pq+1}{q} - \frac{(pq+1)r+p}{qr+1} = \frac{1}{q(qr+1)};$$

$$\frac{(pq+1)r+p}{qr+1} - \frac{(\overline{pq+1}.r+p)s+pq+1}{(qr+1)s+q} = \frac{-1}{(qr+1)(\overline{qr+1}.s+q)}; \text{ and so on.}$$

Thus, in the last example,

$$\frac{3}{1} - \frac{10}{3} = -\frac{1}{3}, \quad \frac{10}{3} - \frac{63}{19} = \frac{1}{57}, \quad \frac{63}{19} - \frac{199}{60} = -\frac{1}{1140}, \text{ \&c.}$$

To prove this generally:

Let $\frac{N_1}{D_1}$, $\frac{N_2}{D_2}$, $\frac{N_3}{D_3}$ be *any* three consecutive convergents,

q_1, q_2, q_3 the corresponding quotients,

and suppose the law above stated to hold for the first two fractions, so that

$$N_1D_2 - N_2D_1 = 1;$$

$$\text{then } N_2D_3 - N_3D_2 = N_2(q_3D_2 + D_1) - (q_3N_2 + N_1)D_2, \text{ (Art. 346)}$$

$$= N_2D_1 - N_1D_2 = -1,$$

which proves that, if the law holds for *any* one fraction and the next preceding one, it also holds for that and the succeeding one. But it has been shewn to hold for two consecutive fractions at the beginning of the series; therefore it holds generally.

350. *To determine the limit of the error in taking any convergent for the true fraction* $\frac{a}{b}$.

Let $\frac{N_1}{D_1}$ and $\frac{N_2}{D_2}$ be any two consecutive convergents, then we know that $N_1D_2 \sim N_2L_1 = 1$, (Art. 349); and of $\frac{N_1}{D_1}$ and $\frac{N_2}{D_2}$ we know that $\frac{a}{b} <$ one, and $>$ the other, that is, lies between them, (Art. 345); therefore the difference between $\frac{a}{b}$ and either of them is less than

$$\frac{N_1}{D_1} \sim \frac{N_2}{D_2} < \frac{N_1D_2 \sim N_2D_1}{D_1D_2} < \frac{1}{D_1D_2}.$$

Or, since $D_2 > D_1$, *à fortiori*, $\frac{a}{b} \sim \frac{N_1}{D_1} < \frac{1}{D_1^2}$.

Thus in Ex. Art. 345, $\frac{22}{7}$ differs from the value of 3·14159 by a quantity less than $\frac{1}{7 \times 106}$, or $\frac{1}{742}$; and $\frac{333}{106}$ differs from the true value by a quantity less than $\frac{1}{(106)^2}$, or $\frac{1}{11236}$.

Cor. 1. *Those convergents, which immediately precede large quotients, approximate especially near to the true value of the continued fraction.*

For, if $\frac{N_1}{D_1}$, $\frac{N_2}{D_2}$, $\frac{N_3}{L_3}$ be three consecutive convergents to $\frac{a}{b}$, corresponding to quotients q_1, q_2, q_3, then

$$\frac{N_1}{D_1},\ \frac{N_3}{D_3},\ \frac{a}{b},\ \frac{N_2}{D_2}$$

are in order of magnitude: and

$$\frac{N_3}{D_3} \sim \frac{N_1}{D_1} = \frac{q_3N_2 + N_1}{q_3D_2 + D_1} \sim \frac{N_1}{D_1} = \frac{q_3(N_2D_1 \sim N_1D_2)}{D_1(q_3D_2 + D_1)} = \frac{q_3}{D_1D_3};$$

$$\text{also } \frac{N_2}{D_2} \sim \frac{N_3}{I_3} = \frac{1}{D_2D_3}.$$

And if q_3 be especially large, since $D_1 < D_2$, $\frac{q_3}{D_1D_3}$ will be much greater than $\frac{1}{D_2D_3}$; $\therefore \frac{N_3}{D_3}$ lies much nearer to $\frac{N_2}{D_2}$ than to $\frac{N_1}{D_1}$, and therefore, *à fortiori*, $\frac{a}{b}$ lies much nearer to $\frac{N_2}{D_2}$ than to $\frac{N_1}{D_1}$.

Thus for 3·14159 the quotients are 3, 7, 15, 1, 243, &c.

and the convergents are $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, &c., of which $\frac{355}{113}$ is especially near to the true value.

∴ **Cor. 2.** ***Any convergent is nearer to the true value than any other fraction with a less denominator.***

For, if possible, let $\frac{n}{d}$ be a fraction nearer to the true value, $\frac{a}{b}$, than its convergent $\frac{N_1}{D_1}$ is, and $d<D_1$. Then since $\frac{n}{d}$ lies between $\frac{N_1}{D_1}$ and $\frac{a}{b}$ and $\frac{a}{b}$ lies between $\frac{N_1}{D_1}$ and $\frac{N_2}{D_2}$, therefore $\frac{n}{d} \sim \frac{N_1}{D_1} < \frac{N_1}{D_1} \sim \frac{N_2}{D_2} < \frac{1}{D_1 D_2} < \frac{1}{D_1^2}$,

$$\therefore nD_1 \sim N_1 d < \frac{d}{D_1},$$

or a whole number < a proper fraction, which is impossible.

[*Exercises Zk.*]

351. *To find the value of a continued fraction, when the quotients* q, r, s, &c. *recur in any certain order.*

Ex. 1. Let $\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{q+\cfrac{1}{r+\text{\&c. in inf.}}}}} = x;$

$$\text{then } \cfrac{1}{q+\cfrac{1}{r+x}} = x, \quad \text{or } \frac{r+x}{qr+qx+1} = x;$$

$$\text{hence } r+x = qrx + qx^2 + x, \quad \text{and } x^2 + rx - \frac{r}{q} = 0,$$

by the solution of which quadratic equation the value of x may be obtained.

Ex. 2. Let $x = \sqrt{a + \cfrac{b}{\sqrt{a + \cfrac{b}{\sqrt{a + \text{\&c.}}}}}}$, *in inf.*

$$\text{squaring both sides, } x^2 = a + \cfrac{b}{\sqrt{a + \cfrac{b}{\sqrt{a+\text{\&c.}}}}} = a + \frac{b}{x};$$

and $x^2 - ax - b = 0$; whence the value of x is to be found.

352. The roots of quadratic equations may sometimes be approximated to by means of continued fractions. Thus,

Let $x^2-px-q=0$; then $x=p+\frac{q}{x}=p+\cfrac{q}{p+\cfrac{q}{x}}$, &c.;

so that $x=p+\cfrac{q}{p+\cfrac{q}{p+\cfrac{q}{p+\&c.}}}$, which, though not a continued fraction of such a form as those to which the preceding theory applies, may yet sometimes be made use of to calculate the value of x.

353. *To find in the form of a continued fraction the value of* x *which satisfies the equation* $a^x=b$.

Substitute for x the numbers 0, 1, 2, 3, &c. until two consecutive numbers are found, n, and $n+1$, such that

$$a^n<b, \text{ and } a^{n+1}>b;$$

then it appears that $x<n+1$, and $>n$; so that $x=n+\frac{1}{y}$, where $y>1$, and $a^{n+\frac{1}{y}}=b$, or $a^n.a^{\frac{1}{y}}=b$, and $a^{\frac{1}{y}}=\frac{b}{a^n}$, or $\left(\frac{b}{a^n}\right)^y=a$. Again, since $\frac{b}{a^n}>1$, and $<a$, by substituting the numbers between 1 and a for y in the last equation, two consecutive numbers, p, and $p+1$, will be found such that $y<p+1$, and $>p$, so that $y=p+\frac{1}{z}$; and $\therefore x=n+\cfrac{1}{p+\cfrac{1}{z}}$; and by continuing the process in the same manner, the fraction expressing the value of x may be continued.

Ex. Required the value of x in $10^x=2$.

By substituting 0 and 1 for x, it appears that $x>0$, and <1; let $x=\frac{1}{y}$, then $10^{\frac{1}{y}}=2$, or $2^y=10$. Again, it appears that $y>3$, and <4; let $y=3+\frac{1}{z}$, then $2^{3+\frac{1}{z}}=2^3.2^{\frac{1}{z}}=10$, or $2^{\frac{1}{z}}=\frac{10}{8}=1{\cdot}25$, $\therefore (1{\cdot}25)^z=2$. Again, it appears that $z>3$, and <4; let $z=3+\frac{1}{u}$, then $(1{\cdot}25)^{3+\frac{1}{u}}=(1{\cdot}25)^3.(1{\cdot}25)^{\frac{1}{u}}=2$, $\therefore (1{\cdot}25)^{\frac{1}{u}}=\frac{2}{(1{\cdot}25)^3}=1{\cdot}024$, $\therefore (1{\cdot}024)^u=1{\cdot}25$, and by trial $u>9$ and <10.

Hence, by successive substitution,

$$x=\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{9+\&c.}}}$$

INDETERMINATE EQUATIONS AND UNLIMITED PROBLEMS.

354. When there are more unknown quantities than independent equations, the number of corresponding values which those quantities admit is indefinite (Art. 198). This number may be lessened by rejecting all the values which are not integers; it may be further lessened by rejecting all the negative values; and still further, by rejecting all values which are not square or cube numbers; &c.

By restrictions of this kind the number of answers may be confined within definite limits; and problems are not wanting, in which such restrictions must be made.

355. Rule. *If a simple equation express the relation of two unknown quantities, and their corresponding* integral *values be required, divide the whole equation by the coefficient which is the less of the two, and suppose that part of the result, which is in a fractional form, equal to some whole number; thus a new simple equation is obtained, with which we may proceed as before; let the operation be repeated, till the coefficient of one of the unknown quantities is* 1, *and the coefficient of the other a whole number; then an integral value of the former may be obtained by substituting* 0, *or any whole number, for the other; and from the preceding equations integral values of the original unknown quantities may be found.*

Ex. 1. Let $5x+7y=29$; to find the corresponding integral values of x and y.

Dividing the whole equation by 5, the less coefficient,

$$x+y+\frac{2y}{5}=5+\frac{4}{5},$$

$$\text{or}\quad x=5-y+\frac{4-2y}{5},\ \text{a whole number};$$

$$\therefore\ \frac{4-2y}{5}\ \text{is a whole number.}$$

Assume $\dfrac{4-2y}{5}=p,$ or $4-2y=5p,$

then $2-y=2p+\dfrac{p}{2};$

$$\therefore\ y = 2 - 2p - \frac{p}{2}, \text{ a whole number.}$$

And $\frac{p}{2}$ is a whole number.

Let $\frac{p}{2} = s$, or $p = 2s$, then $y = 2 - 5s$,

and $x = 5 - y + p = 3 + 5s + 2s = 3 + 7s$*.

If $s = 0$, then $x = 3$, and $y = 2$, the only *positive* whole numbers which answer the conditions of the equation; for, if x and y are positive integers, $5s$ cannot be greater than 2, that is, s cannot be greater than $\frac{2}{5}$, and s cannot be negative, for then x would be negative.

If negative values of x and y are not excluded, then an indefinite number of such solutions may be found by putting 1, 2, 3, &c. -1, -2, -3, &c. for s.

Ex. 2. To find a number which being divided by 3, 4, 5, gives the remainders 2, 3, 4, respectively.

Let x be the number,

then $\frac{x-2}{3} = p$, a whole number, or $x = 3p + 2$;

also, from the second condition,

$$\frac{x-3}{4}, \text{ or } \frac{3p-1}{4} = q, \text{ a whole number,}$$

$$\text{that is, } 3p - 1 = 4q, \text{ or } p = q + \frac{q+1}{3};$$

$$\text{let } \frac{q+1}{3} = r, \text{ or } q = 3r - 1, \text{ then } p = 4r - 1,$$

$$\text{and } x = 3p + 2 = 12r - 1;$$

again, from the third condition, $\frac{x-4}{5}$, or $\frac{12r-5}{5}$, is a whole number, that is, $2r + \frac{2r}{5} - 1$ is a whole number;

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* This operation might have been abridged thus,

$$x = 5 - y + \frac{2(2-y)}{5};\ \text{let } \frac{2-y}{5} = s, \text{ or } y = 2 - 5s,$$

then $x = 5 - y + 2s = 3 + 7s$.—Ed.

$$\therefore \frac{2r}{5} \text{ is a whole number;}$$

$$\text{let } \frac{2r}{5} = 2m; \text{ then } r = 5m,$$

$$\text{and } x = 12r - 1 = 60m - 1.$$

$$\text{If } m = 1,\ x = 59; \text{ if } m = 2,\ x = 119; \text{ \&c.}$$

The artifices employed in the two following examples are deserving of notice.

Ex. 3. Let $11x - 17y = 5$; to find the integral values of x and y.

Here $x = \frac{17y+5}{11} = \frac{22y-5y+5}{11} = 2y - \frac{5(y-1)}{11} =$ a whole number.

$$\text{Let } \frac{y-1}{11} = p, \text{ or } y - 1 = 11p; \ \therefore y = 11p + 1.$$

$$\text{And } x = 2y - 5p = 22p + 2 - 5p = 17p + 2.$$

If $p = 0$, $x = 2$, and $y = 1$; if $p = 1$, $x = 19$, and $y = 12$; &c.

Ex. 4. Let $11x - 18y = 63$; to find the integral values of x and y.

Since 18 and 63 are divisible by 9, let $x = 9z$, then

$11 \times 9z - 18y = 63$; and dividing by 9,

$$11z - 2y = 7;$$

$$\therefore y = 5z - 3 + \frac{z-1}{2}.$$

Let $\frac{z-1}{2} = p$, or $z - 1 = 2p$; $\therefore z = 2p + 1$.

Hence $x = 18p + 9$; and $y = 10p + 5 - 3 + p = 11p + 2$;

and by giving to p the values 0, 1, 2, 3, &c. the integral values of x and y are determined.

356. If the simple equation contain more unknown quantities, their corresponding integral values may be found in the same manner.

Ex. Let $4x + 3y + 10 = 5z$; to find corresponding integral values of x, y, and z.

Dividing the whole equation by 3, the least coefficient,

$$x + y + 3 + \frac{x+1}{3} = z + \frac{2z}{3},$$

$$y = z - x - 3 + \frac{2z - x - 1}{3}, \text{ a whole number.}$$

Assume $\frac{2z-x-1}{3}=p$, or $2z-x-1=3p$,

then $x=2z-3p-1$,

and $y=z-2z+3p+1-3+p=4p-z-2$;

and for p and z substituting 0, or any whole numbers, integral values of x and y are obtained. If $z=3$, and $p=1$, then $x=2$, and $y=-1$; if $z=4$, and $p=0$, then $x=7$, and $y=-6$; &c.

357. If the number of independent equations be less by one than the number of unknown quantities, the equations may be reduced by elimination to one equation only betwixt two unknown quantities, and then the preceding method of solution may be applied.

Ex. Let $\left.\begin{array}{l}2x+5y+3z=108,\\ \text{and } 3x-2y+7z=95,\end{array}\right\}$ to find the integral values of x, y, z.

From 1st equ^n $\left.\begin{array}{l}6x+15y+9z=324,\\ 6x-4y+14z=190,\end{array}\right\}$
... 2nd ...

$\therefore\ 19y-5z=134$;

from which equation the values of y and z may be found by the usual method, and then the values of x may be deduced from either of the original equations.

[*Exercises Zl.*]

A more *complete* Theory of Indeterminate Equations, distinct from the preceding, is contained in the following Articles of this Section:—

358. *To shew that, if* a *is prime to* b, *the equation* ax+by = c *has one integral solution at least*[*].

$$\text{Since } ax=c-by,\ \therefore\ x=\frac{c-by}{a};$$

now give to y the several successive values 0, 1, 2, 3, ... $\overline{a-1}$, and since a is prime to b, the several values of $c-by$, divided by a, will all leave *different remainders.* [For, if not, let y_1 and y_2 be two of the values of y, which make $c-by$, divided by a, if possible, to leave the same remainder r, q_1 and q_2 being the quotients, then

$$c-by_1=aq_1+r, \text{ and } c-by_2=aq_2+r,$$
$$\therefore\ b(y_1-y_2)=a(q_2-q_1),$$

or $b(y_1-y_2)$ is divisible by a without remainder; but b is prime to a, therefore y_1-y_2 must be divisible by a, which is impossible, since $y_1-y_2<a$].

Hence the *remainders* being all different, since the number of them is a, and each one necessarily less than a, it follows that one of them must be 0, that is, x is a whole number for a certain integral value of y less than a, and these values of x and y satisfy the equation $ax+by=c$.

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[*] Not necessarily a solution in *positive* integers: for it is plainly impossible for x and y to be *both positive* integers, whenever $a+b>c$. The integral quotients q_1, q_2, &c., mentioned in the proof, may be *negative* as well as positive.

Cor. It is obvious, that not only is an integral solution proved possible, but also we are directed to a most simple way of obtaining it, in all cases, where the coefficient of either x or y is *small.*

Ex. 1. $4x+29y=150$, find x and y.

Here $x=\frac{150-29y}{4}=$ a whole number for some value of y between 0 and 3 inclusive. Try 0, 1, 2, 3, successively, and we find 2 will answer; in which case

$$\frac{150-29y}{4}=\frac{92}{4}=23; \quad \therefore x=23, \text{ and } y=2.$$

Ex. 2. $17x=7y+1$, find x and y.

$$y=\frac{17x-1}{7}=\text{a whole number for some value of } x \text{ between } 0$$

and 6 inclusive.

By trial $x=5$, and then $y=\frac{84}{7}=12$.

359. *To find all the integral values of* x *and* y *which satisfy an equation of the form* $ax \pm by=c$.

Suppose a and b prime to each other; for, if they are not, (since a, b, x, and y, are all integers by supposition, and every common measure of a and b measures ax and by, and therefore $ax \pm by$, or c,) a, b, and c, must have a common measure, and dividing by this common measure, the equation is reduced to one in which the coefficients of x and y *are* prime to each other. Hence it is only necessary to consider the case when a and b are prime.

Let $x=\alpha$, $y=\beta$, be one solution of the equation $ax+by=c$; then

$$a\alpha+b\beta=c=ax+by,$$

$$\therefore a(\alpha-x)=b(y-\beta),$$

$$\text{and } \frac{a}{b}=\frac{y-\beta}{\alpha-x};$$

but $\frac{a}{b}$ is in its lowest terms, since a is prime to b, $\therefore y-\beta$, and $\alpha-x$, are certain *equimultiples* of a and b, (Art. 258, Cor.), or $y-\beta=at$, and $\alpha-x=bt$, where t is integral and remains undetermined;

$$\left.\begin{aligned}\therefore x&=\alpha-bt,\\ \text{and } y&=\beta+at;\end{aligned}\right\}$$

which values of x and y are found to satisfy the original equation.

Hence, if we know *one* solution, $x=\alpha$, $y=\beta$, (which may often be guessed at sight, or found by Arts. 355, 358,) *all* the required solutions are found by giving different *integral* values, positive or negative, to t in the equations

$$\left.\begin{aligned}x&=\alpha-bt,\\ y&=\beta+at.\end{aligned}\right\}$$

Similarly, if the proposed equation be $ax-by=c$, it may be shewn, (or deduced from the former by writing $-b$ for b,) that the general solution is

$$\left.\begin{aligned}x&=\alpha+bt,\\ y&=\beta+at.\end{aligned}\right\}$$

Ex. **If $11x-17y=5$, find the integral values of x and y.**

Here it is easily *seen* that $x=2$, $y=1$, is *one* solution; therefore the general solution is $x=2+17t$, $y=1+11t$, from which all the others may be obtained, by giving integral values at pleasure to t.

360. *Another Method.* Find the series of fractions converging to $\frac{a}{b}$, and let $\frac{p}{q}$ be the convergent immediately preceding $\frac{a}{b}$, then, by Art. 349, since $\frac{p}{q}$ and $\frac{a}{b}$ are contiguous convergents,

$aq-bp=\pm 1$, (+ or − according as $\frac{a}{b}>$ or $<\frac{p}{q}$, that is, according as $\frac{p}{q}$ occupies an even or an odd place in the series of convergents;)

$$\therefore \text{ either } a.cq+b.(-cp)=c, \qquad \text{or } a.(-cq)+b.cp=c,$$
$$a(cq-bt)+b(at-cp)=c, \text{ or } a(-cq-bt)+b(cp+at)=c.$$

But $ax+by=c$, ∴, according as $\frac{a}{b}>$ or $<\frac{p}{q}$,

$$\left.\begin{aligned} x&=cq-bt,\\ y&=at-cp,\end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} x&=-cq-bt,\\ y&=cp+at.\end{aligned}\right\}$$

Similarly, if the proposed equation be $ax-by=c$, the general solution will be, according as $\frac{a}{b}>$ or $<\frac{p}{q}$,

$$\left.\begin{aligned} x&=cq+bt,\\ y&=cp+at,\end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} x&=-cq+bt,\\ y&=-cp+at.\end{aligned}\right\}$$

N.B. Of course, in any proposed case, where we can *see* at once the values of x and y which will make $ax-by=1$, or -1, there will be no need to form the series of convergents.

Ex. 1. $21x+40y=4000$.

The convergents are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{19}{10}$, $\frac{40}{21}$,

$$\therefore 21\times 19-40\times 10=-1,$$
$$21\times 19\times 4000-40\times 10\times 4000=-4000,$$
$$21(-76000-40t)+40(40000+21t)=4000,$$
$$\therefore x=-76000-40t, \text{ and } y=40000+21t.$$

Ex. 2. $5x+7y=29$.

Here we can *see* that $x=3$, $y=2$, will satisfy the equation $5x-7y=1$, that is, $5\times 3-7\times 2=1$,

$$\therefore 5\times 3\times 29+7\times(-2\times 29)=29,$$
$$\text{or } 5\times 87+7(-58)=29,$$
$$\therefore 5(87-7t)+7(-58+5t)=29,$$
$$\therefore x=87-7t, \text{ and } y=-58+5t.$$

COR. 1. If only *positive* integral values of x and y are required, t will be restricted within certain limits dependent upon the particular Example. Thus, in Ex. 2 above, if x and y must both be *positive*, $7t<87$, and $5t>58$, that is, $t<12\frac{3}{7}$, and $>11\frac{3}{5}$, $\therefore$ t can only be 12; and the only *positive* solution is $x=3$, $y=2$.

COR. 2. Since $x=\alpha-bt$, and $y=\beta+at$, is the general solution from which all the values of x and y are found by substituting for t, 0, ± 1, ± 2, ± 3, &c., it follows that the values of x and y, *taken in order*, form two arithmetic progressions, the one decreasing, and the other increasing; so that when *two* contiguous values of each are obtained, the rest may be *assumed*.

N.B. If an equation of the form $ax+by=c$, or $ax-by=c$, be proposed for solution, in which (when reduced as low as possible) a *and* b *are not prime to each other*, it may at once be affirmed that it admits of no solution in whole numbers. For, if x and y be whole numbers, dividing by the common measure of a and b, which is not a measure of c by supposition, we have a whole number equal to a fraction in lowest terms, which is impossible.

361. *To shew that the number of* positive integral *solutions is limited for* ax+by = c, *but unlimited for* ax−by = c, a *being prime to* b.

1st. It has been shewn (Art. 359) that *all* the solutions of $ax+by=c$ are deducible from

$$x=\alpha-bt, \quad y=\beta+at,$$

where $x=\alpha$, $y=\beta$, is *one* solution, and t any integer, positive or negative. But, if x and y must both be *positive*, then it is plain that bt must be less than α, that is, the *positive* values of t are restricted to those which are less than $\frac{\alpha}{b}$. Again, *every* negative value of t makes $\alpha-bt$, or x, positive; but for *negative* values of t, y will be positive only so long as t, irrespective of sign, is less than $\frac{\beta}{a}$. Hence both positive and negative values of t are limited, that is, the number of *positive* solutions is limited.

2nd. If $ax-by=c$, the *general* solution is $x=\alpha+bt$, $y=\beta+at$; and whatever α, and β, may be, positive or negative, it is plain, that there are an infinite number of positive values for t, which will make both x and y positive. Hence the number of *positive* solutions is unlimited.

362. *To find the* number *of solutions* in positive integers *of the equation* ax+by = c, *where* a *is prime to* b.

Let the series of fractions converging to $\frac{a}{b}$ be found, and let $\frac{p}{q}$ be that which immediately precedes $\frac{a}{b}$, then (Art. 349) either $aq-bp=1$, or $aq-bp=-1$, according as $\frac{a}{b}>$ or $<\frac{p}{q}$.

I. If $\frac{a}{b}>\frac{p}{q}$, then $a.cq-b.cp=c$,

$$\therefore\ a(cq-bt)+b(at-cp)=c\,;$$

and comparing this with the original equation, $ax+by=c$,

$$x=cq-bt, \text{ and } y=at-cp\,;$$

and the several solutions will be found by giving t such values, that $cq-bt$ and $at-cp$ may be positive integral quantities, that is, t may be any *positive integer less than* $\frac{cq}{b}$ and *greater than* $\frac{cp}{a}$, but no other number, (*supposing* 0 *to be excluded as a value of either* x *or* y).

Hence the number of solutions will be the number of integers which lie *between* $\frac{cq}{b}$ and $\frac{cp}{a}$.

II. If $\frac{a}{b}<\frac{p}{q}$, then $aq-bp=-1$, and it may be shewn, as before, that the number of solutions is the number of integers which lie *between* $\frac{cp}{a}$ and $\frac{cq}{b}$.

COR. 1. If $\frac{cp}{a}$, and $\frac{cq}{b}$ are both whole numbers, the number of solutions will be $\frac{cp}{a}\sim\frac{cq}{b}-1$*.

If either $\frac{cp}{a}$, or $\frac{cq}{b}$, be a whole number, and the other fractional, then the number of solutions will be the greatest integer contained in $\frac{cp}{a}\sim\frac{cq}{b}$†.

If $\frac{cp}{a}$ and $\frac{cq}{b}$ be *both* fractional, reducing each to a mixed number, the number of solutions will be the greatest integer contained in $\frac{cp}{a}\sim\frac{cq}{b}$‡, or $\frac{cp}{a}\sim\frac{cq}{b}+1$§, according as the greater of the two quantities, $\frac{cp}{a}$, $\frac{cq}{b}$, has the greater or smaller fractional excess.

COR. 2. Since $\frac{cp}{a}\sim\frac{cq}{b}=c.\frac{bp\sim aq}{ab}=\frac{c}{ab}$, (Art. 349), the greatest integer contained in $\frac{c}{ab}$ cannot differ by more than 1 from the number of solutions; so that the number of solutions may be determined within this error (which may be either in excess or defect) without solving the equation at all. And hence also, if $c<ab$, there cannot be more than *one* solution in *positive* integers, and there may be none.

OBS. In any case where $a+b>c$ it is impossible to have a *positive* solution; for $x=1$, $y=1$, are the *smallest* positive values which x and y admit of, and for these $ax+by>c$, therefore *à fortiori* for any other positive values $ax+by>c$.

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* Ex. $7x+2y=42$.

† Ex. $21x+5y=400$.

‡ Ex. $9x+7y=110$.

§ Ex. $3x+5y=26$.

Ex. In how many different ways may £1000. be paid in crowns *and* guineas?

Let x be the number of guineas, and y the number of crowns, then, reducing all to shillings,

$$21x+5y=20000\,;$$

and it is required to find how many solutions this equation admits of in positive integers.

$$\text{The fractions are } \frac{4}{1},\ \frac{21}{5}\,;\ \therefore\ p=4,\ q=1\,;$$

$$\text{hence } \frac{cq}{b}\sim\frac{cp}{a}=\frac{20000}{5}\sim\frac{20000\times 4}{21},$$

$$=4000-3809\tfrac{11}{21}=190\tfrac{10}{21}\,;$$

and since one of the quantities $\frac{cp}{a}$, $\frac{cq}{b}$, is a whole number,

$\therefore$ the number of solutions required is 190.

Or, if $x=0$ be admitted as a solution, that is, if the payment may be made in crowns *only*, then the number required is 191.

363. (*Another Method*[*]). *To find the number of the positive integral solutions of* ax + by = c.

Let $x=\alpha$, $y=\beta$, be one of them, then all the others are found from $x=\alpha-bt$, $y=\beta+at$, subject to the condition that

$$t<\frac{\alpha}{b}\ \text{and} > -\frac{\beta}{a}\,;$$

and there will be as many *positive* solutions as there are integral values of t between these limits, (reckoning $t=0$ for one of them to include the solution $x=\alpha$, $y=\beta$, which is *positive* by the supposition).

$$\text{But } a\alpha+b\beta=c,\quad \therefore\ \frac{\alpha}{b}=\frac{c}{ab}-\frac{\beta}{a},$$

$$\therefore\ t<\frac{c}{ab}-\frac{\beta}{a},\ \text{and} > -\frac{\beta}{a}.$$

$$\text{Also}\quad c>b\beta,\ \because a\alpha \text{ is } \textit{positive},\ \therefore\ \frac{c}{ab}>\frac{\beta}{a}.$$

Hence the number of *negative* values of t will be the greatest integer less than $\frac{\beta}{a}$, and the number of *positive* values the greatest integer less than $\frac{c}{ab}-\frac{\beta}{a}$, that is, the *total* number will be (excluding *zero* values of x and y) the greatest integer either in $\frac{c}{ab}$, or $\frac{c}{ab}-1$, or $\frac{c}{ab}+1$, according to the particular forms of $\frac{c}{ab}$ and $\frac{\beta}{a}$.

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[*] This method may be employed when *one* solution is either given, or easily discovered by trial.

1st. If $\frac{c}{ab}$ and $\frac{\beta}{a}$ be *both* whole numbers, let $\frac{c}{ab}=m$, and $\frac{\beta}{a}=n$; then the number of solutions for *negative* values of t will be $n-1$; and since $\frac{c}{ab}-\frac{\beta}{a}=m-n$, the number for *positive* values of t will be $m-n-1$. Adding 1 for $t=0$, the total number of solutions is $n-1+(m-n-1)+1$, or $m-1$, that is, $\frac{c}{ab}-1$[*].

2nd. If one be a whole number, and the other fractional, let $\frac{c}{ab}=m$, and $\frac{\beta}{a}=n+f$, f being a proper fraction; then the solutions for *negative* values of t will be n; and since $\frac{c}{ab}-\frac{\beta}{a}=m-n-f$, or $m-n-1+(1-f)$, the solutions for *positive* values of t will be $m-n-1$. Adding 1 for $t=0$, the total number of solutions is $n+(m-n-1)+1$, or m, that is, $\frac{c}{ab}$[†].

Or suppose $\frac{c}{ab}=m+\phi$, ϕ being a proper fraction, and $\frac{\beta}{a}=n$; then the solutions for *negative* values of t will be $n-1$; and since $\frac{c}{ab}-\frac{\beta}{a}=m-n+\phi$, the solutions for *positive* values will be $m-n$. Adding 1 for $t=0$, the total number of solutions will be $n+(m-n)$, or m, that is, $\frac{c}{ab}$[†].

3rd. If both $\frac{c}{ab}$ and $\frac{\beta}{a}$ be fractional, let $\frac{c}{ab}=m+\phi$, and $\frac{\beta}{a}=n+f$; when $\phi>f$, the *negative* values of t give n solutions; and since

$$\frac{c}{ab}-\frac{\beta}{a}=m-n+(\phi-f),$$

the *positive* give $m-n$ solutions. Adding 1 for $t=0$, the total number of solutions is $n+(m-n)+1$, or $m+1$, that is, the greatest integer in $\frac{c}{ab}+1$[‡].

But if ϕ is *not* greater than f, since $\frac{c}{ab}-\frac{\beta}{a}=m-n-1+(1-\overline{f-\phi})$, the *positive* values of t will give only $m-n-1$ solutions, and the result is m, that is, the greatest integer in $\frac{c}{ab}$[§].

Ex. Given $x=4$, $y=9$, one solution of $2x+3y=35$, find the total number of solutions in positive integers.

Here $\frac{c}{ab}=\frac{35}{6}=5\frac{5}{6}$, and $\frac{\beta}{a}=\frac{9}{2}=4\frac{1}{2}=4\frac{3}{6}$, both fractional, and the fractional excess in $\frac{c}{ab} >$ that in $\frac{\beta}{a}$, therefore the number of solutions required is the greatest integer in $\frac{c}{ab}+1$, viz. 6.

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[*] Ex. $3x+4y=24$.

[†] Ex. $3x+4y=39$.

[‡] Ex. $2x+7y=125$.

[§] Ex. $11x+7y=108$.

Or thus, with less burden on the memory:—Since $x=4$, $y=9$, is one solution, $x=4-3t$, and $y=9+2t$, (Art. 359) is the general solution; and in order that both x and y may be *positive* integers, t will be restricted, so as to be less than $\frac{4}{3}$ or $1\frac{1}{3}$, and greater than $\frac{-9}{2}$ or $-4\frac{1}{2}$. Hence the values of t will be $-4, -3, -2, -1, 0, 1$, or the number of positive solutions will be 6, as before.

This is the method most commonly used.

364. *To find the number of solutions in positive integers of the equation* ax+by+cz=d, *each term being positive**.

Transposing one of the terms, cz, the equation becomes

$$ax+by=d-cz,$$

where z may have any integral value greater than 0 and less than $\frac{d}{c}$; *zero* values of the unknown quantities being supposed to be excluded.

Hence, by Art. 362, the number of solutions will be the number of integers which lie between $\frac{(d-cz)p}{a}$, and $\frac{(d-cz)q}{b}$, where p and q are those quantities which satisfy the equation $aq \sim bp=1$.

Thus a certain number of solutions is determined for each value of z, and the sum of these numbers will be the number required.

Obs. The application of this rule is attended with considerable difficulty (see *Barlow's Theory of Numbers*, Art. 162, or *Peacock's Algebra*, Art. 506); but the following Example will shew the student how he may determine the number of solutions in some cases without much trouble.

Ex. Required the number of positive integral solutions of

$$17x+19y+21z=400.$$

Divide by 17, $$x+y+z=23-\frac{2y+4z-9}{17};$$

assume $$2y+4z-9=17t,$$

then $$y+2z=4+8t+\frac{t+1}{2};$$

assume $t+1=2s$, then $t=2s-1$, an odd number.

$$\therefore\ x+y+z=23-t, \text{ an even number.}$$

But $17(x+y+z)<400$, $\therefore\ x+y+z<\frac{400}{17}<23\frac{9}{17}$,

---

* If any term be negative, as by, then the equation being of the form $ax-by=C$, the number of solutions is at once declared infinite. (Art. 361.)

$$\text{and}\quad 21(x+y+z) > 400, \quad \therefore\ x+y+z > \frac{400}{21} > 19\tfrac{1}{21};$$

hence, since $x+y+z$ must be even, it can only be 22, and 20; and $\therefore$ t can only be 1, and 3.

I. If $t=1$, $y+2z=13$, $\therefore\ y=13-2z$.

Also $\overline{x+y+z}+z=13+x$, or $22+z=13+x$, $\therefore\ x=z+9$.

Now z may have the values 1, 2, 3, 4, 5, 6, but no more, each of which makes both x and y positive, giving 6 solutions.

II. If $t=3$, $y+2z=30$, $\therefore\ y=30-2z$, shewing that z cannot be greater than 14. Also $\overline{x+y+z}+z=30+x$, or $20+z=30+x$, $\therefore\ z=x+10$. Hence x cannot be greater than 4, but may be 1, 2, 3, 4, each of which makes y and z positive, giving 4 solutions.

$\therefore$ total number of solutions $=6+4=10$.

365. The theory for the solution of indeterminate equations of more than one *dimension* is too difficult to be admitted into an elementary work, like the present. The reader is referred for further information to *Barlow's Theory of Numbers*, Chaps. III. and IV. But there are two classes of such equations, which admit of easy solution: 1st. Such as do not involve the square of *either* of the unknown quantities; and 2nd. Such as involve the square of *one* of them only.

Ex. 1. Required the positive integral solutions of $3xy-4y+3x=14$.

$$\text{Here } 3x(y+1)=14+4y;$$

$$\therefore\ 3x=\frac{14+4y}{y+1}=\frac{4(y+1)}{y+1}+\frac{10}{y+1},$$

$$\text{or } 3x=4+\frac{10}{y+1};$$

hence $y+1$ must be either 10, or a divisor of 10; that is, it must be either 1, 2, 5, or 10; and therefore the values of y can be only 0, or 1, or 4, or 9; of which the first and last make x fractional.

$$\left.\begin{array}{l}\therefore\ x=3,\ 2\\ \quad\ y=1,\ 4\end{array}\right\}\text{ are the solutions required.}$$

Ex. 2. Required the positive integral solutions of $2xy-3x^2+y=1$.

$$\text{Here } y=\frac{3x^2+1}{2x+1}=\frac{3x}{2}-\frac{3}{4}+\frac{7}{4(2x+1)},\ \text{by division};$$

$$\therefore\ 4y=6x-3+\frac{7}{2x+1},\ \text{a whole number.}$$

Hence, 7 being a prime number, $2x+1$ can be no other number but 7, or 1;

$$\therefore\ 2x=6,\ \text{ or } x=3,\ \text{ and } \therefore\ y=4;$$

$$\text{or } 2x=0,\ \text{ i.e. } x=0,\ \text{ and } \therefore\ y=1;$$

which are the only solutions in positive integers.

[*Exercises Zm.*]

366. In the solution of different kinds of unlimited problems different expedients must be made use of, which expedients, and their application, are chiefly to be learned by practice.

Ex. 1. To find a "perfect number", that is, one which is equal to the sum of all the numbers which divide it without remainder.

Suppose y^nx to be a "perfect number"; its divisors are

$$1,\ y,\ y^2 \ldots y^n,\ x,\ xy,\ xy^2 \ldots xy^{n-1};$$

$$\therefore\ y^nx = 1 + y + y^2 \ldots\ldots + y^n + x + xy + xy^2 \ldots\ldots + xy^{n-1}.$$

Now $$1 + y + y^2 \ldots\ldots + y^n = \frac{y^{n+1} - 1}{y - 1},$$

and $$x + xy + xy^2 \ldots\ldots + xy^{n-1} = \frac{y^n - 1}{y - 1} \times x;$$

$$\therefore\ y^nx = \frac{y^{n+1} - 1 + (y^n - 1) \times x}{y - 1},$$

or $$y^{n+1}x - y^nx = y^{n+1} - 1 + y^nx - x;$$

$$\therefore\ x = \frac{y^{n+1} - 1}{y^{n+1} - 2y^n + 1};$$

and, that x may be a whole number, let $y^{n+1} - 2y^n = 0$, or $y - 2 = 0$, that is, $y = 2$; then $x = 2^{n+1} - 1$.

Also, let n be so assumed that $2^{n+1} - 1$ may have no divisor but unity, which was supposed in taking the divisors of y^nx; then y^nx, or $2^n \times (2^{n+1} - 1)$ is a "perfect number". Thus, if $n = 1$, the number is 2×3, or 6, which is equal to $1 + 2 + 3$, the sum of its divisors. If $n = 2$, the number is $2^2 \times (2^3 - 1) = 4 \times 7 = 28$.

Ex. 2. To find two square numbers, whose sum is a square.

Let x^2 and y^2 be the two square numbers;

assume $$x^2 + y^2 = (nx - y)^2 = n^2x^2 - 2nxy + y^2,$$

then $$x^2 = n^2x^2 - 2nxy,$$

$$x = n^2x - 2ny;$$

hence $$(n^2 - 1)x = 2ny,$$

or $$x = \frac{2ny}{n^2 - 1}.$$

And if n and y be assumed at pleasure, such a value of x is obtained, that $x^2 + y^2$ is a square number.

But if it be required to find *integers* of this description, let $y = n^2 - 1$, then $x = 2n$, and n being taken at pleasure, integral values of x and y, and consequently of x^2 and y^2, will be found. Thus, if $n=2$, then $y=3$, and $x=4$, and the two squares are 9 and 16, whose sum is 25, a square number.

Ex. 3. To find two square numbers, whose difference is a square.

Let x^2 and y^2 be the two squares;

$$\text{assume } x^2-y^2=(x-ny)^2,$$
$$=x^2-2nxy+n^2y^2.$$
$$\text{Then } y^2=2nxy-n^2y^2,$$
$$\text{or } 2nx=(n^2+1)y;$$
$$\therefore\ x=\frac{n^2+1}{2n}.y.$$

And if $y=2n$, then $x=n^2+1$. Thus, if $n=2$, then $y=4$, and $x=5$; hence $x^2-y^2=25-16=9$.

SCALES OF NOTATION.

367. *To explain the different systems of notation.*

DEF. In the common system of notation each figure of any number* increases its value in a tenfold proportion in proceeding from right to left. Thus 3256 may be expressed by

$$6+50+200+3000,$$
$$\text{or } 6+5\times10+2\times10^2+3\times10^3.$$

The figures 3, 2, 5, 6, by which the number is formed, are called its *digits*, and the number 10, according to whose powers their values proceed, is called the *radix* of the scale.

It is purely conventional that 10 should be the *radix;* and therefore there may be any number of different *scales*, each of which has its own *radix*. When the *radix* is 2, the *scale* is called *Binary;* when 3, *Ternary;* when 10, *Denary*, or *Decimal;* when 12, *Duodenary*, or *Duodecimal;* &c.

If 3256 expressed a number in a scale whose radix is 7, that number might be expressed thus,

$$6+5\times7+2\times7^2+3\times7^3.$$

And generally, if the digits of a number be a_0, a_1, a_2, a_3, &c., reckoning from right to left, and the radix r, the number will be properly represented by

$$a_0+a_1r+a_2r^2+a_3r^3+\&c.$$

Or, if there be n digits, the number will be (reversing the order of the terms)

$$a_{n-1}r^{n-1}+a_{n-2}r^{n-2}+a_{n-3}r^{n-3}+\ldots+a_1r+a_0.$$

---

* In this Section and in the following one by *number* a *whole number* is always meant.

Obs. In any scale of notation every digit is necessarily less than r, and the number of them, including 0, is equal to r. Also in any number the highest power of r is less by 1 than the number of digits.

368. *To express a given number in any proposed scale.*

Let N be the number, and r the radix of the proposed scale.

Then if a_0, a_1, a_2, &c. be the unknown digits,

$$N = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \&c.;$$

and if N be divided by r, the remainder is a_0.

If the quotient be again divided by r, the remainder is a_1.

If .... is a_2;

and so on, until there is no further quotient.

Therefore all the digits a_0, a_1, a_2, a_3, &c. are found by these repeated divisions, and consequently the number in the proposed scale.

Ex. To express 1820, written according to the denary scale, in a scale whose radix is 6.

$$\begin{array}{r|rl}
6 & 1820 & \\
\hline
6 & 303,\ 2 & \therefore \text{ 1st rem}^{\text{r}}.\ a_0 = 2, \\
\hline
6 & 50,\ 3 & \text{2d rem}^{\text{r}}.\ a_1 = 3, \\
\hline
6 & 8,\ 2 & \text{3d rem}^{\text{r}}.\ a_2 = 2, \\
\hline
6 & 1,\ 2 & \text{4th rem}^{\text{r}}.\ a_3 = 2, \\
\hline
 & 0,\ 1 & \text{5th rem}^{\text{r}}.\ a_4 = 1;
\end{array}$$

∴ the number required is 12232.

This may easily be verified. Thus if the result be correct,

$$2 + 3 \times 6 + 2 \times 6^2 + 2 \times 6^3 + 1 \times 6^4$$

must amount to 1820; which upon trial it is found to do.

By the same method a number may be transformed from *any* given scale to any other of which the radix is given. It is only necessary to bear in mind throughout the process that the radix is not 10, as usual, but some other number. Or the same thing may be done by first expressing the given number in the *denary* scale, and then proceeding as in the last example.

Ex. 1. Transform 12232 from a scale whose radix is 6 to a scale whose radix is 4.

(Observe, 1st, that in proceeding here to find how often 4 is contained in 12, 12 does not mean *twelve*, but $1 \times 6 + 2$, or 8. So also 23 is *fifteen*, 32 is *twenty*, and so on.)

$$\begin{array}{r|r l}
4 & 12232 & \\
\hline
4 & 2035,\ 0 & \therefore\ \text{1st rem}^{\text{r}}.\ a_0 = 0, \\
\hline
4 & 305,\ 3 & \text{2d rem}^{\text{r}}.\ a_1 = 3, \\
\hline
4 & 44,\ 1 & \text{3d rem}^{\text{r}}.\ a_2 = 1, \\
\hline
4 & 11,\ 0 & \text{4th rem}^{\text{r}}.\ a_3 = 0, \\
\hline
4 & 1,\ 3 & \text{5th rem}^{\text{r}}.\ a_4 = 3, \\
\hline
 & 0,\ 1 & \text{6th rem}^{\text{r}}.\ a_5 = 1,
\end{array}$$

∴ the number required is 130130.

This number expressed in the *denary* scale is

$$1\times4^5+3\times4^4+0\times4^3+1\times4^2+3\times4+0, \text{ or } 1820,$$

which proves the result correct according to the preceding example.

Ex. 2. Transform 3256 from a scale whose radix is 7 to a scale whose radix is twelve.

Bearing in mind that the digits in 3256 increase from right to left in a *sevenfold* proportion, the division by twelve will be performed thus,

$$\begin{array}{r|r l}
\text{twelve} & 3256 & \\
\hline
\text{twelve} & 166,\ 4 & \therefore\ \text{1st rem}^{\text{r}}.\ a_0 = 4, \\
\hline
\text{twelve} & 11,\ 1 & \text{2d rem.}^{\text{r}}\ a_1 = 1, \\
\hline
 & 0,\ 8 & \text{3d rem}^{\text{r}}.\ a_2 = 8;
\end{array}$$

∴ the number required is 814.

369. The most useful scale of notation, after the common denary scale, is the one which has *twelve* for its radix, called Duodecimal. Here it is necessary to have two new symbols, in addition to 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, for the purpose of expressing *ten* and *eleven;* since 10, and 11, in the duodenary scale, signify twelve and thirteen respectively. These new symbols may be any thing distinct from the other digits, and are usually two letters, *t* for ten, and *e* for eleven.

The common *pence-table* will be found of considerable service in the use of the duodecimal notation.

Ex. 1. Multiply 25 ft. 7 in. by 7 ft. 10 in.

The lengths are here expressed in the *denary* scale. In the duodenary the question is, what is the product of 21·7 by 7·*t*?

$$\begin{array}{r}
21{\cdot}7 \\
7{\cdot}t \\
\hline
193t \\
12e1 \\
\hline
148{\cdot}4t \\
\hline
\end{array}$$

∴ the product is 148 ft. 4′. t'' in the duodenary notation,
or $8+4\times12+1\times12^2$ ft. 4′. 10″ in the denary;
that is, 200 ft. 4′. 10″ ....

Ex. 2. A floor in the form of a rectangular parallelogram contains 1532 square feet, 9′. 9″, and is 81 ft. 9′ long; required the breadth.

The whole area÷the length = the breadth; and in the duodenary scale the given quantities are $t78{\cdot}99$ and $69{\cdot}9$ respectively.

$$\begin{array}{l}
69{\cdot}9)\ t78{\cdot}99\ (16{\cdot}9\text{ft. in the duodenary scale,}\\
\qquad\quad \underline{669}\\
\qquad\quad 39e9\\
\qquad\quad \underline{34t6}\\
\qquad\quad\ \ 5139\\
\qquad\quad\ \ \underline{5139}\\
\qquad\quad\ \ \ *\ \ *
\end{array}$$

∴ in the common scale the breadth of the floor is 18 ft. 9 in.

[*Exercises Zn.*]

370. *To find the greatest and least numbers with a given number of digits in any proposed scale.*

Let r be the radix of the scale, and n the number of digits; then it is evident that the number will be *greatest*, when every digit is as great as it can be, that is, $r-1$; in which case the number will be

$$r-1+(r-1)r+(r-1)r^2+\ldots\ldots+(r-1)r^{n-1},$$
$$\text{or } (r-1)(1+r+r^2+\ldots\ldots+r^{n-1});$$
$$\text{or } (r-1)\frac{r^n-1}{r-1}, \text{ or } r^n-1.$$

Again, the number will be *least*, when the extreme digit to the left is 1, and every other is 0; in which case it will be equal to r^{n-1}.

Ex. Thus in the common, or denary, scale, the greatest number having 4 digits is 10^4-1, or 9999; and the least is 10^3, or 1000.

371. *To find the number of digits in the product or quotient of two given numbers.*

I. Let P and Q represent the numbers, having p and q digits respectively; and r the radix of the scale; then

Since P is a number of p digits, $P<1$ followed by p ciphers, that is, $P<r^p$. So also $Q<r^q$. Therefore $P\times Q<r^{p+q}<1$ followed by $\overline{p+q}$ ciphers. But this is the *least* number having $p+q+1$ digits; and hence $P\times Q$ cannot have more than $\overline{p+q}$ digits.

Again P is not *less* than 1 followed by $p-1$ ciphers, or than r^{p-1}; and Q is not *less* than r^{q-1}. Therefore $P\times Q$ is not *less* than r^{p+q-2}, or than 1 followed by $p+q-2$ ciphers, which is the *least* number having $p+q-1$ digits; and hence $P\times Q$ cannot have less than $p+q-1$ digits.

II. To find the number in $P\div Q$, when P is exactly divisible by Q; let $P\div Q=R$, then $P=QR$, and $\therefore$ QR contains p digits. Suppose R to contain x digits; then, by former case, p cannot be greater than $q+x$, nor less than $q+x-1$; $\therefore$ x, the number required, cannot be less than $p-q$, nor greater than $p-q+1$.

COR. Hence the number of digits in P^2 is either $2p$, or $2p-1$; in P^3 either $3p$, or $3p-1$, or $3p-2$; and so on.

Also, if $\sqrt{P}$ have x digits, then $\sqrt{P}\times\sqrt{P}$, or P, has either $2x$, or $2x-1$, that is, $p=2x$, or $2x-1$, and $\therefore$ $x=\frac{1}{2}p$, or $\frac{1}{2}(p+1)$, whichever is integral.

Again, if $\sqrt[3]{P}$ have x digits, then P has $3x$, or $3x-1$, or $3x-2$, that is, $p=3x$, or $3x-1$, or $3x-2$, and $\therefore$ $x=\frac{1}{3}p$, or $\frac{1}{3}(p+1)$, or $\frac{1}{3}(p+2)$.

372. *In every system of notation, of which the radix is* r, *the sum of the digits of any number divided by* $\overline{r-1}$ *will leave the same remainder as the whole number divided by* $\overline{r-1}$.

Let $a_0+a_1r+a_2r^2+a_3r^3+\dots\dots+a_nr^n$ be the number (N),

then $N=a_0+a_1+a_2+\dots+a_n+a_1(r-1)+a_2(r^2-1)+\dots\dots+a_n(r^n-1)$;

$$\therefore\ \frac{N}{r-1}=\frac{a_0+a_1+\dots+a_n}{r-1}+a_1+a_2(r+1)+\dots\dots+a_n.\frac{r^n-1}{r-1}.$$

But $\frac{r^n-1}{r-1}$ is a whole number, whatever positive whole number n may be; (Art. 99, Ex. 6);

$$\therefore\ \frac{N}{r-1}=P+\frac{a_0+a_1+\dots+a_n}{r-1},\ P \text{ being a whole number,}$$

or the number divided by $\overline{r-1}$ leaves the same remainder as the sum of the digits divided by $\overline{r-1}$.

COR. Hence, if the sum of the digits of any number be divisible by $\overline{r-1}$, the number itself is divisible by $\overline{r-1}$.

Similarly, it may be shewn that, if the difference between the sum of the digits in the odd places and the sum of the digits in the even places be divisible by $\overline{r+1}$, the number is divisible by $\overline{r+1}$.

For $N=a_0+a_1(\overline{r+1}-1)+a_2(\overline{r+1}-1)^2+\dots+a_n(\overline{r+1}-1)^n$,

$=a_0-a_1+a_2-a_3+\dots\pm a_n+$ multiples of $r+1$.

PROB. *To find what numbers are divisible by* 3 *and* 9 *without remainders.*

Let a, b, c, d, &c. be the digits, or figures in the units', tens', hundreds', thousands', &c. place of any number, then the number is

$$a+10b+100c+1000d+\&c.$$

this divided by 3 is

$$\frac{a}{3} + 3b + \frac{b}{3} + 33c + \frac{c}{3} + 333d + \frac{d}{3} + \&c.$$

$$\text{or } \frac{a+b+c+d+\&c.}{3} + 3b + 33c + 333d + \&c.$$

which is a whole number when $\frac{a+b+c+\&c.}{3}$ is a whole number; that is, any number is a multiple of 3 if the sum of its digits be a multiple of 3. Thus 111, 252, 7851, &c. are multiples of 3.

In the same manner, any number is a multiple of 9 if the sum of its digits be a multiple of 9.

$$\text{For } \frac{a + 10b + 100c + 1000d + \&c.}{9}$$

$$= \frac{a}{9} + b + \frac{b}{9} + 11c + \frac{c}{9} + 111d + \frac{d}{9} + \&c.$$

$$= \frac{a+b+c+d+\&c.}{9} + b + 11c + 111d + \&c.$$

which is a whole number when $\frac{a+b+c+d+\&c.}{9}$ is a whole number. Thus 684, 6588, &c. are multiples of 9.

Cor. 1. Hence, if any number, and the sum of its digits be respectively divided by 9, the remainders are equal.

Cor. 2. From this property of 9 may be deduced a rule which will *sometimes* detect an error in the multiplication of two numbers, called *casting out the Nines.*

Let the multiplicand be $9a + x$, that is, let it consist of a nines, with a remainder x; and let $9b + y$ be the multiplier; then

$$81ab + 9bx + 9ay + xy \text{ is the product;}$$

and if the sum of the digits in the multiplicand be divided by 9, the remainder is x; if the sum of the digits in the multiplier be divided by 9, the remainder is y; and if the sum of the digits in the product be divided by 9, the remainder is the same as when the sum of the digits in xy is divided by 9, if there be no mistake in the operation*.

---

* It will be easily seen that this method fails to detect an error in any of the following cases:—(1) when one or more cyphers have been omitted in the product; (2) when any of its digits are *misplaced;* and (3) when the error is equal to 9 or any multiple of 9.—Ed.

373. Thus far we have treated of *whole* numbers only; but *fractions* also may be expressed in different scales of notation. Thus

$$23{\cdot}21,\ \text{radix } 10,\ \text{signifies } 2\times10+3+\frac{2}{10}+\frac{1}{10^2},$$

$$\ldots\ldots\ \text{radix } 6,\ \ldots\ldots\ldots\ 2\times6+3+\frac{2}{6}+\frac{1}{6^2},$$

$$\ldots\ldots\ \text{radix } 4,\ \ldots\ldots\ldots\ 2\times4+3+\frac{2}{4}+\frac{1}{4^2},$$

$$\ldots\ldots\ \text{radix } r,\ \ldots\ldots\ldots\ 2\times r+3+\frac{2}{r}+\frac{1}{r^2}.$$

And generally, if there be n digits after the point which separates the integral from the fractional part of a number, the *fractional* part will be expressed by

$$\frac{a_{-1}}{r}+\frac{a_{-2}}{r^2}+\frac{a_{-3}}{r^3}+\ldots\ldots\ldots+\frac{a_{-n}}{r^n},$$

$$\text{or } a_{-1}r^{-1}+a_{-2}r^{-2}+a_{-3}r^{-3}+\ldots\ldots\ldots+a_{-n}r^{-n}.$$

It is to be understood that as any vulgar fraction in the common scale may be converted into a decimal fraction, the same may be done in any other scale, and in the same way, bearing in mind only the difference of radix.

374. *To transform a* fraction *from one scale to another.*

Let N be the given fraction, r the radix of the new scale, and a_{-1}, a_{-2}, a_{-3}, &c. the unknown digits; then

$$N=a_{-1}r^{-1}+a_{-2}r^{-2}+a_{-3}r^{-3}+\&\text{c.}$$

$\therefore\ rN=a_{-1}+a_{-2}r^{-1}+a_{-3}r^{-2}+\&$c., the *integral* part of which is a_{-1} the first digit required, leaving the *fractional* part

$$a_{-2}r^{-1}+a_{-3}r^{-2}+\&\text{c};$$

multiply this by r, and the result is $a_{-2}+a_{-3}r^{-1}+\&$c., the *integral* part of which is a_{-2}, the 2nd digit required; and so on, continually multiplying by r, and separating the *integral* part for a new digit after each operation.

If the proposed fraction be not a *proper* fraction, the *integral* part must be dealt with separately according to the former rules for whole numbers.

Ex. 1. Transform $\frac{4}{9}$ from the denary to the ternary scale.

$$3\times\frac{4}{9}=\frac{12}{9}=1\tfrac{1}{3},\quad 3\times\tfrac{1}{3}=1,\quad \therefore \text{ the No. required is } 0{\cdot}11.$$

Ex. 2. Transform $43{\cdot}7$ or $43\frac{7}{10}$ from the denary to the senary scale.

$$\begin{array}{r|l} 6 & 43 \\ \hline 6 & 7,\ 1 \\ \hline 6 & 1,\ 1 \\ \hline & 0,\ 1 \end{array}$$

$\therefore\ 43=111$ in the senary scale.

And $\frac{7}{10}\times6=\frac{42}{10}=4\frac{1}{5}$, $\frac{1}{5}\times6=\frac{6}{5}=1\frac{1}{5}$, &c.

$\therefore$ the fractional part is $\cdot41111\ldots\ldots$

and the No. required is $111{\cdot}4111\ldots\ldots$

Ex. 3. Transform $23\frac{1}{4}$ from the senary to the septenary scale.

Here $23\frac{1}{4} = 2\times6+3+\frac{1}{4} = 15\frac{1}{4}$ in the denary scale;

$$\begin{array}{r|l} 7 & 15 \\ \hline 7 & 2,\ 1 \\ \hline & 0,\ 2 \end{array} \qquad \frac{1}{4}\times7=\frac{7}{4}=1\frac{3}{4},\ \ \frac{3}{4}\times7=\frac{21}{4}=5\frac{1}{4},\ \&c.$$

$\therefore$ the No. required is $21{\cdot}1515\ldots\ldots$

Or thus, without introducing the *denary* scale at all, but bearing in mind throughout that the radix is 6,

$$23\tfrac{1}{4} = 23{\cdot}13,$$

$$\begin{array}{r|l} 7 & 23 \\ \hline 7 & 2,\ 1 \\ \hline & 0,\ 2 \end{array} \qquad \begin{array}{r} {\cdot}13 \\ 7 \\ \hline 1{\cdot}43 \\ 7 \\ \hline 5{\cdot}13 \end{array}$$

$\therefore$ No. required is $21{\cdot}1515\ldots\ldots$

Ex. 4. Transform $456{\cdot}16$ from the duodenary to the ternary scale.

$$\begin{array}{r|l} 3 & 456 \\ \hline 3 & 15t,\ 0 \\ \hline 3 & 5e,\ 1 \\ \hline 3 & 1e,\ 2 \\ \hline 3 & 7,\ 2 \\ \hline 3 & 2,\ 1 \\ \hline & 0,\ 2 \end{array} \qquad \begin{array}{r} 0{\cdot}16 \\ 3 \\ \hline 0{\cdot}46 \\ 3 \\ \hline 1{\cdot}16 \end{array}$$

$\therefore$ No. required is $212210{\cdot}0101\ldots\ldots$

[*Exercises Zo.*]

PROPERTIES OF NUMBERS.

375. *The product of any two consecutive numbers is divisible by* 1×2.

Of the two numbers one must evidently be even, that is, divisible by 2, therefore their product is divisible by 2, or by 1×2.

376. *The product of any three consecutive numbers is divisible by* $1\times2\times3$, or 6.

Every number must be either of the form $3m$, or $3m\pm1$, or $3m\pm2$, since it must be either divisible by 3 without remainder, or have a remainder 1 or 2; therefore the product of any three consecutive numbers may be represented by one of the forms

$$3m(3m+1)(3m+2),$$
$$(3m-1)3m(3m+1),$$
$$(3m-2)(3m-1)3m.$$

Now, since by last Art. both $(3m+1)(3m+2)$ and $(3m-2)(3m-1)$ are divisible by 1×2, and $3m$ is a multiple of 3, therefore the first and third forms are divisible by $1\times2\times3$. Also, if m be an even number, that is, divisible by 2, it is clear that $3m$, and therefore the second form is divisible by $1\times2\times3$. Or, if m be an odd number, $3m$ is odd, and divisible by 3; also $3m+1$ is even and therefore divisible by 2; consequently the second form is divisible by $1\times2\times3$.

Hence, in all cases, the product of three consecutive numbers is divisible by $1\times2\times3$.

377. *The continued product of any* r *consecutive numbers is divisible by* 1.2.3...r.

[This has been already shewn indirectly in Art. 316; but the following is a more independent proof].

Let n be the least of the numbers, and let $\dfrac{n(n+1)(n+2)\dots(n+r-1)}{1\;.\;2\;.\;3\;\dots\;r}$ be represented by ${}_nP_r$ for all values of n and r.

Then $${}_nP_r=\frac{n(n+1)\dots(n+r-2)}{1\;.\;2\dots\dots\dots(r-1)}\cdot\frac{n+r-1}{r}={}_nP_{r-1}\cdot\left(\frac{n-1}{r}+1\right),$$
$$=\frac{(n-1)n(n+1)\dots(n+r-2)}{1\;.\;2\;.\;3\;\dots\dots\dots\;r}+{}_nP_{r-1},$$
or $${}_nP_r={}_{n-1}P_r+{}_nP_{r-1}.$$

Now *assume* that the product of any $\overline{r-1}$ consecutive integers is divisible by $1.2.3\dots(r-1)$, that is, *suppose* ${}_nP_{r-1}$ is an integer, then

${}_nP_r={}_{n-1}P_r+$ an Integer, for all values of n and r,

Write $n-1$ for n, ${}_{n-1}P_r={}_{n-2}P_r+$ an Integer,

...... $n-2$ ......, ${}_{n-2}P_r={}_{n-3}P_r+$ an Integer,

.... &c. = &c.

.... 3 .... ${}_3P_r={}_2P_r+$ an Integer,

.... 2 .... ${}_2P_r={}_1P_r+$ an Integer $=\dfrac{1.2.3\dots r}{1.2.3\dots r}+$ an Integer,

$=1+$ an Integer $=$ Integer,

∴ adding and cancelling,

${}_nP_r=$ the sum of Integers $=$ an Integer;

which proves that, if ${}_nP_{r-1}$ be an Integer, then also is ${}_nP_r$. But we know that ${}_nP_3$ is an integer, (by the last Art.), *therefore* also is ${}_nP_4$; and if ${}_nP_4$, *therefore* also ${}_nP_5$; and so on generally for ${}_nP_r$; that is $n(n+1)(n+2)\dots\dots(n+r-1)$ is divisible by $1.2.3\dots r$.

Ex. 1. If n be any whole number, then will $n(n^2-1)(n^2-4)$ be divisible by 120.

$$n(n^2-1)(n^2-4)=n(n-1)(n+1)(n-2)(n+2),$$
$$=(n-2)(n-1)n(n+1)(n+2),$$

which is the product of 5 consecutive numbers, and is therefore divisib by 1.2.3.4.5, or 120.

Ex. 2. *If n be any even number, n^3+20n is divisible by 48.*

Let $n=2m$, since it is an even number,

$$\begin{aligned} \text{then } n^3+20n &= 8m^3+40m, \\ &= 8m(m^2+5), \\ &= 8m(m^2-1)+48m, \\ &= 8(m-1)m(m+1)+48m. \end{aligned}$$

Now $(m-1)m(m+1)$, being the product of three consecutive number is divisible by 1.2.3, or 6; therefore n^3+20n is divisible by 48.

378. *Every number which is a perfect square is of one of the form* $5m$ *or* $5m\pm1$.

For every number is of one of the forms $5m$, $5m+1$, $5m+2$, $5m+3$ $5m+4$; all of which are included in the forms $5m$, $5m\pm1$, $5m\pm2$, sinc $5m+3=5(m+1)-2=5m'-2$, and $5m+4=5(m+1)-1=5m'-1$.

But $(5m)^2=5(5m^2)=5m'$, which is of the form $5m$;

$(5m\pm1)^2=25m^2\pm10m+1=5(5m^2\pm2m)+1$, which is of the form $5m+1$;

$(5m\pm2)^2=25m^2\pm20m+4=5(5m^2\pm4m+1)-1$, which is of the form $5m-1$;

$\therefore$ every square is of one of the forms $5m$, $5m+1$, $5m-1$.

379. *Every "prime number" greater than 2 is of one of the form* $4m\pm1$.

For every number is of one of the forms $4m$, $4m+1$, $4m+2$, $4m+3$ but neither $4m$, nor $4m+2$, can represent prime numbers, since each is divisible by 2; therefore all prime numbers greater than 2 are represented by $4m+1$, and $4m+3$. But $4m+3=4(m+1)-1=4m'-1$; therefore the two forms for prime numbers are $4m\pm1$.

Cor. Since m may be odd or even, that is, of the form $2n+1$, or $2n$, all prime numbers are represented by $8n\pm1$, or $8n\pm3$.

380. *Every prime number greater than 3 is of one of the forms* $6m\pm1$.

For every number is of one of the forms $6m$, $6m+1$, $6m+2$, $6m+3$, $6m+4$, $6m+5$, of which the 1st, 3d, 4th, and 5th obviously cannot represent *prime* numbers; and therefore, all prime numbers greater than 3 are represented by $6m+1$, and $6m+5$. But $6m+5=6(m+1)-1=6m'-1$. Therefore, $6m\pm1$ will include all prime numbers greater than 3.

Cor. Since m may be odd or even, that is, of the form $2n+1$, or $2n$, all prime numbers greater than 3, will be included in $12n\pm1$, or $12n\pm5$.

[*Exercises Zp.*]

381. *No Algebraical formula can represent* prime *numbers only.*

Let $p+qx+rx^2+$&c. be a general algebraical formula; and let it be a prime number, when $x=m$; therefore (P) the prime number in that case is

$$p+qm+rm^2+\&c.$$

Now let $x=m+nP$; then

$$p+q(m+nP)+r(m+nP)^2+\&c.$$

is the number; and is equal to

$$p+qm+rm^2+\&c.+MP,$$

(M signifying "some multiple of,") $=P+PM$, which is divisible by P, and therefore not a prime; consequently the formula does not represent prime numbers only.

382. *The number of primes is indefinitely great.*

For, if not, let there be a fixed number of them, and let p be the greatest: then

$1.2.3.5.7.11\ldots p$ is divisible by each of them,

and $1.2.3.5.7.11\ldots p+1$..................not one of them.

If this latter number, then, be divisible by a prime number, it must be one greater than p; if not, it is itself a prime, (since every number is either a prime, or capable of being resolved into factors which are prime) and is greater than p. Therefore, in either case, there is a prime greater than p; that is, we may not assume any prime to be the greatest; or, the number of primes is indefinitely great.

383. *To determine whether a proposed number be a prime or not.*

It is obvious that this may be done by dividing the proposed number by every number less than itself, beginning with 2, until we have either proved it to be divisible by some one of them without remainder, or that it is a prime, from not being divisible by any one of them. But there is no necessity to proceed so far, as may thus be shewn. If the proposed number (p) be not a prime, then $p=ab$, the product of two other numbers. If then $a>\sqrt{p}$, $b<\sqrt{p}$; and if $a<\sqrt{p}$, $b>\sqrt{p}$. Hence in both cases p is divisible by a number less than the square root of itself. Or it may be that the proposed number is an exact square, in which case it is divisible by its square root. If, therefore, a proposed number be not divisible by some number not greater than the square root of itself, it must be a prime.

384. *To find the number of divisors of a given number.*

Let a, b, c, &c. represent the prime factors of which the given number is composed; and let a be repeated p times; b, q times; c, r times; &c.; so that the number $=a^pb^qc^r$, &c.; then it is evident that it is divisible by each of the quantities

$1,\ a,\ a^2,\ a^3,\ \ldots\ldots\ a^p$, $p+1$ in number,

$1,\ b,\ b^2,\ b^3,\ \ldots\ldots\ b^q$, $q+1$ ... ...

$1,\ c,\ c^2,\ c^3,\ \ldots\ldots\ c^r$, $r+1$ ... ...

&c. &c. &c.

and also by the product of any two or more of them, that is, by every term of the continued product of

$$(1+a+a^2+\ldots+a^p)(1+b+b^2+\ldots+b^q)(1+c+c^2+\ldots+c^r).\&c.;$$

for

$$\begin{aligned}(1+a+a^2+\ldots+a^p)(1+b+b^2+\ldots+b^q) = {} & 1+a+a^2+\ldots\ldots+a^p \\ & +b+ab+\ \ldots\ldots+a^pb \\ & +b^2+ab^2+\ldots\ldots+a^pb^2 \\ & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & +b^q+ab^q+\ldots\ldots+a^pb^q,\end{aligned}$$

which are *all* the different divisors (including 1) of a^pb^q.

Similarly, if this result be multiplied by $(1+c+c^2+\ldots+c^r)$, the product will consist of all the different divisors of $a^pb^qc^r$; and so on, if there be more factors of the given number.

Now the *number* of these divisors is obviously $p+1$ in a^p; in a^pb^q the number is $p+1$ taken as many times as there are terms in the second series, that is, $(p+1)(q+1)$; in $a^pb^qc^r$ it is $(p+1)(q+1)$ taken $\overline{r+1}$ times, or $(p+1)(q+1)(r+1)$; and generally

the number of divisors in $a^pb^qc^r.\&c. = (p+1)(q+1)(r+1).\&c.$

including 1 and the number itself.

Ex. Find the number of divisors of 2160.

Here $2160 = 2\times1080 = 2^2\times540 = 2^3\times270 = 2^4\times135 = 2^4\times3\times45 = 2^4\times3^2\times15 = 2^4\times3^3\times5$. Therefore

$$\text{number of divisors} = (4+1)(3+1)(1+1) = 40.$$

COR. 1. The number of divisors will always be even unless each of the quantities p, q, r, &c. be even, that is, unless the proposed number be a perfect square.

COR. 2. It is also obvious from what has been said above that the *sum* of all the different divisors of $a^pb^qc^r.\&c.$ is the sum of all the terms in the continued product of $(1+a+a^2+\ldots+a^p)(1+b+b^2+\ldots+b^q)(1+c+c^2+\ldots+c^r)$ &c.

$$\text{or}\quad \frac{a^{p+1}-1}{a-1}\cdot\frac{b^{q+1}-1}{b-1}\cdot\frac{c^{r+1}-1}{c-1}.\&c.$$

385. *If* m *be any prime number, and* N *a number not divisible by* m, *then* $N^{m-1}-1$ *is divisible by* m. (FERMAT'S THEOREM.)

For, it is easily seen, by the Binomial Theorem, that in the expansion of $(a+b+c+\&c.)^m$ m is a factor of every term except a^m, b^m, c^m, &c.; and that the coefficients are always whole numbers.

But m, being a *prime* number, will not be divisible by any factor in the denominator of a coefficient, and will therefore remain as a factor of each term, when the coefficients are reduced. Therefore we may assume, (m being a prime number),

$$(a+b+c+\&c.)^m = a^m+b^m+c^m+\&c.+mP.$$

Let, then, $a=b=c=\&c.=1$, and the number of them be N; and we have

$$N^m = N + mP,$$

$$\therefore\ N^m - N = mP,$$

$$\text{or}\ N(N^{m-1}-1) = mP.$$

But, by supposition, N is not divisible by m,

$\therefore\ N^{m-1}-1$ is a multiple of m, or is divisible by m.

386. *To prove that for any positive integral value of* n,

$$n^n - n(n-1)^n + \frac{n(n-1)}{1\,.\,2}(n-2)^n - \&c. = 1.2.3\ldots n.$$

By the Binomial Theorem,

$$(\epsilon^x-1)^n = \epsilon^{nx} - n\epsilon^{(n-1)x} + \frac{n(n-1)}{1\,.\,2}\epsilon^{(n-2)x} - \&c.\ldots\ldots(1).$$

Also by the Exponential Theorem,

$$(\epsilon^x-1)^n = \left(1+x+\frac{x^2}{1.2}+\ldots-1\right)^n = \left(x+\frac{x^2}{1.2}+\ldots\right)^n,$$

$$= x^n + Ax^{n+1} + Bx^{n+2} + \ldots \text{ by the Binomial Theor.}\ldots(2).$$

Now, the coefficient of x^n in ϵ^{nx} is $\dfrac{n^n}{1.2\ldots n}$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \epsilon^{(n-1)x} \ldots \dfrac{(n-1)^n}{1.2\ldots n}$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \epsilon^{(n-2)x} \ldots \dfrac{(n-2)^n}{1.2\ldots n}$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

and equating coefficients of x^n in (1) and (2),

$$\frac{n^n}{1.2\ldots n} - \frac{n(n-1)^n}{1.2\ldots n} + \frac{n(n-1)}{1\,.\,2}\cdot\frac{(n-2)^n}{1.2\ldots n} - \&c. = 1,$$

$$\text{and}\ \therefore\ n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1\,.\,2}(n-2)^n - \&c. = 1.2.3\ldots n.$$

Cor. Let p be a prime number, and $p-1=n$, then

$$1.2.3\ldots\overline{p-1} = (p-1)^{p-1} - \frac{p-1}{1}(p-2)^{p-1} + \frac{(p-1)(p-2)}{1\,.\,2}(p-3)^{p-1} - \&c.$$

$$= M_1p + 1 - \frac{p-1}{1}(M_2p+1) + \frac{(p-1)(p-2)}{1\,.\,2}(M_3p+1) - \&c.,$$ by

Fermat's Theorem; $= p\left\{M_1 - (p-1)M_2 + \dfrac{(p-1)(p-2)}{1\,.\,2}M_3 - \&c.\right\}$

$$+1 - \frac{p-1}{1} + \frac{(p-1)(p-2)}{1\,.\,2} - \&c. \text{ to } \overline{p-1} \text{ terms},$$

$$= pQ + (1-1)^{p-1} - 1;$$

$$\therefore\ 1.2.3\ldots\overline{p-1}+1=pQ, \text{ or is divisible by } p;$$

which is *Wilson's Theorem* for determining whether any proposed number be a prime or not.

VANISHING FRACTIONS.

387. *To find the value of a fraction when the numerator and denominator are evanescent.*

Since the value of a fraction depends, not upon the absolute, but the relative, magnitude of the numerator and denominator, if in their evanescent state they have a finite ratio, the value of the fraction will be finite. To determine this value, substitute for the variable quantity its magnitude, when the numerator and denominator vanish, increased by another variable quantity; then, after reduction, suppose this latter to decrease without limit, and the value of the proposed fraction will be known.

Ex. 1. Required the value of $\frac{x^2-a^2}{x-a}$, when $x=a$.

Let $x=a+h$, and the fraction becomes

$$\frac{a^2+2ah+h^2-a^2}{a+h-a}=\frac{2ah+h^2}{h}=2a+h,$$

and when $h=0$, or $x=a$, its value is $2a$.

Ex. 2. Required the value of $\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$, when $x=1$

Let $x=1+h$, and the fraction becomes

$$\frac{1-(n+1)(1+h)^n+n(1+h)^{n+1}}{h^2},$$

$$=\frac{1-(n+1)(1+nh+n.\frac{n-1}{2}h^2+n.\frac{n-1}{2}.\frac{n-2}{3}h^3+\&c.)}{h^2}$$

$$+\frac{n\{1+(n+1)h+(n+1)\frac{n}{2}h^2+(n+1)\frac{n}{2}.\frac{n-1}{3}h^3+\&c.\}}{h^2},$$

$=n.\frac{n+1}{2}+\frac{(n+1)n(n-1)}{3}h+\&c.$ the remaining terms being multiples of h;

and when $h=0$, or $x=1$, the fraction becomes $n.\frac{n+1}{2}$.

388. It is clear that every fraction, whose terms are made evanescent for a particular value (a) of some quantity (x) contained in them, is capable of being reduced to the form $\frac{p(x-a)^m}{q(x-a)^n}$; and, upon dividing the numerator and denominator by their greatest common factor, the fraction will no longer have *both* its terms evanescent, when $x=a$.

Hence by whatever process this common factor can be discovered, and divided out, the value of the fraction will be found.

A simple algebraical reduction is frequently sufficient for this purpose, as will be shewn in the following Examples.

Ex. 1. Required the value of $\frac{1-3x^2+2x^3}{(1-x)^2}$, when $x=1$.

Here
$$\frac{1-3x^2+2x^3}{(1-x)^2}=\frac{1-x^2-2x^2(1-x)}{(1-x)^2},$$
$$=\frac{1+x-2x^2}{1-x}=\frac{1-x^2+x(1-x)}{1-x},$$
$$=1+x+x,$$
$$=3, \quad \text{when } x=1.$$

Ex. 2. Required the value of $\frac{x-a+\sqrt{2ax-2a^2}}{\sqrt{x^2-a^2}}$, when $x=a$.

Here
$$\frac{x-a+\sqrt{2ax-2a^2}}{\sqrt{x^2-a^2}}=\frac{x-a+\sqrt{2a}.\sqrt{x-a}}{\sqrt{x+a}.\sqrt{x-a}},$$
$$=\frac{\sqrt{x-a}+\sqrt{2a}}{\sqrt{x+a}},$$
$$=\frac{\sqrt{2a}}{\sqrt{2a}}=1, \quad \text{when } x=a.$$

Ex. 3. Required the value of $\frac{(x+h)^{\frac{m}{n}}-x^{\frac{m}{n}}}{h}$, when $h=0$.

Here $\frac{(x+h)^{\frac{m}{n}}-x^{\frac{m}{n}}}{h}=\frac{(x+h)^{\frac{m}{n}}-x^{\frac{m}{n}}}{(x+h)-x}$; let $v=(x+h)^{\frac{1}{n}}$, and $w=x^{\frac{1}{n}}$, then

fraction $=\frac{v^m-w^m}{v^n-w^n}$; divide num^r^. and denom^r^. by $v-w$,
$$=\frac{v^{m-1}+v^{m-2}w+\ldots+vw^{m-2}+w^{m-1}}{v^{n-1}+v^{n-2}w+\ldots+vw^{n-2}+w^{n-1}},$$

$$=\frac{w^{m-1}+w^{m-1}+\&\text{c. to } m \text{ terms}}{w^{n-1}+w^{n-1}+\&\text{c. to } n \text{ terms}}, \text{ when } h=0, \text{ and } \therefore v=w,$$

$$=\frac{mw^{m-1}}{nw^{n-1}}=\frac{m}{n}w^{m-n},$$

$$=\frac{m}{n}x^{\frac{m-n}{n}},$$

$$=\frac{m}{n}x^{\frac{m}{n}-1}.$$

[*Exercises Zq.*]

INFINITE SERIES.

389. **Def.** An *infinite series* is a series of terms proceeding according to some law, and continued without limit. Thus the series treated of in Art. 290, Cor. 2, is an "infinite series".

Def. The *sum* of an infinite series is the *limit* to which we approach more nearly by adding more terms, (as in Art. 290, Cor. 2,) but cannot be exceeded by adding any number of terms whatever.

Def. A *convergent* series is one which has a *sum* or *limit*, as here defined. A *divergent* series is one which has no such *sum* or *limit*.

Hence every infinite series in Geometrical progression, in which the common ratio is less than 1, is *convergent*.

390. *To determine in certain cases whether a series is convergent or divergent.*

I. Let $a_1+a_2+a_3+a_4+\ldots\ldots$ be the series, in which all the terms are positive. Then the sum is equal to

$$a_1\left\{1+\frac{a_2}{a_1}+\frac{a_3}{a_1}+\frac{a_4}{a_1}+\ldots\ldots\right\},$$

$$\text{or } a_1\left\{1+\frac{a_2}{a_1}+\frac{a_3}{a_2}\cdot\frac{a_2}{a_1}+\frac{a_4}{a_3}\cdot\frac{a_3}{a_2}\cdot\frac{a_2}{a_1}+\ldots\ldots\right\}.$$

And, if each of the quantities $\frac{a_2}{a_1}$, $\frac{a_3}{a_2}$, $\frac{a_4}{a_3}$, &c. be less than some quantity p, the whole series is less than

$$a_1\{1+p+p^2+p^3+\ldots\ldots\}.$$

But if p be less than 1, the *sum* or *limit* of this latter series is $a_1\cdot\frac{1}{1-p}$; therefore the proposed series also has a sum or limit *less than* this quantity. Hence ***an infinite series of positive terms is always convergent, if the ratio of each term to the preceding term is less than some assignable quantity which is itself less than*** **1.**

II. Next, let $a_1 - a_2 + a_3 - a_4 + \ldots\ldots$ be the series, in which the terms are alternately positive and negative, and go on decreasing without limit.

Then, since the series may be written in the two following ways,

$$\overline{a_1 - a_2} + \overline{a_3 - a_4} + \overline{a_5 - a_6} + \ldots\ldots$$

$$a_1 - \overline{a_2 - a_3} - \overline{a_4 - a_5} - \ldots\ldots\ldots$$

and since $a_3 - a_4$, $a_5 - a_6$, &c., $a_2 - a_3$, $a_4 - a_5$, &c., are severally positive, it is evident that the *sum* or *limit* of the series is greater than $a_1 - a_2$ and less than a_1, that is, the series has a sum or limit; consequently it is convergent:—or, *every series, in which the terms continually decrease and are alternately positive and negative, is convergent.*

Cor. If each of the quantities $\frac{b}{a}$, $\frac{c}{b}$, $\frac{d}{c}$, &c. be less than p, the series $a + bx + cx^2 + dx^3 \ldots$ is *convergent* whenever x is less than $\frac{1}{p}$. And if p be less than each of the quantities $\frac{b}{a}$, $\frac{c}{b}$, $\frac{d}{c}$, &c. then the series $a + bx + cx^2 + \ldots$ is *divergent* for every value of x greater than $\frac{1}{p}$.

For, if kx^n and lx^{n+1} be any two consecutive terms of the series, $\frac{lx^{n+1}}{kx^n} = \frac{l}{k}x$; and if $\frac{l}{k} = p - y$, and $x = \frac{1}{p} - a$, where p and a are constant, and y always less than p, then $\frac{l}{k}x = (p - y)\left(\frac{1}{p} - a\right)$, or $1 - ap - xy$. Now $1 - ap$ is positive, and x and y are both positive, $\therefore 1 - ap - xy$, or $\frac{l}{k}x$, is less than $1 - ap$; that is, the ratio of each term to the preceding term is less than an assigned quantity, $1 - ap$, which is itself less than 1; and therefore the series is *convergent.*

But if p be less than $\frac{l}{k}$, and x greater than $\frac{1}{p}$, which will be expressed by changing the signs of y and a, then $\frac{l}{k}x$ will be $1 + ap + xy$, greater than 1, and the sum of the series infinite; therefore the series is *divergent.*

Ex. To determine whether $1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots$ be a convergent series.

$$\text{Here } \frac{a_2}{a_1} = 1, \quad \frac{a_3}{a_2} = \frac{1}{2}, \quad \frac{a_4}{a_3} = \frac{1}{3}, \quad \frac{a_5}{a_4} = \frac{1}{4}, \text{ \&c.}$$

each of which ratios, after the second, is less than $\frac{1}{2}$, which is itself less than 1. Therefore the series is *convergent.*

391. Another method of determining the convergency or divergency of series is to find the limit of the sum of the series after the first n terms; which also determines the limits of the error arising from taking any number of terms instead of the whole series. Thus,

Ex. 1. In the series $1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots\ldots$ the sum of the series after n terms

$$=\frac{1}{\underline{|n}}+\frac{1}{\underline{|n+1}}+\frac{1}{\underline{|n+2}}+\ldots\ldots\ldots\ldots$$

$$=\frac{1}{\underline{|n}}.\left\{1+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\ldots\ldots\right\},$$

$$<\frac{1}{\underline{|n}}.\left\{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+\ldots\ldots\right\},$$

$$<\frac{1}{\underline{|n}}.\frac{1}{1-\frac{1}{n}},$$

$$<\frac{1}{\underline{|n-1}}.\frac{1}{n-1}, \text{ or } <\frac{1}{1.2.3\ldots(n-1)}.\frac{1}{n-1}.$$

But this quantity decreases as n increases; and, by increasing n without limit, it may be made less than any assignable quantity. Therefore the series is *convergent*. And if n terms be taken for the whole series, the error is less than

$$\frac{1}{1.2.3\ldots(n-1)}.\frac{1}{n-1}.$$

Ex. 2. In the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots$ the sum of the series after n terms

$$=(-1)^n.\left\{\frac{1}{n+1}-\frac{1}{n+2}+\left(\frac{1}{n+3}-\frac{1}{n+4}\right)+\ldots\ldots\right\},$$

$$\text{or }=(-1)^n.\left\{\frac{1}{n+1}-\left(\frac{1}{n+2}-\frac{1}{n+3}\right)-\left(\frac{1}{n+4}-\frac{1}{n+5}\right)-\ldots\right\};$$

and, since the quantities within the inner brackets are all positive, this sum

$$>(-1)^n.\left\{\frac{1}{n+1}-\frac{1}{n+2}\right\},$$

$$\text{and }<(-1)^n.\frac{1}{n+1},$$

both which quantities are diminished without limit as n is increased.

Hence the series is *convergent*; and if n terms be taken for the whole series, the error is less than the $\overline{n+1}^{\text{th}}$ term.

Ex. 3. In the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots$ the sum of the series after n terms

$$=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots\ldots$$

$$=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots\ldots+\frac{1}{2n}$$

$$+\frac{1}{2n+1}+\frac{1}{2n+2}+\frac{1}{2n+3}+\ldots+\frac{1}{4n}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$>\frac{1}{2n}+\frac{1}{2n}+\&\text{c. to } n \text{ terms}+\frac{1}{4n}+\frac{1}{4n}+\&\text{c. to } 2n \text{ terms}+\ldots\ldots$$

$$>n.\frac{1}{2n}+2n.\frac{1}{4n}+4n.\frac{1}{8n}+\ldots\ldots\ldots$$

$$>\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$>$ any assignable quantity.

Hence the series is *divergent.*

392. *In the series* $a_1x+a_2x^2+a_3x^3+\ldots\ldots$ in inf. *such a value may be given to* x, *that the value of the whole series shall be less than any proposed quantity* p.

Let k be the greatest of the coefficients a_1, a_2, a_3, ..., then the whole series is less than

$$kx+kx^2+kx^3+\ldots\ldots \textit{ in inf.},$$

$$<k.\frac{x}{1-x}, \text{ if } x<1;$$

hence that which is required is done, if x be such a value that

$$\frac{kx}{1-x}<p,$$

$$\text{that is, } kx<p-px,$$

$$\text{or } x<\frac{p}{p+k}.$$

Cor. Hence also in the series $a_0+a_1x+a_2x^2+\ldots\ldots$ *in inf.* such a value may always be given to x, that the first term is greater than the sum of all the other terms. This value of x will be any quantity less than $\frac{a_0}{a_0+k}$.

RECURRING SERIES.

393. If each succeeding term of a decreasing Infinite Series bear an invariable relation to a certain number of the preceding terms, the series is called a *Recurring Series,* and its sum may be found*.

Let $a + bx + cx^2 + \dots$ be the proposed series; call its terms A, B, C, D, &c. and let

$$C = fxB + gx^2A, \quad D = fxC + gx^2B, \text{ \&c.}$$

where $f + g$ is called the *scale of relation;* then, by the supposition,

$$\begin{aligned} A &= A, \\ B &= B, \\ C &= fxB + gx^2A, \\ D &= fxC + gx^2B, \\ E &= fxD + gx^2C, \\ \dots &= \dots \end{aligned}$$

and, if the whole sum $A + B + C + D +$ &c. *in inf.* $= S$, we have

$$S = A + B + fx.(S - A) + gx^2.S,$$

$$\text{or} \quad S - fxS - gx^2S = A + B - fxA;$$

$$\therefore S = \frac{A + B - fxA}{1 - fx - gx^2}.$$

In the same manner, if the scale of relation be $f + g +$ &c. to n terms, the sum of the series is

$$\frac{A+B+C\dots\text{to } n \text{ terms} - fx\{A+B\dots\text{to } \overline{n-1} \text{ terms}\} - gx^2\{A+\dots\text{to } \overline{n-2} \text{ terms}\} - \&c}{1 - fx - gx^2 - hx^3\dots\text{to } (n+1) \text{ terms}}$$

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* In practice, perhaps, it will be found best not to rely upon the general theory here given. Every example in this section may be worked out in the most simple manner. Thus, take Ex. 1; and let S be the sum required; then

$$\frac{S-1}{x} = 3(1 + 3x + 9x^2 + \dots) = 3S,$$

$$\therefore S = \frac{1}{1-3x}.$$

Similarly, in Ex. 2, $\frac{S-1}{x} - S = \frac{1}{1-x}$, from which $S = \frac{1}{(1-x)^2}$.

In Ex. 3, $\frac{S-1}{x} - S = \frac{2}{1-x}$, from which $S = \frac{1+x}{(1-x)^2}$. And so on.—Ed.

Ex. 1. To find the sum of the infinite series $1 + 3x + 9x^2 + \ldots$ when x is less than $\frac{1}{3}$.

$$\text{Here } f = 3, \text{ and the sum} = \frac{1}{1-3x}.$$

Ex. 2. To find the sum of the infinite series $1 + 2x + 3x^2 + 4x^3 + \ldots$ when x is less than 1.

Here $f = 2$, $g = -1$, and

$$\text{the sum} = \frac{1 + 2x - 2x}{1 - 2x + x^2} = \frac{1}{(1-x)^2}.$$

If $x =$ or > 1, the sum is infinite; yet we know that the series arises from the division of 1 by $(1-x)^2$, and the sum of n terms may be determined as follows:—

The series after the first n terms becomes

$$(n+1)x^n + (n+2)x^{n+1} + (n+3)x^{n+2} + \&c.$$

in which the scale of relation, as before, is $2-1$; and therefore the series arises from the fraction

$$\frac{(n+1)x^n + (n+2)x^{n+1} - 2(n+1)x^{n+1}}{1 - 2x + x^2},$$

$$\text{or } \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2};$$

$$\therefore\ 1 + 2x + 3x^2 + \&c. \text{ to } n \text{ terms} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

Cor. If the sign of x be changed, $1 - 2x + 3x^2 - \&c.$ to n terms

$$= \frac{1 \mp (n+1)x^n \mp nx^{n+1}}{(1+x)^2},$$

where the upper or lower sign is to be used, according as n is an even or odd number.

Ex. 3. To find the sum of n terms of the series

$$1 + 3x + 5x^2 + 7x^3 + \ldots$$

Suppose $f + g$ to be the scale of relation;

then $3f + g = 5$, and $5f + 3g = 7$; hence $f = 2$, and $g = -1$;

and, by trial, it appears that the scale of relation is properly determined;

$$\text{hence } S = \frac{1+3x-2x}{1-2x+x^2} = \frac{1+x}{(1-x)^2}.$$

After n terms the series becomes

$$(2n+1)x^n + (2n+3)x^{n+1} + (2n+5)x^{n+2} + \ldots,$$

which arises from the fraction

$$\frac{(2n+1)x^n + (2n+3)x^{n+1} - 2(2n+1)x^{n+1}}{1-2x+x^2};$$

$$\text{or } \frac{(2n+1)x^n - (2n-1)x^{n+1}}{(1-x)^2};$$

hence $1 + 3x + 5x^2 + 7x^3 + \&c.$ to n terms

$$= \frac{1 + x - (2n+1)x^n + (2n-1)x^{n+1}}{(1-x)^2}.$$

Ex. 4. To find the sum of $1 + 2x + 3x^2 + 5x^3 + 8x^4 + \&c.$ *in inf.* when the series converges.

In this case the scale of relation is $1 + 1$, and consequently the sum is

$$\frac{1+2x-x}{1-x-x^2} = \frac{1+x}{1-x-x^2}.$$

If x becomes negative,

$$1 - 2x + 3x^2 - 5x^3 + \&c. \textit{ in inf.} = \frac{1-x}{1+x-x^2}.$$

Ex. 5. To find the sum of n terms of the series

$$(n-1)x + (n-2)x^2 + (n-3)x^3 + \ldots$$

The scale of relation is $2 - 1$; therefore the sum *in inf.* is

$$\frac{(n-1)x + (n-2)x^2 - 2(n-1)x^2}{(1-x)^2}, \text{ or } \frac{(n-1)x - nx^2}{(1-x)^2}.$$

After n terms the series becomes $-x^{n+1} - 2x^{n+2} - \ldots$, the sum of which is found in the same manner to be $\dfrac{-x^{n+1}}{(1-x)^2}$;

$$\therefore\ (n-1)x+(n-2)x^2+(n-3)x^3+\text{\&c. to } n \text{ terms}$$

$$=\frac{(n-1)x-nx^2+x^{n+1}}{(1-x)^2}.$$

COR. Hence $\frac{(n-1)x}{n}+\frac{(n-2)x^2}{n}+$ &c. to n terms

$$=\frac{(n-1)x-nx^2+x^{n+1}}{n(1-x)^2}.$$

Ex. 6. To find the sum of n terms of the series

$$1^2+2^2x+3^2x^2+4^2x^3+\ldots$$

Let the scale of relation be $f+g+h$; then

$$9f+4g+h=16,$$

$$16f+9g+4h=25,$$

$$25f+16g+9h=36.$$

From these equations we obtain $f=3$, $g=-3$, $h=1$, which values, when substituted, produce the successive terms of the proposed series; therefore

$$S=\frac{1+4x+9x^2-3x-12x^2+3x^2}{1-3x+3x^2-x^3}=\frac{1+x}{(1-x)^3},$$ the sum of the series *in inf.* when x is less than 1.

After the first n terms the series becomes

$$(n+1)^2x^n+(n+2)^2x^{n+1}+(n+3)^2x^{n+2}+\ldots,$$

of which the sum is

$$\frac{(n+1)^2x^n-(2n^2+2n-1)x^{n+1}+n^2x^{n+2}}{(1-x)^3};$$

and consequently the sum of n terms of the series is

$$\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}.$$

On this subject the reader may consult De Moivre's *Misc. Analyt.* p. 72; and Euler's *Analys. Infinit.* Chap. XIII.

LOGARITHMS.

394. Def. If there be a series of magnitudes

$$a^0,\ a^1,\ a^2,\ a^3,\ldots a^x;\ \ a^{-1},\ a^{-2},\ a^{-3},\ldots a^{-y},$$

the indices, $0,\ 1,\ 2,\ 3,\ldots x;\ \ -1,\ -2,\ -3,\ldots -y,$

are called the measures of the ratios of those magnitudes to 1, or the *Logarithms* of the magnitudes, for the reason assigned in Art. 230. Thus x, the *Logarithm* of any number n, is such a quantity, that $a^x = n$.

Here a may be assumed at pleasure, and is called the *base;* and for every different value so assumed a different *system* of logarithms will be formed. In the common Tabular logarithms a is 10, and consequently $0, 1, 2, 3,\ldots x$, are the logarithms of $1, 10, 100, 1000,\ldots 10^x$, in that system.

395. Cor. 1. Since the tabular logarithm of 10 is 1, the logarithm of a number between 1 and 10 is less than 1; and, in the same manner, the logarithm of a number between 10 and 100 is between 1 and 2; of a number between 100 and 1000 is between 2 and 3; &c.

These logarithms are also real quantities, to which approximation, sufficiently accurate for all practical purposes, may be made.

Thus, if x be the logarithm of 5, then $10^x = 5$; let $\frac{2}{3}$ be substituted for x, and $10^{\frac{2}{3}}$ is found to be less than 5, therefore $\frac{2}{3}$ is less than the logarithm of 5; but $10^{\frac{3}{4}}$ is greater than 5, or $\frac{3}{4}$ is greater than the logarithm of 5; thus it appears that there is a value of x between $\frac{2}{3}$ and $\frac{3}{4}$, such that $10^x = 5$; the value set down in the Tables* is 0·69897, and $10^{0\cdot69897} = 5$, nearly.

396. Cor. 2. Since $a^0 = 1$, $b^0 = 1$, &c. in *any* system the logarithm of 1 is 0. Also since $a^1 = a$, the logarithm of the base is always 1.

The method of finding the logarithms of the natural numbers, or forming a Table*, is explained in Treatises on Trigonometry.

Def. If n be any number, $\log_a n$ signifies the logarithm of n to base a; and $\log n$ the logarithm of n to *any* base.

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* Tables of Logarithms have been published in a very cheap and convenient form by Taylor and Walton, London, under the superintendence of the Society for the Diffusion of Useful Knowledge.—Ed.

397. *In the same system the sum of the logarithms of two numbers is the logarithm of their product; and the difference of the logarithms is the logarithm of their quotient.*

Let $x = \log_a n$, and $y = \log_a n'$; then $a^x = n$, and $a^y = n'$; hence $a^{x+y} = nn'$, and $a^{x-y} = \frac{n}{n'}$; or $x + y$ is $\log_a \overline{nn'}$, and $x - y$ is $\log_a \frac{n}{n'}$;

that is, $\log_a \overline{nn'} = \log_a n + \log_a n'$; and $\log_a \frac{n}{n'} = \log_a n - \log_a n'$.

Ex. 1. $\text{Log}\, \overline{3 \times 7} = \log 3 + \log 7$.

Ex. 2. $\text{Log}\, \overline{pqr} = \log \overline{pq} + \log r = \log p + \log q + \log r$.

Ex. 3. $\text{Log}\, \frac{5}{7} = \log 5 - \log 7$.

Ex. 4. $\text{Log}_{10} 0{\cdot}6 = \log_{10} \frac{6}{10} = \log_{10} 6 - \log_{10} 10 = 0{\cdot}77815 - 1$.

Ex. 5. $\text{Log}_{10} 0{\cdot}006 = \log_{10} 6 - \log_{10} 10^3 = 0{\cdot}77815 - 3$.

The last two results are usually written $\bar{1}{\cdot}77815$, $\bar{3}{\cdot}77815$.

398. *If the logarithm of a number be multiplied by* n, *the product is the logarithm of that number raised to the* nth *power.*

Let N be the number whose logarithm is x, or $a^x = N$; then $a^{nx} = N^n$; that is, nx is the log. of N^n, or $\log_a N^n = n . \log_a N$.

Exs. $\text{Log}\, (13)^5 = 5 \times \log 13$. $\text{Log}\, b^z = z \times \log b$.

Cor. $\text{Log}\, (a^m b^n c^p \ldots) = m \log a + n \log b + p \log c + \ldots$

Ex. $\text{Log} \sqrt{a^2 - x^2} = \log (\sqrt{a+x} . \sqrt{a-x}) = \frac{1}{2} \log \overline{a+x} + \frac{1}{2} \log \overline{a-x}$.

399. *If the logarithm of a number be divided by* n, *the quotient is the logarithm of the* nth *root of that number.*

Let $a^x = N$, then $a^{\frac{x}{n}} = N^{\frac{1}{n}}$, or $\frac{x}{n}$ is the log. of $N^{\frac{1}{n}}$, that is, $\log_a N^{\frac{1}{n}} = \frac{1}{n} . \log_a N$.

Exs. $\text{Log}\, 5^{\frac{1}{4}} = \frac{1}{4} \times \log 5$. $\text{Log} \sqrt{\frac{a}{b}} = \frac{1}{2} \log a - \frac{1}{2} \log b$.

400. The utility of a Table of logarithms in arithmetical calculations will from hence be manifest; the multiplication and division of numbers being performed by the addition and sub-

traction of these artificial representatives; and the involution or evolution of numbers by multiplying or dividing their logarithms by the indices of the powers or roots required.

Also the value of x, which satisfies an equation of the form $a^x = b$, may be found, since $x.\log a = \log b$, and $\therefore\ x = \dfrac{\log b}{\log a}$.

But much practice will be needed before the student will be able to make a satisfactory use of the *Tables*, and he will be required to take the subject in hand with great earnestness and determination. Every requisite direction will be found in any of the most approved treatises on Trigonometry. A few easy examples are subjoined.

Ex. 1. Required the value of $\sqrt[7]{3047}$.

Here $\log_{10}\sqrt[7]{3047} = \frac{1}{7}\log_{10}3047 = \dfrac{3\cdot48387}{7}$,

$= 0\cdot49769 = \log_{10}3\cdot1456$;

$\therefore$ $3\cdot1456$ is the root required.

Ex. 2. Let the value of $\sqrt[5]{7\sqrt{2}\times\sqrt[3]{3}}$ be required.

The log. of the proposed quantity to base 10 is

$\frac{1}{5}\{\log_{10}7 + \frac{1}{2}\log_{10}2 + \frac{1}{3}\log_{10}3\}$ (Arts. 397, 399).

And by the Tables $\log_{10}7 = 0\cdot845098$

$\frac{1}{2}\log_{10}2 = 0\cdot150515$

$\frac{1}{3}\log_{10}3 = 0\cdot1590404$

$5)\ 1\cdot1546534$

$0\cdot2309306 = \log_{10}1\cdot70188$ &c.

$\therefore$ the value required is $1\cdot70188$ &c.

Ex. 3. Find a fourth proportional to the 6th power of 9, the 4th power of 7, and the 5th power of 5.

Let x be the required number; then

$$9^6 : 7^4 :: 5^5 : x, \text{ and } x = \frac{7^4\times5^5}{9^6};$$

$$\begin{aligned}\therefore\ \log x &= 4\log 7 + 5\log 5 - 6\log 9,\\ &= 3\cdot38040 + 3\cdot49485 - 5\cdot72544, \text{ to base 10},\\ &= 1\cdot14981 = \log_{10}14\cdot12 \text{ nearly};\end{aligned}$$

$\therefore\ x = 14\cdot12$ nearly.

Ex. 4. Given $s = a.\dfrac{r^n - 1}{r - 1}$; required the value of n.

$$\text{Here} \quad ar^n = s(r-1)+a\,;$$

$$\therefore \log a + \log r^n = \log \overline{s(r-1)+a},$$

$$n.\log r = \log \overline{s(r-1)+a} - \log a,$$

$$\therefore n = \frac{\log(s.\overline{r-1}+a)-\log a}{\log r}\,;$$

from which we may obtain the number of terms in any Geometric Progression, when the first term, common ratio, and sum, are given.

Ex. 5. Given $2^x = 1976$; find the value of x.

Here $x.\log 2 = \log 1976$,

$$\therefore x = \frac{\log 1976}{\log 2} = \frac{3{\cdot}29579}{0{\cdot}30103} = 10{\cdot}9 \text{ nearly}.$$

Ex. 6. Given $a^{b^x} = c$; find x*.

$$\text{Let } b^x = y, \text{ then } x.\log b = \log y, \text{ and } x = \frac{\log y}{\log b}.$$

$$\text{Also } a^y = c, \ \therefore y.\log a = \log c, \text{ and } y = \frac{\log c}{\log a}\,;$$

$$\therefore x = \frac{\log(\log c) - \log(\log a)}{\log b}.$$

[*Exercises Zr.*]

INTEREST AND ANNUITIES.

DEF. *Interest* is the consideration paid for the use of money which belongs to another. The *rate* of interest is the consideration paid for the use of a certain sum for a certain time, as of £1 for one year.

When the interest of the *Principal* alone, or sum lent, is taken, it is called *Simple Interest;* but if the interest, as soon as it becomes due, be added to the principal, and interest be charged upon the whole, it is called *Compound Interest.*

The *Amount* is the whole sum due at the end of any time, Interest and Principal together.

Discount is the abatement made for the payment of money before it becomes due.

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* In this Ex. by a^{b^x} is meant a raised to the power expressed by b^x, and not a^b raised to the x^{th} power.

SIMPLE INTEREST.

401. *To find the Amount of a given sum, in any time, at simple interest.*

Let P be the principal, in pounds,

n the No. of years for which the interest is to be calculated*.

r the interest of 1£ for one year†.

M the amount.

Then, since the interest of a given sum, at a given rate, must be proportional to the time, 1 (year) : n (years) :: r : nr, the interest of 1£ for n years; and the interest of P£ must be P times as great, or Pnr; therefore the amount $M = P + Pnr$.

402. From this simple equation, any three of the quantities P, n, r, M being given, the fourth may be found; thus

$$P = \frac{M}{1+nr}; \quad n = \frac{M-P}{Pr}; \quad r = \frac{M-P}{Pn}.$$

Ex. What sum must be paid down to receive 600£, at the end of nine months, allowing 5 per cent. abatement? Or, which is the same thing, what principal P will in nine months amount to 600£, allowing interest at the rate of 5 per cent. per annum?

In this case $M = 600$, $n = \frac{3}{4} = 0{\cdot}75$, $r = \frac{5}{100} = 0{\cdot}05$; hence

$$P = \frac{M}{1+nr} = \frac{600}{1+0{\cdot}75 \times 0{\cdot}05} = £578{\cdot}313 = £578.\ 6s.\ 3d.$$

COMPOUND INTEREST.

403. *To find the amount of a given sum in any time at compound interest.*

Let $R = 1£$ together with its interest for a year; then at the end of the first year, R becomes the principal, or sum due.

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* When days, weeks, or months, not making an exact number of years, enter the calculation, n is fractional.—ED.

† It must always be borne in mind that r is not the rate *per cent.* but only the hundredth part of it. Thus for 4 per cent. $r = 0{\cdot}04£$, for 5 per cent. $r = 0{\cdot}05£$; and so on.—ED.

The amount at the end of the 2d year = amount of R£ in 1 year,

$$= R \times R = R^2.$$

The amount at the end of the 3d year = amount of R^2 in 1 year,

$$= R^2 \times R = R^3;$$

and so on; so that R^n is the amount of 1£ in n years. And if P£ be the principal, the amount must be P times as great, or

$$M = PR^n.$$

Cor. 1. From this equation we have

$$P = \frac{M}{R^n}, \quad n = \frac{\log M - \log P}{\log R}, \quad \text{and } R = \left(\frac{M}{P}\right)^{\frac{1}{n}}.$$

Cor. 2. The interest $= M - P = PR^n - P = P\{R^n - 1\}$.

Ex. 1. What must be paid down to receive 600£ at the end of 3 years, allowing 5 per cent. per annum compound interest?

In this case $R = 1{\cdot}05$, $n = 3$, $M = 600$;

$$\therefore\ P = \frac{M}{R^n} = \frac{600}{(1{\cdot}05)^3} = £518{\cdot}302 = £518.\ 6s.\ 0\tfrac{1}{2}d.$$

Ex. 2. Find the amount of 5£ in $2\frac{1}{2}$ years at 3 per cent., compound interest.

Here $P = 5$, $R = 1{\cdot}03$, $n = \frac{5}{2}$,

$$\therefore\ R^n = (1{\cdot}03)^{\frac{5}{2}} = (1 + 0{\cdot}03)^{\frac{5}{2}},$$

$$= 1 + \frac{5}{2} \times 0{\cdot}03 + \frac{5}{2} \cdot \frac{\frac{5}{2} - 1}{2} \times (0{\cdot}03)^2 + \ldots\ldots$$

$$= 1 + 0{\cdot}075 + 0{\cdot}0017 + \ldots\ldots$$

$$= 1 + 0{\cdot}0767 \text{ nearly};$$

$\therefore\ PR^n = 5 \times 1{\cdot}0767 = 5{\cdot}3835 = £5.\ 7s.\ 8d.$, the amount required.

404. When compound interest is named, it is usually meant that interest is payable only at the end of each year; but there may be cases in which the interest is due half-yearly, quarterly, &c.; and then the amount found in the last Article will be altered. Thus, if r be the interest of 1£ paid at the end of the year, it has been shewn that the amount of P£ at the end of n years $= P(1 + r)^n$.

But if $\frac{r}{2}$ be the interest of 1£ paid at the end of each half-year, then

$$1+\frac{r}{2} = \text{the amount of } 1£ \text{ for half a year,}$$

$$\left(1+\frac{r}{2}\right)^2 = \ldots\ldots\ldots\ldots\ldots\ldots 1 \text{ year,}$$

$$\left(1+\frac{r}{2}\right)^3 = \ldots\ldots\ldots\ldots\ldots\ldots 1\tfrac{1}{2} \text{ years,}$$

$$\left(1+\frac{r}{2}\right)^4 = \ldots\ldots\ldots\ldots\ldots\ldots 2 \text{ years;}$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left(1+\frac{r}{2}\right)^{2n} = \ldots\ldots\ldots\ldots\ldots\ldots n \text{ years;}$$

$$\therefore \text{ amount of } P£ = P\left(1+\frac{r}{2}\right)^{2n}.$$

Similarly, if $\frac{r}{4}$ be the interest of 1£ paid at the end of each quarter,

$$\text{amount of } P£ \text{ in } n \text{ years} = P\left(1+\frac{r}{4}\right)^{4n}.$$

And, generally, if interest be considered due q times a year, at equal intervals, each payment for 1£ being $\frac{r}{q}$,

$$\text{amount of } P£ \text{ in } n \text{ years} = P\left(1+\frac{r}{q}\right)^{nq}.$$

Ex. Find the amount of 100£ in 1 year at 5 per cent. *per annum*, when the interest is due, and converted into principal, at the end of each *half-year*.

$$\text{Here } P=100, \quad r=0{\cdot}05, \quad n=1,$$

$$\therefore \text{ Amount required} = 100\times(1+0{\cdot}025)^2 = 105{\cdot}0625£,$$

$$= 105£.\ 1s.\ 3d.$$

405. *Required the amount of a given sum at compound interest, the interest being supposed due every instant.*

The interest being paid q times per annum, by last Art., the amount $M=P\left(1+\frac{r}{q}\right)^{nq}$,

$$= P\left\{1+nq.\frac{r}{q}+\frac{nq(nq-1)}{1\ .\ 2}.\frac{r^2}{q^2}+\ldots\right\},$$

$$= P\left\{1+nr+\frac{n\left(n-\frac{1}{q}\right)}{1\ .\ 2}r^2+\ldots\right\}.$$

Let q be indefinitely great, that is, the intervals between the payments indefinitely small, then, neglecting $\frac{1}{q}$ and its powers,

$$M = P\left\{1 + nr + \frac{n^2r^2}{1.2} + \frac{n^3r^3}{1.2.3} + \ldots\right\},$$

$$= P\epsilon^{nr},\ \epsilon \text{ being } 2{\cdot}7182818.\quad (\text{Art. } 325.)$$

406. *To determine the advantage, when compound interest is reckoned, of having interest paid half-yearly, quarterly, &c., instead of yearly.*

It appears, from Art. 404, that the advantage per 1£ *for a year*, when interest is paid half-yearly, and the half-yearly payment is half the yearly one,

$$= \left(1 + \frac{r}{2}\right)^2 - (1 + r) = 1 + r + \frac{r^2}{4} - (1 + r),$$

$$= \frac{r^2}{4}.$$

In the case of *quarterly* payments of interest, the advantage

$$= \left(1 + \frac{r}{4}\right)^4 - (1 + r),$$

$$= 1 + r + \frac{3r^2}{8} + \frac{r^3}{16} + \frac{r^4}{256} - (1 + r),$$

$$= \frac{3r^2}{8} \text{ nearly, } \because r \text{ is a small fraction.}$$

And generally, when the interest is paid q times a year, the advantage

$$= \left(1 + \frac{r}{q}\right)^q - (1 + r),$$

$$= 1 + r + \frac{q(q-1)}{1\ .\ 2} \cdot \frac{r^2}{q^2} + \&c. - (1 + r),$$

$$= \frac{q-1}{2q} \cdot r^2 \text{ nearly.}$$

Hence it appears, that, for a single year, the advantage of having interest paid frequently is very small. But it increases as the number of years increases, and is expressed in n years, for every 1£, (when interest is paid q times a year at equal intervals, $\frac{r}{q}$ being the payment per 1£,) by

$$\left(1 + \frac{r}{q}\right)^{nq} - (1 + r)^n.$$

407. It must be observed always, when interest is paid q times per num, each payment being $\frac{r}{q}$ for every 1£, that the true *annual* rate of nterest is not r, but $\left(1 + \frac{r}{q}\right)^q - 1$, since this expresses the value of the interest for 1£ for a year.

Prob. In what time will any sum of money double itself, at any given rate of interest simple or compound?

I. In the case of simple interest $M=P+Pnr$,

$$\therefore \text{ here } 2P=P+Pnr,$$

$$\text{or } n=\frac{1}{r}, \text{ the number of years required.}$$

II. In the case of compound interest paid yearly $M=P(1+r)^n$,

$$\therefore \text{ here } 2P=P(1+r)^n,$$

$$\text{or } (1+r)^n=2\,;$$

$$\therefore n=\frac{\log 2}{\log \overline{1+r}}, \text{ the number of years required.}$$

The following Table, calculated from these two results, and shewing the several times in which any sum will double itself at the rates of interest there given, is taken from Baily's *Doctrine of Interest and Annuities*, and will furnish good practice to the learner, who will verify it by means of the Logarithmic Tables. But an *exact* verification to *eight* decimal places must not be expected, when a Table is used, which gives the logarithms to a few decimal places only.

Rate of Interest.	No. of Years at	
	Simple Interest.	Compound Interest.
2	50·	35·00278878
2½	40·	28·07103453
3	33·33333333	23·44977225
3½	28·57142857	20·14879168
4	25·	17·67298769
4½	22·22222222	15·74730184
5	20·	14·20669908
6	16·66666667	11·89566105
7	14·28571429	10·24476835
8	12·5	9·00646834
9	11·11111111	8·04323173
10	10·	7·27254090

In general practice, compound interest is only reckoned for an integral number of years, so that if there be any fractional part of a year remaining, for this simple interest is taken.

DISCOUNT.

408. Discount is defined to be "the abatement made for the payment of money before it becomes due", and although in ordinary business the *quantity* of such abatement is generally according to private contract, there is besides a true mathematical discount which affords exact justice both to the payer and to the receiver.

It is clear, that if A receives from B a sum of money n years before it is due, n being whole or fractional, A is benefited by the interest of that money for the time; and therefore, in justice, B ought to receive an abatement such, that the sum thus diminished, paid to A, would, if put out to interest until the proper time of payment arrives, amount to the sum due. This sum is called the *Present Value* of the debt.

Hence if D be the discount, and V the present value, of a debt of £P due at the expiration of n years, $V=P-D$, and V would amount at the end of n years to P, *i.e.* $V(1+nr)=P$, reckoning simple interest.

$$\text{Hence } V=\frac{P}{1+nr}, \text{ and } D=P-V=\frac{Pnr}{1+nr}.$$

Cor. 1. If n be sufficiently great that compound interest may be reckoned, then

$$V(1+r)^n=P, \quad i.e.\ V=\frac{P}{(1+r)^n};$$

$$\text{and } D=P-V=P-\frac{P}{(1+r)^n}.$$

Cor. 2. If I be the interest on £P for the given time, we shall have

$$P+I=P(1+nr), \text{ or } P(1+r)^n,$$

according as simple or compound interest is reckoned;

$$\therefore \text{ in both cases } V=P.\frac{P}{P+I};$$

$$\therefore D=P-P.\frac{P}{P+I}=\frac{PI}{P+I},$$

$$\text{or } \frac{1}{D}=\frac{1}{P}+\frac{1}{I}.$$

From this result we see that the discount on any sum is always less than the interest.

Cor. 3. Since $D=P-V$, if these be put out to interest for the time in question, we shall have

$$\begin{aligned}\text{amount of } D &= \text{amount of } P - \text{amount of } V,\\ &= \text{amount of } P-P,\\ &= \text{interest on } P.\end{aligned}$$

EQUATION OF PAYMENTS.

409. When various sums of money due at different times are to be paid, it may be required to know the time at which they may all be paid together, without injury to either debtor or creditor. To determine this time, which is called the *Equated Time*, it is clear that we must suppose the interest of the sums paid after they are due to be together equal to the discount of the sums paid before they are due, the debtor being entitled to discount for that which is paid before, and the creditor to interest for that which is paid after, it becomes due.

410. *To find the equated time of payment of two sums due at different times, reckoning simple interest.*

Let P, p, be two sums due at the end of times T, t, respectively; r the rate of interest, and x the equated time; then supposing $T > t$, the interest of p for the time $x-t$ must be equal to the discount of P for the time $T-x$, or

$$p(x-t)r = \frac{P(T-x)r}{1+(T-x)r}, \text{ (Arts. 401, 408),}$$

whence we may obtain the quadratic equation

$$x^2 - \frac{pr(T+t)+P+p}{pr}.x + \frac{prTt+PT+pt}{pr} = 0,$$

from which x may be found by the usual method.

411. Since the above method does not furnish any simple Rule, and is more complicated as the number of payments is increased, another method is generally used, although incorrect, which is founded on the supposition that the interest of the sums paid after they are due should be equal to the *interest* (not the *discount*) of those which are paid before they are due.—Thus, if P and p be the sums due at the end of times T and t, and x the equated time required,

$$p(x-t)r = P(T-x)r;$$
$$\therefore\ x = \frac{PT+pt}{P+p}.$$

Or, more generally, let P_1, P_2, P_3, &c. be the sums due after the equated time, at the end of times T_1, T_2, T_3, &c. and p_1, p_2, p_3, &c. the sums due before, at the end of times t_1, t_2, t_3, &c. then we have

$$p_1(x-t_1)r + p_2(x-t_2)r + p_3(x-t_3)r + \&c.$$
$$= P_1(T_1-x)r + P_2(T_2-x)r + P_3(T_3-x)r + \&c.$$

$$\text{and }\therefore\ x = \frac{P_1T_1+P_2T_2+P_3T_3+\&c.+p_1t_1+p_2t_2+p_3t_3+\&c.}{P_1+P_2+P_3+\&c.+p_1+p_2+p_3+\&c.},$$

which furnishes a simple rule easy of application.

By this rule a small advantage is given to the payer, because he reckons on his side the interest, instead of the discount, of those sums which he pays before they are due, whilst the opposite side of the account is confined to strict accuracy; and it has been shewn in Art. 408, Cor. 2, that the interest of any sum is greater than the discount.

412. Another method of finding the Equated time is to find the *Present Value* of each payment, and make the sum of them equal to the Present Value of the sum of the several payments supposed due at the Equated time. Thus, if P, p be due at the end of times T, t, respectively, r the rate of interest, and x be the equated time,

$$\text{Present value of } P = \frac{P}{1+Tr}, \text{ (Art. 408)},$$

$$\text{.................. of } p = \frac{p}{1+tr},$$

$$\text{.............. of } P+p = \frac{P+p}{1+xr},$$

$$\text{and } \frac{P}{1+Tr} + \frac{p}{1+tr} = \frac{P+p}{1+xr},$$

from which simple equation with respect to x we get

$$x = \frac{PT+pt+r(P+p)Tt}{P+p+r(Pt+pT)}.$$

COR. If the quantities multiplied by r be neglected, since r is generally a very small fraction, we have

$$x = \frac{PT+pt}{P+p}, \text{ which is the common rule.}$$

ANNUITIES.

413. ***To find the Amount of an annuity, or pension, left unpaid any number of years, allowing simple interest upon each sum, or pension, from the time it becomes due.***

Let A be the annuity; then at the end of the first year A becomes due, and at the end of the second year the interest of the first annuity is rA (Art. 401); at the end of this year the principal becomes $2A$, therefore the interest due at the end of the third year is $2rA$; in the same manner, the interest due at the end of the fourth year is $3rA$; &c. Hence the whole interest at the end of n years is

$$rA + 2rA + 3rA \ldots\ldots + \overline{n-1}.rA = n.\frac{n-1}{2}rA \text{ (Art. 282)};$$

and the sum of the annuities is nA, therefore the whole amount

$$M = nA + n.\frac{n-1}{2}rA.$$

420. COR. 2. If the number of years be infinite, R^n is infinite, and R^{-n} vanishes; therefore $P=\frac{A}{R-1}$, or $\frac{A}{r}$.

Ex. If the annual rent of a freehold estate be 1£, what is its value, allowing 5 per cent. per ann. compound interest?

In this case, $A=1$, $R-1=0{\cdot}05$; therefore the Present Value $P=\frac{1}{0{\cdot}05}=$ £20, or 20 years' purchase.

COR. 3. The Present Value of an Annuity of A£ payable m times per annum for n years, each of the payments being $\frac{A}{m}$, and ρ the *annual* rate of interest, will be

$$\frac{A}{m}\cdot\frac{1-(1+\rho)^{-n}}{(1+\rho)^{\frac{1}{m}}-1};$$

and if the interest also be payable q times a year, each payment of interest for every 1£ being $\frac{r}{q}$, the Present Value will be

$$\frac{A}{m}\cdot\frac{1-\left(1+\frac{r}{q}\right)^{-nq}}{\left(1+\frac{r}{q}\right)^{\frac{q}{m}}-1}.$$

COR. 4. If the annuity is to continue for ever, this Present Value becomes

$$\frac{A}{m}\cdot\frac{1}{\left(1+\frac{r}{q}\right)^{\frac{q}{m}}-1}.$$

421. The Present Value of an annuity, to commence at the expiration of p years, and to continue q years, is the difference between its present value for $\overline{p+q}$ years, and its present value for p years,

$$=\frac{A}{R-1}-\frac{AR^{-\overline{p+q}}}{R-1}-\left\{\frac{A}{R-1}-\frac{AR^{-p}}{R-1}\right\},$$

$$=\frac{AR^{-p}}{R-1}\{1-R^{-q}\}.$$

COR. If the annuity commences after p years, and continues for ever, the Present Value will be $\frac{AR^{-p}}{R-1}$.

Ex. What is the Present Value of an annuity of 1£, for 14 years, to commence at the expiration of 7 years, allowing 5 per cent. per ann. compound interest?

$$\text{The Present Value for 21 years} = \frac{(1\cdot05)^{21}-1}{(1\cdot05)^{21}\times0\cdot05} = 12\cdot82£;$$

$$\text{and the Present Value for 7 years} = \frac{(1\cdot05)^{7}-1}{(1\cdot05)^{7}\times0\cdot05} = 5\cdot79£;$$

hence the value of the annuity for fourteen years after the expiration of 7 is 7·03£, or 7 years' purchase, nearly.

The preceding Article contains the whole Theory of the

RENEWAL OF LEASES.

422. *To determine the fine which ought to be paid for renewing any number of years lapsed in a lease.*

Let $p+q$ be the number of years for which the lease was originally granted; p the number lapsed; and A the clear annual value of the estate, after deducting reserved rent (if any), taxes, and all other fixed annual charges.

Then it is clear, that the lessee has to purchase an annuity of A£ to commence at the expiration of q years, and to continue p years, the Present Value of which is the Present Value for $\overline{p+q}$ years − Present Value for q years,

$$= \frac{A}{R-1}\cdot\{R^{-q}-R^{-(p+q)}\}.$$

Cor. The number of years' purchase is

$$\frac{1}{R-1}\cdot\{R^{-q}-R^{-(p+q)}\}.$$

Ex. In a lease of 21 years 7 years lapsed are to be renewed, the reserved rent is 10£, and the estate is really worth 150£ a year, what fine ought to be paid for the renewal, reckoning interest at 5 per cent.?

In this case the lessee has to pay for an annuity of 140£ to commence at the end of 14 years and to continue 7 years; therefore the fine required is

$$\frac{140}{0\cdot05}\{\overline{1\cdot05}|^{-14}-\overline{1\cdot05}|^{-21}\}£.$$

$$\text{Now } \frac{140}{0\cdot05} = 2800,$$

$$\log_{10}\overline{1\cdot05}|^{-14} = -0\cdot29666 = \bar{1}\cdot70334 = \log_{10}0\cdot50505,$$

$$\log_{10}\overline{1\cdot05}|^{-21} = -0\cdot44499 = \bar{1}\cdot55501 = \log_{10}0\cdot35895.$$

$$\therefore \text{ the required fine} = 2800\times\{0{\cdot}50505-0{\cdot}35895\},$$
$$= 2800\times 0{\cdot}1461,$$
$$= 409£, \text{ very nearly.}$$

[*Exercises Zs.*]

SCHOLIUM.

423. The method of determining the present value of an annuity at simple interest, given in Art. 414, has been decried by several eminent Arithmeticians, and in its stead a solution of the question has been proposed upon the following principle; "If the present value of each payment be determined separately, the sum of these values must be the value of the whole annuity."

Let x be the value or price paid down for the annuity, a the yearly payment, n the number of years for which it is to be paid, r the interest of 1£ for one year. The present value of the first payment is $\frac{a}{1+r}$ (Art. 402); the present value of the second payment, or of a£ to be paid at the end of two years, is $\frac{a}{1+2r}$; and so on: therefore $x=\frac{a}{1+r}+\frac{a}{1+2r}\dots+\frac{a}{1+nr}$.

These different conclusions arise from a circumstance which the opponents seem not to have attended to. According to the former solution, no part of the interest of the price paid down is employed in paying the annuity, till the principal is exhausted.

Let the annuity be always paid out of the principal x as long as it lasts, and afterwards out of the interest which has accrued; then x, $x-a$, $x-2a$, $x-3a$, &c. are the sums in hand, during the first, second, third, fourth, &c. years, the interest arising from which rx, $rx-ra$, $rx-2ra$, $rx-3ra$, &c., that is the whole interest, is $nrx-\{1+2+3\dots(n-1)\}\times ra$, or, $nrx-n.\frac{n-1}{2}ra$, which, together with the principal x, is equal to the sum of all the annuities; therefore

$$(1+nr)x-n.\frac{n-1}{2}ra=na, \text{ and } x=\frac{na+n.\frac{n-1}{2}ra}{1+nr} \text{ (Art. 414).}$$

According to the other calculation, part of the interest, as it arises, is employed in paying the annuity, but not the whole. Thus, the first payment is made by a part of the principal, and the interest of that part, which together amount to the annuity; and the other payments are made in the same manner; this is, in effect, allowing interest upon that part of the whole interest which is incorporated with the principal. According to either calculation, the seller has the advantage, since the whole or a part of the interest will remain at his disposal till the last annuity is paid off.

If the whole interest, as it arises, be incorporated with the principal, and employed in paying the annuity, *compound interest* is, in effect, allowed upon the whole. Let x be the price paid for the annuity, n the number of years for which it is granted, and $R = 1$£ together with its interest for one year. Then x in one year amounts to Rx, out of which the annuity being paid, $Rx - a$ is the sum in hand at the end of the first year; $R^2x - Ra$ is the amount of this sum at the end of the second year, therefore $R^2x - Ra - a$ is the sum in hand at the end of the second year; in the same manner, $R^nx - R^{n-1}a - R^{n-2}a \ldots - a$ is the sum left, after paying the last annuity, which ought to be nothing; hence

$$R^nx = R^{n-1}a + R^{n-2}a + \ldots + a = \frac{R^na - a}{R - 1}.$$

$$\therefore\ x = \frac{(R^n - 1)a}{R^n \times (R - 1)}. \quad \text{(See Art. 418).}$$

CHANCES OR PROBABILITIES.

424. *Chance*, or *Probability*, has two meanings; the one a popular meaning, without any very distinct signification; the other a mathematical meaning, pointing out *a real value* existing in the circumstances.

Def. Most questions of probabilities will fall under one of two classes, called *direct* and *inverse* probabilities.

A question of probability is termed *direct*, when, certain causes being given as existent, from which a certain event may proceed, the probability of that event happening is required.

A question of probability is termed *inverse*, when, an event being given as existent, and proceeding from one of several causes, the probability of one proposed cause being the true one is required.

Some more complex questions may partake of the nature of both kinds of probability.

I. DIRECT PROBABILITIES.

425. *If an event may take place in* n *different ways, and each of these be equally likely to happen, the probability that it will take place in a specified way is properly represented by* $\frac{1}{n}$, *certainty being represented by* 1. *Or, which is the same thing, if the value of certainty be* 1, *the* value of the expectation *that the event will happen in a specified way is* $\frac{1}{n}$.

For the sum of all the probabilities is certainty, or 1, because the event must take place in some one of the ways; and the probabilities are equal: therefore each of them is $\frac{1}{n}$.

426. Cor. If the value of certainty be a, the value of the expectation is $\frac{a}{n}$. But in the following Articles we suppose the value of certainty to be 1.

427. *If an event may happen in* a *ways, and fail in* b *ways, any of these being equally probable, the chance of its happening is* $\frac{a}{a+b}$, *and the chance of its failing is* $\frac{b}{a+b}$.

The chance of its happening must, from the nature of the supposition, be to the chance of its failing, as $a : b$; therefore the chance of its happening : chance of its happening together with the chance of its failing $:: a : a+b$. And the event must either happen or fail; consequently the chance of its happening together with the chance of its failing is certainty. Hence the chance of its happening : certainty (1) $:: a : a+b$; or the chance of its happening $= \frac{a}{a+b}$.

Also, since the chance of its happening together with the chance of its failing is certainty, which is represented by 1, $1 - \frac{a}{a+b}$, that is, $\frac{b}{a+b}$ is the chance of its failing.

428. Ex. 1. The probability of throwing an ace with a single die, in one trial, is $\frac{1}{6}$; the probability of not throwing an ace is $\frac{5}{6}$; the probability of throwing either an ace or a deuce is $\frac{2}{6}$; &c.

429. Ex. 2. If n balls, a, b, c, d, &c. be thrown promiscuously into a bag, and a person draw out one of them, the probability that it will be a is $\frac{1}{n}$; the probability that it will be either a or b is $\frac{2}{n}$.

430. Ex. 3. The same supposition being made, if two balls be drawn out, the probability that these will be a and b is $\frac{2}{n(n-1)}$.

For there are $n.\frac{n-1}{2}$ combinations of n things taken two and two together (Art. 300); and each of these is equally likely to be taken; therefore the probability that a and b will be taken is $\frac{1}{n.\frac{n-1}{2}}$, or $\frac{2}{n(n-1)}$.

431. Ex. 4. If 6 white and 5 black balls be thrown promiscuously into a bag, and a person draw out one of them, the probability that this will be a white ball is $\frac{6}{11}$; and the probability that it will be a black ball is $\frac{5}{11}$.

From the Bills of Mortality in different places Tables* have been constructed which shew how many persons, upon an average, out of a certain number born, are left at the end of each year, to the extremity of life. From such Tables the probability of the continuance of a life, of any proposed age, is known.

432. Ex. 1. *To find the probability that an individual of a given age will live one year.*

Let A be the number, in the Tables, of the given age, B the number left at the end of the year; then $\frac{B}{A}$ is the probability that the individual will live one year; and $\frac{A-B}{A}$ the probability that he will die in that time (Art. 427). In Dr. Halley's Tables, out of 586 of the age of 22, 579 arrive at the age of 23; hence the probability that an individual aged 22 will live one year is $\frac{579}{586}$, or $\frac{1}{1+\frac{7}{579}}$, or $\frac{1}{1+\frac{1}{82+\frac{5}{7}}}$, or $\frac{82}{83}$ nearly; and $\frac{7}{586}$, or $\frac{1}{84}$ nearly, is the probability that he will die in that time.

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* Some of these Tables will be found at the end of the Section, pp. 299—301.—Ed.

433. Ex. 2. *To find the probability that an individual of a given age will live any number of years.*

Let A be the number in the Tables of the given age; B, C, D,...X, the number left at the end of 1, 2, 3,...x, years; then $\frac{B}{A}$ is the probability that the individual will live 1 year; $\frac{C}{A}$ the probability that he will live 2 years; and $\frac{X}{A}$ the probability tha he will live x years. Also $\frac{A-B}{A}$, $\frac{A-C}{A}$, $\frac{A-X}{A}$, are the probabilities that he will die in 1, 2, x years, respectively.

These conclusions follow immediately from Art. 427.

434. *If two events be independent of each other, the probability that they will* both *happen is the* product *of their separate probabilities.*

Suppose one event may happen in a ways, and fail in b ways,

and the other .... a_1 .... b_1 ......;

the chance of the 1st happening $=\frac{a}{a+b}$, (Art. 427),

.... 2nd .... $=\frac{a_1}{a_1+b_1}$.

Now since *any* of the $a+b$ cases may be coincident with *any* of the a_1+b_1, there will be $(a+b)(a_1+b_1)$ ways in which the cases may be combined, two together, and one from each set, all equally likely to happen.

And out of these each of the a cases may be combined with each of the a_1 cases, and so give $a\times a_1$ cases favourable to the happening of both events.

Hence the probability that both events will happen is

$$\frac{a \times a_1}{(a+b)(a_1+b_1)}, \text{ or } \frac{a}{a+b} \times \frac{a_1}{a_1+b_1},$$

the product of the separate probabilities.

COR. 1. If we put m for $1+\frac{b}{a}$, and n for $1+\frac{b_1}{a_1}$, the separate chances may be denoted by $\frac{1}{m}$, and $\frac{1}{n}$, and their product by $\frac{1}{mn}$.

435. COR. 2. The probability that both do not happen is $1-\frac{1}{mn}$, or $\frac{mn-1}{mn}$. For the probability that they both happen, together with the probability that they do not both happen, is certainty; therefore, if from 1 the probability that they both happen be subtracted, the remainder is the probability that they do not both happen.

436. COR. 3. The probability that they will both fail is $\frac{(m-1)(n-1)}{mn}$. For the probability that the first will fail is $\frac{m-1}{n}$, and the probability that the second will fail is $\frac{n-1}{n}$; therefore the probability that they will both fail is $\frac{m-1}{m} \times \frac{n-1}{n}$, or $\frac{(m-1)(n-1)}{mn}$.

437. COR. 4. The probability that one will happen and the other fail is $\frac{m+n-2}{mn}$. For the probability that the first will happen and the second fail is $\frac{1}{m} \times \frac{n-1}{n}$; and the probability that the first will fail and the second happen is $\frac{m-1}{m} \times \frac{1}{n}$; and the sum of these, or $\frac{m+n-2}{mn}$ is the probability that one will happen and the other fail.

438. COR. 5. If there be any number of independent events, and the probabilities of their happening be $\frac{1}{m}$, $\frac{1}{n}$, $\frac{1}{r}$, &c. respectively, the probability that they will all happen is $\frac{1}{mnr \text{ \&c.}}$. For

the probability that the first two will happen is $\frac{1}{mn}$, and the probability that the first two and third will happen is $\frac{1}{mnr}$; and the same proof may be extended to any number of events.

When $m=n=r=$ &c. the probability is $\frac{1}{m^v}$, v being the number of events.

439. Ex. 1. Required the probability of throwing an ace and then a deuce with one die.

The chance of throwing an ace is $\frac{1}{6}$, and the chance of throwing a *deuce in the* second trial is $\frac{1}{6}$; therefore the chance of both happening is $\frac{1}{36}$.

440. Ex. 2. If 6 white and 5 black balls be thrown promiscuously into a bag, what is the probability that a person will draw out first a white and then a black ball?

The probability of drawing a white ball first is $\frac{6}{11}$ (Art. 431), and this being done, the probability of drawing a black ball is $\frac{5}{10}$, or $\frac{1}{2}$, because there are 5 white and 5 black balls left; therefore the probability required is $\frac{6}{11}\times\frac{1}{2}$ or $\frac{3}{11}$.

Or we may reason thus:—unless the person draw a white ball first, the whole is at an end; therefore the probability that he will have a chance of drawing a black ball is $\frac{6}{11}$, and when he has this chance, the probability of its succeeding is $\frac{5}{10}$, or $\frac{1}{2}$; therefore, the probability that both these events will take place is $\frac{6}{11}\times\frac{1}{2}$ or $\frac{3}{11}$.

441. Ex. 3. The same supposition being made as in the last example, what is the chance of drawing a white ball and then two black balls?

The probability of drawing a white ball and then a black one is $\frac{3}{11}$ (Art. 440); when these two are removed, there are 5 white and 4 black balls left; and the probability of drawing a black ball, out of these, is $\frac{4}{9}$; therefore the probability required is $\frac{3}{11}\times\frac{4}{9}$, or $\frac{4}{33}$.

442. Ex. 4. Required the probability of throwing an ace, with a single die, once at least, in two trials.

The chance of failing the first time is $\frac{5}{6}$, and the chance of failing the next is $\frac{5}{6}$; therefore the chance of failing twice together is $\frac{25}{36}$; and the chance of not failing both times is $1-\frac{25}{36}$, or $\frac{11}{36}$.

443. Ex. 5. In how many trials may a person undertake, for an even wager, to throw an ace with a single die?

Let x be the number of trials; then, as in the last Art., the chance of failing x times together is $\left(\frac{5}{6}\right)^x$, and this, by the question, is equal to the chance of happening, or

$$\left(\frac{5}{6}\right)^x=\frac{1}{2};$$

$$\text{hence}\quad x\times\log\frac{5}{6}=\log\frac{1}{2};$$

$$\text{or}\quad x\times(\log 5-\log 6)=\log 1-\log 2,$$

$$x=\frac{\log 1-\log 2}{\log 5-\log 6}=\frac{\log 2}{\log 6-\log 5};\quad \text{since }\log 1=0;$$

$$\therefore\ x=3{\cdot}8,\ \text{nearly};$$

that is, a person might safely undertake to throw the ace in four trials, but not less, and then have some probability to spare in his favour.

444. Ex. 6. *To find the probability that two individuals* P *d* Q, *whose ages are known, will live a year.*

Let the probability that P will live a year, determined by Art. 432, be $\frac{1}{m}$; and the probability that Q will live a year be $\frac{1}{n}$; then the probability that they will both be alive at the end of that time is $\frac{1}{m} \times \frac{1}{n}$, or $\frac{1}{mn}$.

445. Ex. 7. *To find the probability that one of them, at least, will be alive at the end of any number of years.*

The probability that P will die in a year is $\frac{m-1}{m}$, and the probability that Q will die is $\frac{n-1}{n}$; therefore the probability that they will both die is $\frac{(m-1)(n-1)}{mn}$; and the probability that they will not both die is

$$1 - \frac{(m-1)(n-1)}{mn}, \text{ or } \frac{m+n-1}{mn}.$$

In the same manner, if $\frac{1}{p}$ be the probability that P will live t years, and $\frac{1}{q}$ the probability that Q will live the same time (Art. 433), the probability that one of them, at least, will be alive at the end of the time is

$$1 - \frac{(p-1)(q-1)}{pq}, \text{ or } \frac{p+q-1}{pq}.$$

446. *If the probability of an event's happening in one trial be represented by* $\frac{\mathrm{a}}{\mathrm{a}+\mathrm{b}}$ (Art. 427), *to find the probability of its happening once, twice, three times, &c. exactly, in* n *trials.*

The probability of its happening in any one particular trial being $\frac{a}{a+b}$, the probability of its failing in all the other $n-1$ trials is $\frac{b^{n-1}}{(a+b)^{n-1}}$ (Arts. 427, 438); therefore the probability of its happening in one particular trial, and failing in the rest, is $\frac{ab^{n-1}}{(a+b)^n}$; and since there are n trials, the probability that it will happen in some one of these, and fail in the rest, is n times as great, or $\frac{nab^{n-1}}{(a+b)^n}$.

The probability of its happening in any two particular trials, and failing in all the rest, is $\frac{a^2b^{n-2}}{(a+b)^n}$, and there are $n.\frac{n-1}{2}$ ways in which it may happen twice in n trials and fail in all the rest (Art. 300); therefore the probability that it will happen twice in n trials is $\dfrac{n.\frac{n-1}{2}a^2b^{n-2}}{(a+b)^n}$.

In the same manner, the probability of its happening exactly three times is $\dfrac{n.\frac{n-1}{2}.\frac{n-2}{3}a^3b^{n-3}}{(a+b)^n}$; and the probability of its happening exactly t times is

$$\frac{n.\frac{n-1}{2}.\frac{n-2}{3}\,\ldots\ldots\,\frac{n-t+1}{t}a^tb^{n-t}}{(a+b)^n}.$$

447. Cor. 1. The probability of the event's failing exactly t times in n trials may be shewn, in the same way, to be

$$\frac{n.\frac{n-1}{2}.\frac{n-2}{3}\,\ldots\ldots\,\frac{n-t+1}{t}a^{n-t}b^t}{(a+b)^n}.$$

448. Cor. 2. The probability of the event's happening *at least* t times in n trials is

$$\frac{a^n+na^{n-1}b+n.\frac{n-1}{2}a^{n-2}b^2+\ldots\ldots\text{to }(n-t+1)\text{ terms}}{(a+b)^n}.$$

For, if it happen every time, or fail only once, twice,...... $\overline{n-t}$, times, it happens t times; therefore the whole probability of its happening, at least t times, is the sum of the probabilities of its happening every time, of failing only once, twice,...... $\overline{n-t}$ times; and the sum of these probabilities is

$$\frac{a^n + na^{n-1}b + n\,.\,\frac{n-1}{2}\,a^{n-2}b^2 + \ldots\ldots \text{to } (n-t+1) \text{ terms}}{(a+b)^n}.$$

449. Ex. 1. What is the probability of throwing an ace twice at least, in three trials, with a single die?

In this case $n = 3$, $t = 2$, $a = 1$, $b = 5$; and the probability required is $\frac{1 + 3 \times 5}{6 \times 6 \times 6} = \frac{16}{216} = \frac{2}{27}$.

450. Ex. 2. What is the probability that out of five individuals, of a given age, three at least will die in a given time?

Let $\frac{1}{m}$ be the probability that any one of them will die in the given time (Art. 433, and 434, Cor. 1); then we have given the probability of an event's happening in one instance, to find the probability of its happening three times, at least, in five instances.

In this case $a = 1$, $b = m - 1$, $n = 5$, $t = 3$; therefore the probability required is

$$\frac{1 + 5(m-1) + 10(m-1)^2}{m^5}.$$

II. INVERSE PROBABILITIES.

451. Axiom. When an event can proceed from one of a system of causes, the probabilities of these causes having produced the event are proportional to the numbers of ways in which they can severally produce the event.

Cor. Hence the probabilities of the several causes having produced the event are proportional to the chances of the event happening on the assumption of their being severally existent.

452. *To shew that the probability of the event having proceeded from an individual cause of the system is the chance of the event, which that cause, if existent, would give, divided by the sum of the several chances which each, if existent, would give.*

If p_1, p_2, p_3, &c. be the probabilities of several causes from which an event may proceed; a_1, a_2, a_3, &c. the chances of that event happening on supposition of these causes severally existing; we have

$$\frac{p_1}{a_1} = \frac{p_2}{a_2} = \frac{p_3}{a_3} = \text{\&c.}$$

$$\therefore \frac{p_1}{a_1} = \frac{\Sigma(p)}{\Sigma(a)}. \quad \text{(Art. 195*.)}$$

But as *some one* of the system of causes is known to be the true one, $\Sigma(p)$ is certainty, or 1,

$$\therefore\ p_1 = \frac{a_1}{\Sigma(a)}.$$

$$\text{So also } p_2 = \frac{a_2}{\Sigma(a)},\quad p_3 = \frac{a_3}{\Sigma(a)},\ \&\text{c.}$$

Ex. An urn contains 3 balls which may be white or black. A ball is drawn out and replaced three times, and in each case a white ball is drawn. What are the probabilities of the urn containing (1) Three white balls, (2) Two white and one black, (3) One white and two black, (4) Three black?

(1) Supposing the 1st state of the urn, the chance of the event happening which did happen is $\frac{3}{3}\times\frac{3}{3}\times\frac{3}{3}$, or $\frac{27}{27}$, that is, certainty.

(2) Supposing the 2nd state, the chance is $\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$, or $\frac{8}{27}$.

(3) Supposing the 3rd state, the chance is $\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}$, or $\frac{1}{27}$.

(4) Supposing the 4th state, the chance is $\frac{0}{3}\times\frac{0}{3}\times\frac{0}{3}$, or 0.

$\therefore$ the chance *for* the 1st state of the urn is $\frac{27}{27}\div\frac{27+8+1}{27}$, or $\frac{27}{36}$.

The chance *for* the 2nd is $\frac{8}{27}\div\frac{36}{27}$, or $\frac{8}{36}$.

.... 3rd is $\frac{1}{27}\div\frac{36}{27}$, or $\frac{1}{36}$.

.... 4th is $\frac{0}{27}\div\frac{36}{27}$, or 0.

The following Proposition involves the nature of both *direct and inverse* probabilities :—

453. *If* a_1, a_2, a_3, *&c. be the chances of an* observed *event on supposition of each of the system of causes being the true one;* a_1', a_2', a_3', *&c. the probabilities of another* proposed *event on the same separate suppositions; the chance of this event happening* $= \frac{\Sigma(aa')}{\Sigma(a)}$.

For, as has been seen, a_1, a_2, a_3, &c. are proportional to the chances given by the observed event of the separate causes existing. Hence the number of ways of the first cause existing and the second event happening from it is proportional to a_1a_1'; and the number of ways of that event happening from some one of the causes is proportional to $\Sigma(aa')$. But since some one of the causes exists, $\Sigma(a)$ is in the same proportion to the number of ways in which one of the causes may exist, and the event either happen or fail from it. Hence the chance of the event happening $=\frac{\Sigma(aa')}{\Sigma(a)}$.

Ex. 1. An urn contains two balls, but whether white or black, uncertain—we draw one ball, and find it is white. The ball is then replaced; what is the chance of next drawing a black one?

Before the drawing takes place the two states or causes may be, (1) Two white, and (2) One white and one black.

Now the chance of the *observed* event under (1) is $\frac{2}{2}$, or 1,

.... (2) is $\frac{1}{2}$.

Again, the chance of the *proposed* event under (1) is 0,

.... (2) is $\frac{1}{2}$.

$$\therefore \text{ the Chance required} = \frac{1\times 0+\frac{1}{2}\times\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{6}.$$

Ex. 2. Taking the case supposed in Ex. Art. 452, what is the chance of a white ball coming out at a fourth such drawing?

Here before the drawing takes place the states or causes may be, (1) Three white, (2) Two white and one black, (3) One white and two black.

Now the chance of the *observed* event under (1) is 1,

.... (2) is $\frac{8}{27}$,

.... (3) is $\frac{1}{27}$.

Again, the chance of the *proposed* event under (1) is 1,

.... (2) is $\frac{2}{3}$,

.... (3) is $\frac{1}{3}$;

$$\therefore \text{ the Chance required} = \frac{1\times 1+\frac{8}{27}\times\frac{2}{3}+\frac{1}{27}\times\frac{1}{3}}{1+\frac{8}{27}+\frac{1}{27}} = \frac{98}{108} = \frac{49}{54}.$$

SCHOLIUM.

Much more might be said on a subject so extensive as the doctrine of Chances; the Learner will however find the principal grounds of calculation in Articles 425, 427, 434, 446, 448, 452, and 453; and if he wish for further information, he may consult De Moivre's work on this subject*. It may not be improper to caution him against applying principles which, on the first view, may appear self-evident; as there is no subject in which he will be so likely to mistake as in the calculation of probabilities. A single instance will shew the danger of forming a hasty judgment, even in the most simple case. The probability of throwing an ace with one die is $\frac{1}{6}$, and since there is an equal probability of throwing an ace in the second trial, it might be supposed that the probability of throwing an ace in two trials is $\frac{2}{6}$.

This is not a just conclusion (Art. 442); for, it would follow, by the same mode of reasoning, that in six trials a person could not fail to throw an ace. The error, which is not easily seen, arises from a tacit supposition that there must necessarily be a second trial, which is not the case if an ace be thrown in the first.

LIFE ANNUITIES.

454. *To find the Present Value of an annuity of £1, to be continued during the life of an individual of a given age, allowing compound interest for the money.*

Let R be the amount of £1 in one year; A the number of persons in the Tables of the given age; B, C, D, &c. the number left at the end of 1, 2, 3, &c. years; then $\frac{B}{A}$ is the value of the life for one year, $\frac{C}{A}$, $\frac{D}{A}$, &c. its value for 2, 3, &c. years respectively; and the series must be continued to the end of the Tables. Now the Present Value of £1, to be paid at the end of one year, is $\frac{1}{R}$ (Art. 408); but it is only to be paid on condition that the

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* The more modern writers on this subject are Laplace, Galloway, and De Morgan.—ED.

460. **Cor.** If the annuity be M£, the Present Value is M times as great as in the former case, or

$$M\times\left(\frac{A}{R}+\frac{B}{R^2}+\frac{C}{R^3}+\text{\&c.}\right).$$

461. These are the mathematical principles on which the values of annuities for lives are calculated, and the reasoning may easily be applied to every proposed case. But, in practice, these calculations, as they require the combination of every year of each life with the corresponding years of every other life concerned in the question, will be found extremely laborious, and other methods must be adopted when expedition is required. Writers on this subject are De Moivre, Masères, Simpson, Price, Morgan, and Waring.

Other writers on the subject are Milne, Baily, and De Morgan, of which the last mentioned is now most accessible.

462. *To find the Present Value of the next presentation to a Living.*

Let i£ be the average annual net income of the living; c£ the cost of a curate, that is, the money value of the work to be done; and a£ the unavoidable expenses of the admission of a new incumbent. Then the Present Value of the next presentation will obviously be the present value of an annuity of $(i-c)$£, to commence at the death of the present incumbent, and to continue during a life *then* 24 years of age, deducting the present value of a£ payable on admission.

Let n be the number of years which the Tables give to the present incumbent, p the number for a person 24 years of age; then the Present Value required will be that of an annuity of $(i-c)$£ to commence at the expiration of n years, and to continue p years, deducting the Present Value of a£ to be paid after n years*,

$$=\frac{i-c}{R-1}\cdot\left\{R^{-n}-R^{-\overline{n+p}}\right\}-aR^{-n}.$$

Cor. The Present Value of an *Advowson*, or perpetual nomination to a living, will be that of an Annuity of $(i-c)$£ commencing after n years, and continuing for ever, deducting the present value of a£, to be paid at the end of n years, and also of the same sum to be paid at intervals of p years for ever afterwards,

$$=\frac{i-c}{R-1}\cdot R^{-n}-aR^{-n}-aR^{-\overline{n+p}}-aR^{-\overline{n+2p}}-\ldots\ldots \textit{in inf.},$$

$$=\frac{i-c}{R-1}\cdot R^{-n}-aR^{-n}\cdot\frac{1}{1-R^{-p}}.$$

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* Of course this is on the supposition, that the laws, which permit such traffic in spiritual cures, remain in force, and that the values of i, c, and a remain unaltered, for $\overline{n+p}$ years at least.

TABLE I.

For determining the Probabilities of the Duration of Life, from Observations on the Bills of Mortality of BRESLAW, made in the years 1687...1691, by Dr. Halley.

Age.	Persons living.	Decr. of Life.	Age.	Persons living.	Decr. of Life.	Age.	Persons living.	Decr. of Life.
1	1000	145	31	523	8	61	232	10
2	855	57	32	515	8	62	222	10
3	798	38	33	507	8	63	212	10
4	760	28	34	499	9	64	202	10
5	732	22	35	490	9	65	192	10
6	710	18	36	481	9	66	182	10
7	692	12	37	472	9	67	172	10
8	680	10	38	463	9	68	162	10
9	670	9	39	454	9	69	152	10
10	661	8	40	445	9	70	142	11
11	653	7	41	436	9	71	131	11
12	646	6	42	427	10	72	120	11
13	640	6	43	417	10	73	109	11
14	634	6	44	407	10	74	98	10
15	628	6	45	397	10	75	88	10
16	622	6	46	387	10	76	78	10
17	616	6	47	377	10	77	68	10
18	610	6	48	367	10	78	58	9
19	604	6	49	357	11	79	49	8
20	598	6	50	346	11	80	41	7
21	592	6	51	335	11	81	34	6
22	586	7	52	324	11	82	28	5
23	579	6	53	313	11	83	23	4
24	573	6	54	302	10	84	19	4
25	567	7	55	292	10	85	15	4
26	560	7	56	282	10	86	11	3
27	553	7	57	272	10	87	8	3
28	546	7	58	262	10	88	5	2
29	539	8	59	252	10	89	3	2
30	531	8	60	242	10	90	1	1

TABLE II.

For determining the Probabilities of Life at NORTHAMPTON, as deduced by Dr. Price from the mortality of that town in the years 1741...1780.

Age.	Persons living.	Decr. of Life.	Age.	Persons living.	Decr. of Life.	Age.	Persons living.	Decr. of Life.
0	1149	300	31	428	7	62	187	8
1	849	127	32	421	7	63	179	8
2	722	50	33	414	7	64	171	8
3	672	26	34	407	7	65	163	8
4	646	21	35	400	7	66	155	8
5	625	16	36	393	7	67	147	8
6	609	13	37	386	7	68	139	8
7	596	10	38	379	7	69	131	8
8	586	9	39	372	7	70	123	8
9	577	7	40	365	8	71	115	8
10	570	6	41	357	8	72	107	8
11	564	6	42	349	8	73	99	8
12	558	5	43	341	8	74	91	8
13	553	5	44	333	8	75	83	8
14	548	5	45	325	8	76	75	8
15	543	5	46	317	8	77	67	7
16	538	5	47	309	8	78	60	7
17	533	5	48	301	8	79	53	7
18	528	6	49	293	9	80	46	7
19	522	7	50	284	9	81	39	7
20	515	8	51	275	8	82	32	6
21	507	8	52	267	8	83	26	5
22	499	8	53	259	8	84	21	4
23	491	8	54	251	8	85	17	4
24	483	8	55	243	8	86	13	3
25	475	8	56	235	8	87	10	2
26	467	8	57	227	8	88	8	2
27	459	8	58	219	8	89	6	2
28	451	8	59	211	8	90	4	2
29	443	8	60	203	8	91	2	1
30	435	7	61	195	8	92	1	1

The preceding Tables require but little explanation. The former commences by stating that out of 1000 persons who were born at the same time and attain the age of 1 year, 145 die before they attain the age of 2 years.

Consequently at 2 years of age there are left 855 out of 1000. Of these 57 die between 2 and 3 years of age; and so on. Thus, of 1000 persons who attain the age of one year, the Table indicates that 346 live to be 50 years of age; &c.

The latter Table commences a year earlier by taking 1149 persons born together, that is, at the age 0; and then proceeds in the same manner as the former. Thus, we have given by this Table, that after 50 years, out of 1149 persons born together, 284 are then alive, and that of these 9 die before attaining the age of 51; and so on.

It has been objected to both the preceding Tables, although the latter is very generally used by the Assurance offices, that they make no distinction between male and female life, and yet that a very material distinction can be proved to exist.

TABLE III.

Shewing the Expectation of Life, as deduced by Professor De Morgan from the Statistical Returns of the whole of Belgium made by M. Quetelet and Smits.

Age.	Town.		Both.	Country.		Age.
	Males.	Females.	Both.	Males.	Females.	
0	29·2	33·3	32·2	32·0	32·9	0
5	45·0	47·1	45·7	46·1	44·8	5
10	42·9	45·0	43·9	44·4	42·9	10
15	39·0	41·3	40·5	41·2	40·0	15
20	35·4	38·0	37·3	38·1	37·0	20
25	33·1	35·0	34·7	35·7	34·2	25
30	30·4	32·1	32·0	33·0	31·5	30
35	27·5	29·2	28·9	29·7	28·7	35
40	24·4	26·5	25·8	26·0	25·9	40
45	21·5	23·3	22·7	22·5	23·2	45
50	18·3	20·1	19·5	19·1	20·0	50
55	15·5	17·1	16·4	16·2	16·9	55
60	12·8	14·0	13·4	13·3	13·7	60
65	10·4	11·2	10·8	10·6	10·9	65
70	8·2	8·6	8·4	8·2	8·5	70
75	6·3	6·6	6·4	6·3	6·5	75
80	4·8	5·1	5·0	5·0	5·1	80
85	3·7	4·0	3·8	3·8	3·8	85
90	2·9	3·0	3·1	3·1	3·2	90
95	1·8	2·0	2·1	2·2	1·9	95
100	0·0	0·5	1·3	0·5	0·5	100

The extent of the error which arises from not distinguishing between the sexes may be seen in Table III. constructed by Professor De Morgan from the statistical returns of *the whole of Belgium* for three successive years, as given by M. Quetelet and Smits, in the *Recherches sur la Reproduction, &c.* Brussels, 1832. This Table is calculated to shew the "expectation of life," that is, the average number of years remaining to any individual, at intervals of five years, from the age of 0 to 100. It distinguishes not only between male and female, but between town life and rural life; and the middle column gives the general average for the whole kingdom, male and female, town and country.

DISCUSSION AND INTERPRETATION OF ANOMALOUS RESULTS.

463. *Negative Results.* It often happens, that the result of our operations for the solution of a proposed question or problem appears in a negative form, although strictly speaking, there can be no such thing existent as an essentially negative quantity. But it will always be found, when such a result occurs, that there is something in the nature of the question which will either dispose of, or supply a meaning to, the negative result. Thus, to take a simple example; suppose a man wishes to ascertain the amount of his property—he puts down what he has, together with what is due to him, as *positive*, and all his debts with a *negative* sign. If then he finds that by taking the sum of both positive and negative quantities, the result is *negative*, its meaning will be sufficiently obvious, viz. that his property is so much less than nothing, that is, he is so much in debt. See Scholium, p. 44; and Art. 214.

Also, see Art. 282. Ex. 4. In this and like cases it is true that two solutions may be found for the *equation*, that is, two values of n; but when either of those values is *fractional* or *negative*, it is clearly inadmissible as *a solution of the question proposed.*

It may be observed also here generally, that when in solving a *problem*, expressed algebraically, we find it necessary to extract the square root of a quantity, the double sign ±, (that is, + *or* –), need not to be prefixed to the root, at least for the object before us, if we have sufficient *data* beforehand for determining *which* sign the problem requires. Is it to be wondered at, that we produce an anomalous and unintelligible result, if we wilfully make a quantity negative which we know to be positive, or *vice versâ?*

Oftentimes, however, the negative solution, whether it results from carelessness or necessity, will satisfy *another problem* cognate with the proposed one, which may be determined by substituting the negative quantity for the positive in that step of the process which most clearly expresses the conditions of the question; and then interpreting the resulting equation with the assistance of the given problem. This has been done in the cases above referred to.

464. ***Interminable Quotients.*** Strictly speaking a *Quotient* can only exist when after the division by which it is determined there is no *Remainder;* but the term is applied to those cases also where a remainder is left which cannot be got rid of. Thus we say generally, that the quotient of $1 \div \overline{1-x}$ is $1+x+x^2+x^3+\ldots\ldots$; whereas the true quotient is $1+x+x^2+x^3+\ldots\ldots+x^{r-1}+\dfrac{x^r}{1-x}$. Thus, *whatever be the value of x,*

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots\ldots+x^{r-1}+\frac{x^r}{1-x},$$

$$=1+\frac{x}{1-x}, \text{ or } 1+x+\frac{x^2}{1-x}, \text{ or } 1+x+x^2+\frac{x^3}{1-x}; \text{ and so on.}$$

N.B. By taking the *Remainder* into account no unintelligible result can arise from substituting any particular value for x.

COR. 1. If $x<1$, then the *Remainder* may be neglected, if a sufficient number of terms of the series are taken. (Art. 290, Cor. 2.)

If $x=1$, then $\frac{1}{0}=1+1+1+1+\&c.$ *in inf.* = an infinitely great number.

COR. 2. If x be negative, we have $\dfrac{1}{1+x}=1-x+x^2-x^3+\&c.$ *in inf.;* in which if we put 1 for x, we get $\frac{1}{2}=1-1+1-1+\&c.=0$, or 1, according as an even or an odd number of terms is taken; both of which results are obviously impossible.

Now, taking the *Remainder* into account, we have

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots\ldots(-x)^r+\frac{(-x)^{r+1}}{1+x},$$

$$\text{and } \frac{1}{2}=1-1+1-1+\ldots\ldots\ldots\pm\frac{1}{2}=\frac{1}{2}, \text{ as it ought.}$$

Again, as it was shewn in Art. 323,

$$\frac{1}{(1-x)^2}=1+2x+3x^2+\ldots\ldots+rx^{r-1}+\frac{(r+1)x^r-rx^{r+1}}{(1-x)^2},$$

without which fractional *Remainder* no arithmetical equality subsists between the series and $\dfrac{1}{(1-x)^2}$. And it may be observed generally, that no equality subsists, for purposes of calculation, betwixt any infinite series without the "*Remainder*", and the primitive quantity from which it was derived, unless the series is *convergent*, so that we can make the Remainder after r terms, by increasing r, as small as we please.

465. *To explain generally the results which assume the forms* $a\times0$, $\frac{0}{a}$, a^0, 0^0, $\frac{a}{0}$, $\frac{0}{0}$, $\underline{|0}$.

(1). Since $a \times b$ signifies a taken b times, it is clear that, if 0 is to follow the same rule as other multipliers, $a \times 0$ signifies a taken 0 times, and is therefore equal to 0.

(2). Since $\frac{b}{a}$ expresses the number of times a is contained in b, $\frac{0}{a}$ will signify the number of times a is contained in 0, that is, 0 times, or $\frac{0}{a} = 0$.

(3). By the general Rule of Indices, $a^m \times a^n = a^{m+n}$, and $a^m \div a^n = a^{m-n}$. Now in the first of these let $m = 0$, then $a^0 . a^n = a^n$, if the same rule holds when one of the indices is 0; therefore a^0, as far as regards the rule for multiplication of powers, is equivalent to 1, or $a^0 = 1$.

Also, since $\frac{a^m}{a^m} = a^{m-m}$, $1 = a^0$, if the rule for division of powers holds when the powers are equal; therefore it also accords with the general rule for division that $a^0 = 1$.

Hence a^0, b^0, c^0, &c. are separately equal to 1, if 0 be admitted as an index subject to the same rule as other indices.

(4). Since it has been already explained, that *any* quantity raised to a power represented by 0 may be safely expressed by 1, it follows that $(a-b)^0 = 1$, whatever a and b may be. If then $a = b$, we have $0^0 = 1$.

(5). When an algebraical quantity is made to assume the form $\frac{a}{0}$, it is said to be infinitely great, and its value is expressed by the symbol ∞. All that is meant is, that if the denominator be made less than any assignable or appreciable quantity the fraction becomes greater than any assignable quantity. This is easily shewn by taking any fraction, as $\frac{a}{0 \cdot 1}$, which $= 10a$: for if the denominator be successively diminished one-tenth, we obtain the series of quantities $100a$, $1000a$, $10000a$, &c., proving that as the denominator of the fraction is diminished, the value of the fraction is increased, and without limit.

(6). Suppose that an expression involving x assumes the form $\frac{0}{0}$ when some particular value (a) is substituted for x, then it is clear that the expression is capable of being reduced to the form $\frac{p(x-a)^m}{q(x-a)^n}$, where p and q have no factor $x-a$ in them; and by dividing numerator and denominator by their highest common factor, the value of the fraction may be found when $x = a$. (See Art. 387.)

Thus it appears that a quantity which assumes the form $\frac{0}{0}$ may have a determinate value. And, conversely, since p is equal to $p . \frac{x-a}{x-a}$, whatever be the values of x and a; if $x = a$, $p = \frac{0}{0}$, that is, any quantity may be made to assume the form $\frac{0}{0}$. But this is, in fact, multiplying and dividing by 0, on the supposition that the rules which apply to finite quantities apply also to 0 as a multiplier.

It may be said, generally, that to speak of absolute nothing as the subject of mathematical calculation is absurd; and that it can only become so when it is taken to represent some finite quantity in that state when by indefinite diminution it has become less than any appreciable quantity. The mathematical symbol 0 has, then, always reference to some other quantity from which it is derived; and it is this primitive quantity which must be the subject of our investigations when by becoming 0 it produces an anomalous result that requires to be explained.

That mistakes will constantly arise from considering 0 as an absolute quantity is easily seen: Thus, it has been shewn that $a^0 = 1 = b^0$, therefore we might say, if 0 is a quantity, that $a = b$; or, since $2^0 = 4^0$, that $2 = 4$, both of which conclusions are manifestly wrong.

Again, if $x - 1 = 0$, then

$$x(x-1) = x \times 0 = 0, \text{ also } x = 1, \text{ and } \therefore x^2 = 1, \text{ or } x^2 - 1 = 0;$$

$$\therefore x(x-1) = 0, \text{ and } (x+1)(x-1) = 0,$$

$$\therefore x = x + 1,$$

$$\text{or } 1 = 0, \text{ which is absurd.}$$

This amounts to saying that, because $a \times 0 = 0$, and $b \times 0 = 0$, therefore $a = b$, which is obviously incorrect.

(7) By definition $\underline{|n} = n(n-1)(n-2)\ldots3.2.1 = n.\underline{|n-1}$, for all positive integral values of n; suppose, then, $n = 1$, we have $\underline{|1} = 1 \times \underline{|0}$. But $\underline{|1} = 1$; therefore the only *consistent* value we can give to $\underline{|0}$ is 1. And it is to be observed, that, with this conventional meaning of $\underline{|0}$, the expression for ${}_nC_r$ in Art. 303, when $r = n$, becomes $\frac{\underline{|n}}{\underline{|n}.\underline{|0}}$, or 1, as it ought. In like manner the *general term* in the Binomial and Multinomial Theorems will thus include the *first* and *last* terms, as well as all the rest.

466. *Given* $\mathrm{ax + b = cx + d}$, and $\therefore \mathrm{x} = \frac{\mathrm{d-b}}{\mathrm{a-c}}$, *to explain the result* (1) *when* $\mathrm{a = c}$; *and* (2) *when* $\mathrm{a = c}$, *and* $\mathrm{b = d}$.

(1) When $a = c$, $x = \frac{d-b}{0} = \infty$. In this case the proposed equation becomes $ax + b = ax + d$, which can only hold true on the supposition of x being such, that ax is not affected to any appreciable amount whichsoever of the two different quantities b or d be added to it, that is, x must be immeasurably great, agreeing with the result already found.

(2) When $a = c$, and $b = d$, $x = \frac{b-b}{a-a} = \frac{0}{0}$. In this case the original equation becomes $ax + b = ax + b$, an identity which is clearly satisfied by *any value whatever of* x; and this is the meaning of $\frac{0}{0}$ in this case.

467. *Given* $\left.\begin{array}{r}\mathrm{ax+by=c}\\ \mathrm{a'x+b'y=c'}\end{array}\right\}$, *so that* $\mathrm{x}=\dfrac{\mathrm{b'c-bc'}}{\mathrm{ab'-a'b}}$, $\mathrm{y}=\dfrac{\mathrm{ac'-a'c}}{\mathrm{ab'-a'b}}$; *explain the results when* $\mathrm{a'=ma,\ b'=mb,\ c'=mc}$.

Here $\dfrac{a'}{a}=m=\dfrac{b'}{b}=\dfrac{c'}{c}$, or $a'b=ab'$, $ac'=a'c$, $b'c=bc'$; $\therefore x=\dfrac{0}{0}$, and $y=\dfrac{0}{0}$. From the original equation we have $ax+by=c=\dfrac{c'}{m}=\dfrac{a'}{m}x+\dfrac{b'}{m}y=ax+by$, an identity which is satisfied by any values whatever of x and y; agreeing with the results before obtained.

468. *Given* $\mathrm{ax^2+bx+c=0}$, *and* $\therefore \mathrm{x}=\dfrac{\mathrm{-b\pm\sqrt{b^2-4ac}}}{\mathrm{2a}}$; *explain this result when* $\mathrm{a=0}$.

When $a=0$, one value of x becomes $\dfrac{0}{0}$, the other $\dfrac{-2b}{0}$ or ∞. From the original equation, putting $\dfrac{1}{y}$ for x, we have

$$a+by+cy^2=0,$$

$$\begin{aligned} \text{and if } a=0,\ by+cy^2&=0, \quad\text{or } y(b+cy)=0,\\ \therefore y&=0, \quad\text{or } b+cy=0,\\ \frac{1}{x}&=0, \quad\text{or } b+\frac{c}{x}=0,\\ x&=\infty; \quad\text{or } x=-\frac{c}{b},\end{aligned}$$

which is the value of $\dfrac{0}{0}$ in this case, as may be easily shewn. Thus

$$\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{4ac}{2a(-b-\sqrt{b^2-4ac})}=\frac{2c}{-b-\sqrt{b^2-4ac}}=-\frac{c}{b},\ \text{when } a=0.$$

469. *A* and *B* are travelling along the same road, and in the same direction, at a uniform pace of a miles and b miles per hour respectively. Given that at a known time *B* is d miles before *A*, find the time when they are together; and explain the result (1) when $a=b$, and (2) when $a<b$.

Let x be the number of hours from the known time to the time when they are together. Then in that time *A* travels ax miles, and *B* travels bx miles, and by the supposition

$$ax-d=bx,$$

$$\therefore x=\frac{d}{a-b}.$$

(1) Let $a=b$, then $x=\dfrac{d}{0}=\infty$; that is, *A* can never overtake *B*; which is also evident from the circumstances; because, if *A* and *B* are once d miles apart and travel in the same direction *at the same pace*, they must always be d miles apart, and can *never* come together.

(2) Let $a<b$, then x is negative, which signifies that the time of their being together is past. For as A travels more slowly than B, it is evident they cannot at any *future* time come together, because the farther they go the farther they are apart. But as by looking forward in time the distance between them keeps increasing, so by looking backward (supposing the journey continuous in that direction) that distance continually diminishes, and $\frac{d}{b-a}$ hours *ago* it was 0, that is, A and B were then together.

MAXIMA AND MINIMA.

470. There is a class of problems which require for their solution to determine the greatest or smallest values which an algebraical expression will admit of by the variation of some quantity or quantities contained in it. These problems are called *Maxima* and *Minima* Problems. Thus,

PROB. 1. Required to divide a given magnitude, $2a$, into two such parts that the product of the two parts may be the greatest possible.

Let $a+x$ be one part,

then $a-x$ is the other.

The product is a^2-x^2.

Let y be the greatest value required,

$$\text{then } y=a^2-x^2,$$

$$\text{and } x=\pm\sqrt{a^2-y}.$$

Now, that x may have a real value, y cannot be greater than a^2, but may be equal to a^2, which is therefore its greatest value. Hence, in that case, $x=0$, and the two parts of $2a$ required are *equal* to each other.

PROB. 2. Required the minimum of $\frac{a^2x^2+b^2}{(a^2-b^2)x}$, when $a>b$.

$$\text{Let } \frac{a^2x^2+b^2}{(a^2-b^2)x}=y, \text{ then}$$

$$a^2x^2+b^2=(a^2-b^2)yx,$$

$$x^2-\frac{(a^2-b^2)}{a^2}yx+\frac{(a^2-b^2)^2}{4a^4}y^2=\frac{(a^2-b^2)^2}{4a^4}y^2-\frac{b^2}{a^2};$$

$$\therefore\ x=\frac{a^2-b^2}{2a^2}.y\pm\frac{1}{2a^2}\sqrt{(a^2-b^2)^2y^2-4a^2b^2}.$$

Now, that x may have a real value, $(a^2-b^2)^2y^2$ must not be less than $4a^2b^2$, but it may be equal to it, or $y=\frac{2ab}{a^2-b^2}$, which is therefore its *minimum*, or least value.

Hence also, in this case, $x=\frac{a^2-b^2}{2a^2}\cdot\frac{2ab}{a^2-b^2}=\frac{b}{a}$.

471. If the quantity under the radical sign remain positive whatever value be given to y, then the proposed quantity will admit of neither a *maximum* nor a *minimum*.

Ex. Required to determine whether $\frac{4x^2+4x-3}{6(2x+1)}$ admits of either a *maximum* or a *minimum*.

$$\text{Let } \frac{4x^2+4x-3}{6(2x+1)}=y, \text{ then}$$

$$4x^2-4(3y-1)x=6y+3,$$

$$\text{and } x=\frac{3y-1}{2}\pm\frac{1}{2}\sqrt{9y^2+4}.$$

Now, whatever value may be given to y, the quantity under the root will always be positive; therefore the proposed expression does not admit of any *maximum* or *minimum*.

472. If the quantity under the root be of the form my^2+ny+p, then by solving the equation $my^2+ny+p=0$, we can find the greatest or least value of y which will permit $\sqrt{my^2+ny+p}$ to be real, and therefore the required *maximum* or *minimum*.

Ex. 1. Let a and b be two quantities of which $a>b$; required the greatest value of $\frac{(x+a)(x-b)}{x^2}$. Ans. *Maximum* $=\frac{(a+b)^2}{4ab}$; and $x=\frac{2ab}{a-b}$.

Ex. 2. Required the smallest value of $\frac{(a+x)(b+x)}{x}$.

Ans. *Minimum* $=(\sqrt{a}+\sqrt{b})^2$; and $x=\sqrt{ab}$.

472*. Sometimes the introduction of another symbol may be dispensed with: thus, to find the minimum value of x^2+px+q, we see that it may be written as $\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}$, and therefore it has its least value when $\left(x+\frac{p}{2}\right)^2$ is the smallest possible. But this is when $\left(x+\frac{p}{2}\right)^2=0$, *i.e.* when $x=-\frac{p}{2}$, and then the given expression becomes $q-\frac{p^2}{4}$, which is consequently the minimum sought.

Problems of this kind, however, are usually solved by the [illegible] of the Differential Calculus, although ordinary Algebra is mostly sufficient for the purpose. See a remarkably ingenious treatise on the subject by Ramchundra, late Teacher of Science at Delhi College*.

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* *Maxima* and *Minima* by Ramchundra, late Teacher of Science at Delhi College, translated into English, with a Preface by Professor De Morgan, 8vo. London, 1859.

APPLICATION OF ALGEBRA TO GEOMETRY.

473. The signs made use of in algebraical calculations being general, the conclusions obtained by their assistance may, with great ease and convenience, be transferred from abstract magnitudes to every class of particular quantities; thus, the relations of lines, surfaces, or solids, may generally be deduced from the principles of Algebra, and many properties of these quantities discovered, which could not have been derived from principles purely geometrical.

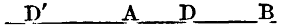
474. *Simple algebraical quantities may be represented by lines.*

Any line, AB, may be taken at pleasure to represent one quantity a; but if we have a second quantity, b, to represent, we must take a line which has to the former line the same ratio that b has to a.

Instead of saying AB represents a, we may say $AB = a$, supposing AB to contain as many linear units as a contains numeral ones.

475. *When a series of algebraical quantities is to be represented on one line, and each of them measured from the same point, the positive quantities being represented by lines taken in one direction, the negative quantities must be represented by lines taken in the opposite direction.*

Let a be the greatest of these quantities, then $a - x$ may, by the variation of x, become equal to each of them in succession. Let AB be the given line, and A the point from which the quantities are to be measured; take AB equal to a; and since $a - x$ must be measured from A,



BD must be taken in the contrary direction equal to x, then $AD = a - x$; and that $a - x$ may successively coincide with each quantity in the series, beginning with the greatest positive quantity, x must increase; therefore BD, which is equal to x, must increase; and when x is greater than a, BD is greater than AB, and AD', which represents the negative quantity $a - x$, lies in the opposite direction from A.

Cor. 1. If the algebraical value of a line be found to be negative, the line must be measured in a direction opposite to that which, in the investigation, we supposed to be positive.

Cor. 2. If quantities be measured upon a line from its intersection with another, the positive quantities being taken in one direction, the negative quantities must be taken in the other.

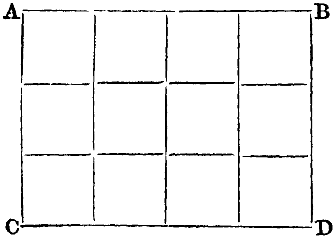
476. If a fourth proportional to lines representing p, q, r, be taken, it will represent $\frac{qr}{p}$; and if $p = 1$, it will represent qr; if also q and r be equal, it will represent q^2.

477. If a mean proportional between lines representing a and b be taken, it will represent $\sqrt{ab}$, which, when $a = 1$, becomes $\sqrt{b}$.

Hence it appears that any possible algebraical quantities may be represented by lines; and conversely, lines may be expressed algebraically; and if the relations of the algebraical quantities be known, the relations of the lines are known.

478. *The area of a rectangular parallelogram may be measured by the product of the two numbers which measure two adjacent sides.*

Let the sides AB, AC of the rectangular parallelogram AD be measured by the lineal quantities a, b, respectively; then $a \times b$ will express the number of superficial units in the area, that is, the number of squares it contains, each of which is described upon a lineal unit. For instance, if the lineal unit be a foot, of which AB contains a, and AC contains b, the parallelogram AD contains $a \times b$ square feet.



For, 1st, if AB, AC be divided into lineal units, and straight lines be drawn through the points of division parallel to the sides, the whole figure is made up of squares, which are equal to each other, and to the square upon the lineal unit; and the number of them is evidently a taken b times, or $a \times b$.

2ndly. If AB and AC do not contain the lineal unit an exact number of times, that is, if a and b be fractional, let $a = \alpha + \frac{\mu}{m}$, and $b = \beta + \frac{\nu}{n}$.

Then take another lineal unit which is $\frac{1}{mn}$th part of the former; and by what has been shewn the square described upon the larger unit contains $mn \times mn$ of that described upon the smaller. Again, the sides *AB*, *AC* respectively contain $mn\alpha + \mu n$, $mn\beta + \nu m$ lineal units of the smaller kind, and therefore, by the first case, the whole figure contains $(mn\alpha + \mu n) \times (mn\beta + \nu m)$ square units of the smaller kind; that is, the area

$$= m^2n^2\left\{\alpha + \frac{\mu}{m}\right\}.\left\{\beta + \frac{\nu}{n}\right\} \text{ of the smaller units,}$$

$$= \left(\alpha + \frac{\mu}{m}\right)\left(\beta + \frac{\nu}{n}\right) \text{ of the larger units,}$$

$$= ab, \text{ as before.}$$

3dly. If the sides *AB*, *AC* be *incommensurable* with the lineal unit, a unit may be found which is commensurable with certain lines that approach *as near as we please* to *AB*, and *AC*, and therefore the product of such lines will represent the area of a rectangle differing from the rectangle *AD* by a quantity less than any that can be assigned, that is, we may, in this case also, without error express *AD* by $AB \times AC$. (See Art. 260.)

479. COR. 1. Since, by *Euclid*, Book I. Prop. 35, the area of an oblique-angled parallelogram is equal to that of the rectangular parallelogram upon the same base and between the same parallels, therefore

area of *any* parallelogram = base × altitude.

480. COR. 2. Also, since by *Euclid*, Book I. Prop. 41, the area of a triangle is half that of the parallelogram upon the same base and between the same parallels, therefore

area of *any* triangle $= \frac{1}{2} \times$ base × altitude.

481. COR. 3. Since any rectilineal figure may be divided into triangles, its area may be found by taking the sum of all the triangles.

482. *The solid content, or volume, of a rectangular parallelopiped may be measured by the product of the three numbers which measure its length, breadth, and height.*

Let the base of the parallelopiped be divided into its component squares, as in the preceding Proposition, and through each of the parallels suppose planes drawn at right angles to the base; and let the same thing be done with one of the faces adjacent to the base. Then it is evident, that the whole figure is divided into a certain number of equal cubes, each cube having for its face one of the squares described upon the lineal unit; (that is, if the lineal unit be a foot, each of these cubes will have its length, breadth, and height equal to a foot, and is called a cubic foot). Now the *number* of these cubes is manifestly equal to the number of squares in the base taken as many times as there are lineal units in the height; therefore

content, or volume = base×height,
= length×breadth×height. (Art. 478.)

COR. *Any* three of these quantities being given, the fourth may be found. Thus, if C be the content, l the length, b the breadth, and h the height, we have

$$l=\frac{C}{bh},\quad b=\frac{C}{hl},\quad h=\frac{C}{bl}.$$

The following Examples will sufficiently illustrate the preceding Theory, for our present purpose :—

Ex. 1. *If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.* (EUCLID, Book II. Prop. 7.)

Let a and b represent the two parts into which the given line is divided; then $a+b$ is the whole line,

and $(a+b)^2+a^2=$ the squares of the whole line, and of one of the parts;

$$=a^2+2ab+b^2+a^2,$$

$$=2a^2+2ab+b^2,$$

$$=2(a+b)a+b^2,$$

= twice the rectangle &c., together with the square of the other part. Q.E.D.

Ex. 2. *To find the radius of a circle inscribed in a given triangle.*

See Euclid's diagram, Book IV. Prop. 4; let r be the radius of the inscribed circle, and a, b, c, the sides of the triangle respectively opposite to the angles A, B, C. Then (Art. 480)

$$\frac{1}{2}r.a+\frac{1}{2}r.b+\frac{1}{2}r.c=\text{whole area of the triangle},$$

or $\frac{1}{2}r(a+b+c)=\frac{1}{2}a\times$ the perpend^r (p) upon a from the opposite angle;

$$\text{and } \therefore\ r=\frac{a}{a+b+c}\cdot p.$$

To find p, let the segments into which a is divided by it be x and $a-x$; then (EUCLID, Book I. Prop. 47)

$$c^2-x^2=p^2=b^2-(a-x)^2,$$

$$c^2=b^2-a^2+2ax;$$

$$\therefore\ x=\frac{a^2+c^2-b^2}{2a},$$

$$\text{and } p^2=c^2-\left(\frac{a^2+c^2-b^2}{2a}\right)^2=\frac{(2ac)^2-(a^2+c^2-b^2)^2}{4a^2},$$

$$=\frac{(a+b+c)(a+c-b)(a+b-c)(b+c-a)}{4a^2};$$

$$\therefore\ p=\frac{1}{2a}\sqrt{(a+b+c)(a+c-b)(a+b-c)(b+c-a)},$$

$$\text{and } r=\frac{1}{2}\sqrt{\frac{(a+c-b)(a+b-c)(b+c-a)}{a+b+c}}.$$

Ex. 3. *To find the area of the square inscribed in a given circle; and also of the square circumscribed about a given circle.*

(1) Let r be the radius of the circle; then (see EUCLID, Book IV. Prop. 6)

$$r^2+r^2=AD^2,$$

$$\text{or } 2r^2=\text{inscribed square.}$$

(2) Again, $r^2+r^2+r^2+r^2$, or $4r^2$ = circumscribed square,
= twice the inscribed square.

Ex. 4. *To find the area of the equilateral and equiangular hexagon inscribed in a given circle.*

Let r be the radius of the given circle, then (EUCLID, Book IV. Prop. 15) also r = the side of the inscribed hexagon; and the area of the hexagon = the sum of the areas of the six equal equilateral triangles of which it is composed,

$$=6\times\frac{1}{2}r.\sqrt{r^2-\frac{r^2}{4}}=\frac{3\sqrt{3}}{2}.r^2.$$

Ex. 5. *The depth of water in a cistern (whose form is a rectangular parallelopiped) is* h *feet, and the base contains* a *square feet. Find* (1) *the number of cubic feet of water; and* (2) *the depth of the same quantity of water in another cistern whose base contains* b *square inches.*

(1) The quantity of water = base × depth = ah cubic feet.

(2) The depth for 2nd cistern $=\dfrac{\text{quantity of water (in cubic feet)}}{\text{base (in square feet)}}$,

$$=ah\div\frac{b}{144}=\frac{144ah}{b}\text{ feet.}$$

It is not necessary here to multiply Examples, because the subject is now of sufficient importance to form a separate treatise, to which, in the regular course of reading, the student's attention will be hereafter directed.

APPENDIX.

EQUATIONS.

In the infinite variety of Equations which ingenious persons may put together, it is not to be supposed that any general Rules can be laid down for every operation necessary to their solution. Indeed the best method of solution is frequently that which lies under the least obligation to general Rules; as may be seen at once by comparing Exs. 1, 2, and 3 below with Art. 194. Numberless are the artifices by which algebraic calculation may be abridged, and their successful application can be learnt by practice only. There are, however, peculiar artifices of more frequent occurrence than others, which shall be exhibited in the following Examples.

Ex. 1. $\frac{4x-17}{9}-\frac{3\frac{2}{3}-22x}{33}=x-\frac{6}{x}\left(1-\frac{x^2}{54}\right)$; find x.

$$\frac{4x}{9}-\frac{17}{9}-\frac{1}{9}+\frac{2x}{3}=x-\frac{6}{x}+\frac{x}{9},\ \because 3\tfrac{2}{3}\div 33=\frac{1}{9},$$

$$\left(\frac{4}{9}+\frac{6}{9}-\frac{1}{9}\right)x=x-\frac{6}{x}+\frac{18}{9},$$

$$\frac{9x}{9},\ \text{or}\ x=x-\frac{6}{x}+2,$$

$$\frac{6}{x}=2,$$

$$\therefore\ x=3.$$

Ex. 2. $\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}=\frac{8x-30}{2x-7}+\frac{5x-4}{x-1}$; find x.

$$\frac{4(x-4)-1}{x-4}+\frac{5(2x-3)+2}{2x-3}=\frac{4(2x-7)-2}{2x-7}+\frac{5(x-1)+1}{x-1},$$

$$4-\frac{1}{x-4}+5+\frac{2}{2x-3}=4-\frac{2}{2x-7}+5+\frac{1}{x-1},$$

$$\frac{2}{2x-3}-\frac{1}{x-4}=\frac{1}{x-1}-\frac{2}{2x-7},$$

$$\frac{-5}{2x^2-11x+12}=\frac{-5}{2x^2-9x+7},$$

$$2x^2-9x+7 = 2x^2-11x+12,$$

$$2x = 5,$$

$$\therefore\ x = \frac{5}{2} = 2\tfrac{1}{2}.$$

Ex. 3. $\dfrac{3x}{2}-\dfrac{81x^2-9}{(3x-1)(x+3)} = 3x-\dfrac{3}{2}\cdot\dfrac{2x^2-1}{x+3}-\dfrac{57-3x}{2}$; find x.

$$\frac{3x}{2}-\frac{9(3x+1)}{x+3} = 3x-\frac{3x^2-\frac{3}{2}}{x+3}-\frac{57}{2}+\frac{3x}{2},$$

transpos^g. and divid^g. by 3 $\Big\}$, $\dfrac{x^2-\frac{1}{2}-9x-3}{x+3} = x-9\tfrac{1}{2}$,

$$x^2-9x-3\tfrac{1}{2} = x^2-6\tfrac{1}{2}.x-28\tfrac{1}{2},$$

$$2\tfrac{1}{2}.x = 25;$$

$$\therefore\ x = \frac{25}{2\frac{1}{2}} = 10.$$

Ex. 4. $\dfrac{2}{19}\left(\sqrt{x^2+39x+374}-\sqrt{x^2+20x+51}\right) = \sqrt{\dfrac{x+22}{x+17}}$; find x.

$$\sqrt{x^2+39x+374}-\frac{19}{2}\sqrt{\frac{x+22}{x+17}} = \sqrt{x^2+20x+51}.$$

Squaring, and observing that $x^2+39x+374 = (x+22)(x+17)$,

$$x^2+39x+374-19(x+22)+\frac{361}{4}\cdot\frac{x+22}{x+17} = x^2+20x+51,$$

$$\frac{361}{4}\cdot\frac{x+22}{x+17} = 51+(19-17)\times 22 = 95,$$

$$\frac{x+22}{x+17} = \frac{4}{361}\times 95 = \frac{20}{19},$$

Art. 195, $\dfrac{2x+39}{5} = \dfrac{39}{1}$,

$$2x = 4\times 39,$$

$$\therefore\ x = 2\times 39 = 78.$$

Ex. 5. $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}} = b^2$; find x.

By Art. 195, $\dfrac{a+x}{\sqrt{2ax+x^2}} = \dfrac{b^2+1}{b^2-1}$,

$$\frac{a^2+2ax+x^2}{2ax+x^2} = \frac{a^2}{2ax+x^2}+1 = \left(\frac{b^2+1}{b^2-1}\right)^2,$$

$$\frac{a^2}{2ax+x^2}=\left(\frac{b^2+1}{b^2-1}\right)^2-1=\frac{4b^2}{(b^2-1)^2},$$

$$\frac{2ax+x^2}{a^2}=\frac{(b^2-1)^2}{4b^2},$$

$$\left(\frac{a+x}{a}\right)^2=\frac{(b^2-1)^2}{4b^2}+1=\frac{(b^2+1)^2}{4b^2},$$

$$\frac{a+x}{a}=\frac{b^2+1}{2b},$$

$$\frac{x}{a}=\frac{b^2+1-2b}{2b}=\frac{(b-1)^2}{2b};$$

$$\therefore\ x=\frac{a}{2b}(b-1)^2.$$

Ex. 6. $\dfrac{a-x+\sqrt{2ax-x^2}}{a-x}=b$; find x.

$$1+\frac{\sqrt{2ax-x^2}}{a-x}=b,$$

$$\frac{2ax-x^2}{a^2-2ax+x^2}=(b-1)^2,$$

$$\left(\frac{a}{a-x}\right)^2-1=(b-1)^2,$$

$$\left(\frac{a}{a-x}\right)^2=1+(b-1)^2,$$

$$\frac{a}{a-x}=\sqrt{1+(b-1)^2},$$

$$\frac{a-x}{a}=\frac{1}{\sqrt{1+(b-1)^2}},$$

$$\frac{x}{a}=1-\frac{1}{\sqrt{1+(b-1)^2}};$$

$$\therefore\ x=a-\frac{a}{\sqrt{1+(b-1)^2}}.$$

Ex. 7. $\dfrac{243+324\sqrt{3x}}{16x-3}=(4\sqrt{x}-\sqrt{3})^2$; find x.

$$\frac{81\sqrt{3}.(\sqrt{3}+4\sqrt{x})}{(4\sqrt{x}+\sqrt{3})(4\sqrt{x}-\sqrt{3})}=(4\sqrt{x}-\sqrt{3})^2,$$

$$\frac{81\sqrt{3}}{4\sqrt{x}-\sqrt{3}}=(4\sqrt{x}-\sqrt{3})^2,$$

$$(4\sqrt{x}-\sqrt{3})^3 = 81\sqrt{3},$$
$$4\sqrt{x}-\sqrt{3} = 3\sqrt{3},$$
$$4\sqrt{x} = 4\sqrt{3},$$
$$\therefore\ x = 3.$$

Ex. 8. $\sqrt[3]{a+x}+\sqrt[3]{a-x} = b$; find x.

$$(\sqrt[3]{a+x}+\sqrt[3]{a-x})^3 = 2a+3\sqrt[3]{a^2-x^2}.(\sqrt[3]{a+x}+\sqrt[3]{a-x}),$$

and $\sqrt[3]{a+x}+\sqrt[3]{a-x}$ by the supposition is equal to b;

$\therefore$ cubing both sides of the proposed equation,

$$2a+3\sqrt[3]{a^2-x^2}.b = b^3,$$
$$3b\sqrt[3]{a^2-x^2} = b^3-2a,$$
$$\sqrt[3]{a^2-x^2} = \frac{b^2}{3}-\frac{2a}{3b},$$
$$a^2-x^2 = \left(\frac{b^2}{3}-\frac{2a}{3b}\right)^3,$$
$$x^2 = a^2-\left(\frac{b^2}{3}-\frac{2a}{3b}\right)^3,$$
$$\therefore\ x = \sqrt{a^2-\left(\frac{b^2}{3}-\frac{2a}{3b}\right)^3}.$$

Ex. 9. $x-1 = 2+\frac{2}{\sqrt{x}}$; find x.

$$x-1 = \frac{2}{\sqrt{x}}(\sqrt{x}+1),$$

$\therefore$ dividing by $\sqrt{x}+1$, $\sqrt{x}-1 = \frac{2}{\sqrt{x}}$,

$\therefore$ $\sqrt{x}+1 = 0$, or $\sqrt{x} = -1$, and $\therefore$ $x = 1$. (Art. 208.)

Also $x-\sqrt{x} = 2$,

$$x-\sqrt{x}+\frac{1}{4} = 2+\frac{1}{4} = \frac{9}{4},$$
$$\sqrt{x}-\frac{1}{2} = \pm\frac{3}{2},$$
$$\sqrt{x} = \frac{1\pm 3}{2} = 2,\ \text{or } -1,$$
$$\therefore\ x = 4,\ \text{or } 1.$$

Ex. 10. $x^3-3x=2$; find x.

$$x^3-x=2x+2,$$

$$x(x^2-1)=2(x+1),$$

$$\therefore\ x+1=0,\quad \text{or } x=-1.\quad \text{(Art. 208.)}$$

$$\text{Also } x(x-1)=2,\quad \text{or } x^2-x=2,$$

$$x^2-x+\frac{1}{4}=\frac{9}{4},$$

$$x-\frac{1}{2}=\pm\frac{3}{2},$$

$$\therefore\ x=\frac{1\pm3}{2}=2,\ \text{or } -1.$$

Ex. 11. $x^2-\dfrac{2}{3x}=1\tfrac{4}{9}$; find x.

$$x^2-\frac{4}{9}=1+\frac{2}{3x},$$

$$\left(x+\frac{2}{3}\right)\left(x-\frac{2}{3}\right)=\frac{1}{x}\left(x+\frac{2}{3}\right),$$

$$\therefore\ x+\frac{2}{3}=0,\quad \text{or } x=-\frac{2}{3}\ldots\ldots\ldots\ldots\text{(I)}.$$

$$\text{Also } x-\frac{2}{3}=\frac{1}{x},$$

$$x^2-\frac{2}{3}x=1,$$

$$x^2-\frac{2}{3}x+\left(\frac{1}{3}\right)^2=1+\frac{1}{9}=\frac{10}{9},$$

$$x-\frac{1}{3}=\pm\frac{\sqrt{10}}{3},$$

$$\therefore\ x=\frac{1\pm\sqrt{10}}{3}\ldots\ldots\ldots\ldots\text{(II)}.$$

Ex. 12. $(x+a+\sqrt{x^2+2ax+b^2})^3+(x+a-\sqrt{x^2+2ax+b^2})^3=14(x+a)^3$.

Observing that $(A+B)^3+(A-B)^3=2A^3+6AB^2$, we have

$$2(x+a)^3+6(x+a)(x^2+2ax+b^2)=14(x+a)^3;$$

$$\therefore\ x+a=0,\quad \text{or } x=-a.\quad \text{(Art. 208.)}\ldots\ldots\ldots\ldots\text{(I)}.$$

$$\text{Also}\quad 6(x^2+2ax+b^2)=12(x+a)^2,$$

$$x^2+2ax+b^2=2x^2+4ax+2a^2,$$

$$x^2+2ax+a^2 = b^2-a^2,$$

$$x+a = \pm\sqrt{b^2-a^2};$$

$$\therefore\ x = \pm\sqrt{b^2-a^2}-a\ldots\ldots\ldots(\text{II}).$$

Ex. 13. $(1+x+x^2)^2 = \dfrac{a+1}{a-1}\cdot(1+x^2+x^4)$; find x.

$$1+x+x^2 = \frac{a+1}{a-1}\cdot\frac{1+x^2+x^4}{1+x+x^2},$$

$$= \frac{a+1}{a-1}\cdot\frac{1+x+x^2-x(1-x^3)}{1+x+x^2},$$

$$= \frac{a+1}{a-1}(1-x.\overline{1-x});$$

$$\therefore\ \frac{1+x+x^2}{1-x+x^2} = \frac{a+1}{a-1},$$

$$\frac{1+x^2}{x} = \frac{a}{1},$$

$$x^2-ax = -1,$$

$$x^2-ax+\frac{a^2}{4} = \frac{a^2}{4}-1;$$

$$\therefore\ x = \frac{a}{2}\pm\sqrt{\frac{a^2}{4}-1}.$$

Ex. 14. $b\left(\dfrac{a-b}{x}+1\right)\left(\dfrac{a-2b}{x}+1\right) = \dfrac{a^2}{x}-a$; find x.

$$b(a-b+x)(a-2b+x) = a^2x-ax^2,$$

$$b(a-b)(a-2b)+b(2a-3b)x+bx^2 = a^2x-ax^2,$$

$$(a+b)x^2-(a^2-2ab+3b^2)x+b(a-b)(a-2b) = 0,$$

$$(a+b)x^2-\{(a-b)(a-2b)+b(a+b)\}x+b(a-b)(a-2b) = 0,$$

$$x\{(a+b)x-(a-b)(a-2b)\}-b\{(a+b)x-(a-b)(a-2b)\} = 0,$$

$$(a+b)x-(a-b)(a-2b) = 0,\ (\text{Art. } 208)$$

$$\therefore\ x = \frac{a-b}{a+b}(a-2b)\ldots\ldots\ldots\ldots\ldots(\text{I}).$$

$$\text{Also } x = b\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{II}).$$

Ex. 15. $\left(\dfrac{x+a}{x+b}\right)^{\frac{1}{2}}-\dfrac{a-b}{2(x+c)} = 1$; find x.

$$\frac{x+a}{x+b} = 1+\frac{a-b}{x+c}+\frac{(a-b)^2}{4(x+c)^2},$$

$$\frac{x+a}{x+b}-1, \text{ or } \frac{a-b}{x+b}=\frac{a-b}{x+c}+\frac{(a-b)^2}{4(x+c)^2},$$

$$\frac{1}{x+b}-\frac{1}{x+c}, \text{ or } \frac{c-b}{(x+b)(x+c)}=\frac{a-b}{4(x+c)^2},$$

$$\frac{c-b}{x+b}=\frac{a-b}{4(x+c)},$$

$$\{a-b-4(c-b)\}x=4c(c-b)-b(a-b);$$

$$\therefore\ x=\frac{4c(c-b)-b(a-b)}{a+3b-4c}.$$

Ex. 16. $\dfrac{x+a}{x+b}=\left(\dfrac{2x+a+c}{2x+b+c}\right)^2$; find x.

$$1+\frac{a-b}{x+b}=\left(1+\frac{a-b}{2x+b+c}\right)^2,$$

$$=1+\frac{2(a-b)}{2x+b+c}+\frac{(a-b)^2}{(2x+b+c)^2},$$

$$\frac{1}{x+b}-\frac{2}{2x+b+c}=\frac{a-b}{(2x+b+c)^2},$$

$$\frac{c-b}{x+b}=\frac{a-b}{2x+b+c},$$

$$2cx-2bx-b^2+c^2=ax-bx+ab-b^2,$$

$$(a+b-2c)x=c^2-ab;$$

$$\therefore\ x=\frac{c^2-ab}{a+b-2c}.$$

Ex. 17. $\sqrt[m]{(1+x)^2}-\sqrt[m]{(1-x)^2}=\sqrt[m]{1-x^2}$; find x.

Converting the roots into fractional indices, the equation is

$$(1+x)^{\frac{2}{m}}-(1-x)^{\frac{2}{m}}=(1-x^2)^{\frac{1}{m}}=(1+x)^{\frac{1}{m}}(1-x)^{\frac{1}{m}}.$$

Dividing by $(1-x)^{\frac{2}{m}}$, $\left(\dfrac{1+x}{1-x}\right)^{\frac{2}{m}}-1=\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{m}}$,

$$\therefore\ \left(\frac{1+x}{1-x}\right)^{\frac{2}{m}}-\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}+\frac{1}{4}=1+\frac{1}{4}=\frac{5}{4},$$

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}-\frac{1}{2}=\pm\frac{\sqrt{5}}{2},$$

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}=\frac{1\pm\sqrt{5}}{2},$$

$$\frac{1+x}{1-x}=\frac{(1\pm\sqrt{5})^m}{2^m};$$

$$\therefore\ x=\frac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}.\quad \text{(Art. 195.)}$$

Ex. 18. $\frac{1+x^4}{(1+x)^4}=a$; find x.

$$1+x^4=a(1+x)^4,$$
$$=a(1+4x+6x^2+4x^3+x^4),$$
$$(1-a)(1+x^4)=4a(x+x^3)+6ax^2,$$

dividing by x^2, $(1-a)\left(x^2+\frac{1}{x^2}\right)=4a\left(x+\frac{1}{x}\right)+6a,$

$$x^2+\frac{1}{x^2}-\frac{4a}{1-a}\left(x+\frac{1}{x}\right)=\frac{6a}{1-a},$$

$$\left(x+\frac{1}{x}\right)^2-\frac{4a}{1-a}\left(x+\frac{1}{x}\right)=\frac{6a}{1-a}+2=\frac{2+4a}{1-a},$$

$$\left(x+\frac{1}{x}\right)^2-\frac{4a}{1-a}\left(x+\frac{1}{x}\right)+\left(\frac{2a}{1-a}\right)^2=\frac{2+4a}{1-a}+\frac{4a^2}{(1-a)^2}=\frac{2(1+a)}{(1-a)^2},$$

$$\therefore\ x+\frac{1}{x}-\frac{2a}{1-a}=\pm\frac{\sqrt{2(1+a)}}{1-a},$$

$$x+\frac{1}{x}=\frac{2a\pm\sqrt{2(1+a)}}{1-a}=2p,\ \text{suppose};$$

$$\therefore\ x^2-2px=-1,$$
$$x^2-2px+p^2=p^2-1,$$
$$x-p=\pm\sqrt{p^2-1};$$
$$\therefore\ x=p\pm\sqrt{p^2-1}.$$

Ex. 19. $x+a+3\sqrt[3]{abx}=b$; find x.

Assume $\sqrt[3]{x}+\sqrt[3]{a}=\sqrt[3]{y}$,

then cubing these equals, $x+a+3\sqrt[3]{ax}.(\sqrt[3]{x}+\sqrt[3]{a})=y$;

$$\therefore\ x+a+3\sqrt[3]{ayx}=y.$$

Now, comparing this with the original equation, it appears that $y=b$;

$$\therefore\ \sqrt[3]{x}+\sqrt[3]{a}=\sqrt[3]{b},$$
$$\sqrt[3]{x}=\sqrt[3]{b}-\sqrt[3]{a},$$
$$\therefore\ x=(\sqrt[3]{b}-\sqrt[3]{a})^3.$$

Ex. 20. $x^3-1=0$; find all the values of x.

$$x^3-1=(x-1)(x^2+x+1)=0,$$

$$\therefore\ x-1=0, \text{ or } x=1 \ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}.$$

Also $x^2+x+1=0$,

$$x^2+x+\frac{1}{4}=\frac{1}{4}-1=-\frac{3}{4}$$

$$x+\frac{1}{2}=\pm\frac{\sqrt{-3}}{2};$$

$$\therefore\ x=\frac{-1\pm\sqrt{-3}}{2}\ldots\ldots\ldots\ldots\text{(II)}.$$

Ex. 21. $x^4+1=0$; find all the values of x.

Dividing by x^2, $x^2+\frac{1}{x^2}=0$,

$$x^2+2+\frac{1}{x^2}=2,$$

$$x+\frac{1}{x}=\pm\sqrt{2},$$

$$x^2\mp\sqrt{2}.x=-1,$$

$$x^2\mp\sqrt{2}.x+\frac{1}{2}=\frac{1}{2}-1=-\frac{1}{2},$$

$$x\mp\frac{1}{\sqrt{2}}=\frac{\pm\sqrt{-1}}{\sqrt{2}};$$

$$\therefore\ x=\frac{\pm1\pm\sqrt{-1}}{\sqrt{2}}.$$

In a similar manner to that of the last two Examples may the roots of $x^5-1=0$, $x^5+1=0$, $x^6+1=0$, and like equations, be easily found.

Ex. 22. $(x-a)\sqrt{x}-(x+a)\sqrt{b}=b(\sqrt{x}-\sqrt{b})$; find x.

$$x\sqrt{x}-a\sqrt{x}-x\sqrt{b}-a\sqrt{b}=b(\sqrt{x}-\sqrt{b}),$$

$$a(\sqrt{x}+\sqrt{b})=(x-b)(\sqrt{x}-\sqrt{b})=(\sqrt{x}+\sqrt{b})(\sqrt{x}-\sqrt{b})^2,$$

$$\therefore\ \sqrt{x}+\sqrt{b}=0,\ \sqrt{x}=-\sqrt{b}, \text{ and } x=b\ldots\ldots\ldots\ldots\ldots\text{(I)}.$$

Also $\sqrt{x}-\sqrt{b}=\sqrt{a}$,

$$\sqrt{x}=\sqrt{a}+\sqrt{b};\ \therefore\ x=(\sqrt{a}+\sqrt{b})^2\ldots\ldots\ldots\ldots\text{(II)}.$$

Ex. 23. $a+x+\sqrt{2ax+x^2}=\sqrt{ax-x^2}+\sqrt{2a^2-ax-x^2}$; find x.

$$a+x-\sqrt{ax-x^2}=\sqrt{2a^2-ax-x^2}-\sqrt{2ax+x^2},$$

$$=\sqrt{2a+x}.\{\sqrt{a-x}-\sqrt{x}\},$$

$$a^2+3ax-2(a+x)\sqrt{ax-x^2}=(2a+x)\{a-2\sqrt{ax-x^2}\},$$

$$2a\sqrt{ax-x^2}=a^2-2ax,$$

$$2\sqrt{ax-x^2}=a-2x,$$

$$4ax-4x^2=a^2-4ax+4x^2,$$

$$8x^2-8ax=-a^2,$$

$$x^2-ax=-\frac{a^2}{8},$$

$$x^2-ax+\frac{a^2}{4}=\frac{2a^2}{16},$$

$$x-\frac{a}{2}=\pm\frac{a}{4}\sqrt{2};$$

$$\therefore\ x=\frac{a}{2}\{1\pm\frac{1}{2}\sqrt{2}\}.$$

Ex. 24. $a+(b+\sqrt{x})\sqrt{x}=(b-\sqrt{x})\sqrt{2a+x}$; find x.

$$a+x+\sqrt{2ax+x^2}=b(\sqrt{2a+x}-\sqrt{x}),$$

$$(a+x+\sqrt{2ax+x^2})^2=b^2(2a+2x-2\sqrt{2ax+x^2}),$$

$$=2b^2(a+x-\sqrt{2ax+x^2}),$$

$$(a+x+\sqrt{2ax+x^2})^3=2b^2.\{\overline{a+x}|^2-\overline{2ax+x^2}\},$$

$$=2a^2b^2,$$

$$a+x+\sqrt{2ax+x^2}=\sqrt[3]{2a^2b^2},$$

$$2ax+x^2=ab\sqrt[3]{4ab}-2\sqrt[3]{2a^2b^2}(a+x)+a^2+2ax+x^2,$$

$$a+x=\frac{a^2}{2\sqrt[3]{2a^2b^2}}+\frac{ab}{2}\sqrt[3]{\frac{2}{ab}};$$

$$\therefore\ x=\frac{a}{2}\sqrt[3]{\frac{a}{2b^2}}+\frac{ab}{2}\sqrt[3]{\frac{2}{ab}}-a.$$

Ex. 25. $\dfrac{2a\sqrt{1+x^2}}{1-x+\sqrt{1+x^2}}=a+b$; find x.

$$2a\sqrt{1+x^2}=(a+b)(1-x)+(a+b)\sqrt{1+x^2},$$

$$(a-b)\sqrt{1+x^2} = (a+b)(1-x),$$

$$\frac{1-x}{\sqrt{1+x^2}} = \frac{a-b}{a+b},$$

$$\frac{1+x^2-2x}{1+x^2} = \left(\frac{a-b}{a+b}\right)^2,$$

$$\frac{2x}{1+x^2} = 1-\frac{(a-b)^2}{(a+b)^2} = \frac{4ab}{(a+b)^2},$$

$$\frac{(1+x)^2}{(1-x)^2} = \frac{(a+b)^2+4ab}{(a+b)^2-4ab} = \frac{(a-b)^2+8ab}{(a-b)^2},$$

$$\frac{1+x}{1-x} = \sqrt{1+\frac{8ab}{(a-b)^2}};$$

$$\therefore\ x = \frac{\sqrt{1+\frac{8ab}{(a-b)^2}}-1}{\sqrt{1+\frac{8ab}{(a-b)^2}}+1} = \frac{(a-b)^2}{8ab}.\left\{\sqrt{1+\frac{8ab}{(a-b)^2}}-1\right\}^2.$$

Ex. 26. $\dfrac{x-\sqrt{x^2-a^2}}{\sqrt{x+\sqrt{x^2-a^2}}} = \sqrt[4]{x^2-a^2}.\{\sqrt{x^2+ax}-\sqrt{x^2-ax}\}$; find x.

Squaring, $\dfrac{(x-\sqrt{x^2-a^2})^2}{x+\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}.2x\{x-\sqrt{x^2-a^2}\}$,

$$\frac{x-\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}} = 2x\sqrt{x^2-a^2},$$

$$\frac{(x-\sqrt{x^2-a^2})^2}{a^2} = 2x\sqrt{x^2-a^2},$$

$$2x^2-a^2-2x\sqrt{x^2-a^2} = 2a^2x\sqrt{x^2-a^2},$$

$$2x^2-a^2 = 2x(a^2+1)\sqrt{x^2-a^2},$$

$$4x^4-4a^2x^2+a^4 = 4a^4x^4+8a^2x^4+4x^4-4a^6x^2-8a^4x^2-4a^2x^2,$$

$$a^4 = 4a^4x^4+8a^2x^4-4a^6x^2-8a^4x^2,$$

$$a^2 = 4(a^2+2)x^4-4a^2(a^2+2)x^2,$$

$$x^4-a^2x^2 = \frac{a^2}{4(a^2+2)},$$

$$x^4-a^2x^2+\frac{a^4}{4} = \frac{a^2}{4}.\frac{a^4+2a^2+1}{a^2+2},$$

$$x^2 = \frac{a}{2}\left\{a\pm\frac{a^2+1}{\sqrt{a^2+2}}\right\},$$

$$= \frac{a}{4}.\frac{\pm 2a\sqrt{a^2+2}+2a^2+2}{\sqrt{a^2+2}},$$

$$= \frac{a}{4}.\frac{(a\pm\sqrt{a^2+2})^2}{\sqrt{a^2+2}};$$

$$\therefore\ x = \pm\frac{\sqrt{a}}{2}.\left\{\frac{a}{\sqrt[4]{a^2+2}} \pm \sqrt[4]{a^2+2}\right\}.$$

Ex. 27. $x-2\sqrt{x+2} = 1+\sqrt[4]{x^3-3x+2}$; find x.

$$x-1-2\sqrt{x+2} = (x^2-2x+1)^{\frac{1}{4}}.(x+2)^{\frac{1}{4}},$$

$$= \sqrt{x-1}.\sqrt[4]{x+2},$$

$$x-1-\sqrt{x+2} = \sqrt[4]{x+2}.\{\sqrt{x-1}+\sqrt[4]{x+2}\};$$

$$\therefore\ \sqrt{x-1}+\sqrt[4]{x+2} = 0;\ \text{and}\ \sqrt{x-1}-\sqrt[4]{x+2} = \sqrt[4]{x+2};$$

$$\text{or}\ \sqrt{x-1} = -\sqrt[4]{x+2};\ \text{and}\ \sqrt{x-1} = 2\sqrt[4]{x+2},$$

$$x^2-2x+1 = x+2,$$

$$x^2-3x+\left(\frac{3}{2}\right)^2 = \frac{13}{4},$$

$$x-\frac{3}{2} = \frac{\pm\sqrt{13}}{2};$$

$$\therefore\ x = \frac{3\pm\sqrt{13}}{2}\ldots(\text{I}).$$

$$x^2-2x+1 = 16x+32,$$

$$x^2-18x+81 = 112,$$

$$x-9 = \pm 4\sqrt{7};$$

$$\therefore\ x = 9\pm 4\sqrt{7}\ldots(\text{II}).$$

Ex. 28. $\dfrac{2x^2+1+x\sqrt{4x^2+3}}{2x^2+3+x\sqrt{4x^2+3}} = a$; find x.

$$1-\frac{2x^2+1+x\sqrt{4x^2+3}}{2x^2+3+x\sqrt{4x^2+3}} = \frac{2}{2x^2+3+x\sqrt{4x^2+3}} = 1-a,$$

$$2x^2+3+x\sqrt{4x^2+3} = \frac{2}{1-a},$$

$$x\sqrt{4x^2+3} = \frac{2}{1-a}-3-2x^2 = \frac{3a-1}{1-a}-2x^2,$$

$$4x^4+3x^2 = \left(\frac{3a-1}{1-a}\right)^2-4.\frac{3a-1}{1-a}x^2+4x^4,$$

$$\left(\frac{12a-4}{1-a}+3\right)x^2 = \left(\frac{3a-1}{1-a}\right)^2,$$

$$(9a-1)x^2 = \frac{(3a-1)^2}{1-a};$$

$$\therefore\ x = \pm\frac{3a-1}{\sqrt{(1-a)(9a-1)}}.$$

Ex. 29. $\dfrac{(a^2-1)a+a^2x-x\sqrt{2a^2-1}}{(a^2-1)a+a^2x+x\sqrt{2a^2-1}}=(1-a^2)(a+x)^2-2ax$; find x.

Multiplying the numerator and denominator of the fraction by the denominator, the numerator becomes

$$\{(a^2-1)a+a^2x\}^2-x^2(2a^2-1),$$

$$\text{or } (a^2-1)^2a^2+2a^3x(a^2-1)+a^4x^2-2a^2x^2+x^2,$$

$$\text{or } (a^2-1)^2a^2+2a^3x(a^2-1)+x^2(a^2-1)^2,$$

$$\text{or } (a^2-1)\{a^2(a^2-1)+2a^3x+x^2(a^2-1)\},$$

$$\text{or } (a^2-1)\{a^4+2a^3x+a^2x^2-a^2-x^2\},$$

$$\text{or } (a^2-1)\{a^2(a+x)^2-(a^2+x^2)\}.$$

$$\begin{aligned}\text{Also}\quad (1-a^2)(a+x)^2-2ax &= (a+x)^2-2ax-a^2(a+x)^2,\\ &= a^2+x^2-a^2(a+x)^2,\\ &= -\{a^2(a+x)^2-(a^2+x^2)\}.\end{aligned}$$

$$\text{Hence}\quad \frac{a^2-1}{(\text{denom}^{\text{r}}.)^2}=-1;$$

$$\therefore\ \text{denom}^{\text{r}}.=\pm\sqrt{1-a^2},$$

$$\text{or } (a^2-1)a+a^2x+x\sqrt{2a^2-1}=\pm\sqrt{1-a^2},$$

$$(a^2+\sqrt{2a^2-1})x=\pm\sqrt{1-a^2}.\{1\pm a\sqrt{1-a^2}\};$$

$$\therefore\ x=\pm\sqrt{1-a^2}.\frac{1\pm a\sqrt{1-a^2}}{a^2+\sqrt{2a^2-1}}.$$

Ex. 30. $\dfrac{1-ax+\sqrt{1+a^2}-a\sqrt{1+x^2}}{1-ax+\sqrt{1+x^2}-x\sqrt{1+a^2}}=a$; find x.

$$1-ax+\sqrt{1+a^2}-a\sqrt{1+x^2}=a(1-ax)+a\sqrt{1+x^2}-ax\sqrt{1+a^2},$$

$$(ax-1)(a-1)+(ax+1)\sqrt{1+a^2}=2a\sqrt{1+x^2},$$

$$\begin{aligned}(a^2x^2-2ax+1)(a-1)^2+2(a-1)\sqrt{1+a^2}.(a^2x^2-1)&\\ +(a^2x^2+2ax+1)(1+a^2)&=4a^2+4a^2x^2,\end{aligned}$$

$$2(a^2x^2+1)(1+a^2)-2a(ax-1)^2+2(a-1)\sqrt{1+a^2}.(a^2x^2-1)=4a^2+4a^2x^2,$$

$$(a^2x^2+1)(1+a^2)-a(ax-1)^2+(a-1)\sqrt{1+a^2}(a^2x^2-1)=2a^2+2a^2x^2,$$

$$a^2x^2+1+a^4x^2+a^2-a(ax-1)^2+(a-1)\sqrt{1+a^2}(a^2x^2-1)=2a^2+2a^2x^2,$$

$$a^2x^2(a^2-1)-(a^2-1)-a(ax-1)^2+(a-1)\sqrt{1+a^2}(a^2x^2-1)=0,$$

$$(a^2-1)(a^2x^2-1)-a(ax-1)^2+(a-1)\sqrt{1+a^2}.(a^2x^2-1)=0;$$

$$\therefore\ ax-1=0,\ \text{or } x=\frac{1}{a}\text{.........................(1).}$$

Also $(a^2-1)(ax+1)-a(ax-1)+(a-1)\sqrt{1+a^2}(ax+1) = 0$,

or $\{a^2-1+(a-1)\sqrt{1+a^2}-a\}ax = -\{a^2-1+(a-1)\sqrt{1+a^2}+a\}$,

$$-ax = \frac{a^2+a-1+(a-1)\sqrt{1+a^2}}{a^2-a-1+(a-1)\sqrt{1+a^2}},$$

$$= 1+\frac{2a}{a^2-a-1+(a-1)\sqrt{1+a^2}},$$

$$= 1+\frac{2a\{a^2-a-1-(a-1)\sqrt{1+a^2}\}}{(a^2-a-1)^2-(a-1)^2(a^2+1)},$$

$$= 1+\frac{2a\{a^2-a-1-(a-1)\sqrt{1+a^2}\}}{a(4-3a)};$$

∴ by reduction, $x = \dfrac{(a-1-\sqrt{1+a^2})^2-3a}{a(3a-4)}$ .... (II).

Ex. 31. $2x\sqrt{1-x^4} = a(1+x^4)$; find x.

Squaring, $4x^2-4x^6 = a^2+2a^2x^4+a^2x^8$,

dividing by a^2x^4, $\dfrac{4}{a^2x^2}-\dfrac{4x^2}{a^2} = \dfrac{1}{x^4}+2+x^4$,

$$\text{or } -\frac{4}{a^2}\left(x^2-\frac{1}{x^2}\right) = \left(x^2+\frac{1}{x^2}\right)^2,$$

$$\left(x^2+\frac{1}{x^2}\right)^2+\frac{4}{a^2}\left(x^2-\frac{1}{x^2}\right) = 0,$$

$$\left(x^2-\frac{1}{x^2}\right)^2+\frac{4}{a^2}\left(x^2-\frac{1}{x^2}\right) = -4,$$

$$\left(x^2-\frac{1}{x^2}\right)^2+\frac{4}{a^2}\left(x^2-\frac{1}{x^2}\right)+\left(\frac{2}{a^2}\right)^2 = \frac{4}{a^4}-4,$$

$$x^2-\frac{1}{x^2}+\frac{2}{a^2} = \pm\frac{2}{a^2}\sqrt{1-a^4},$$

$x^2-\dfrac{1}{x^2} = -\dfrac{2}{a^2}\{1\mp\sqrt{1-a^4}\} = 2b^2$, suppose; then

$$x^4-2b^2x^2 = 1,$$

$$x^4-2b^2x^2+b^4 = 1+b^4,$$

$$x^2 = b^2\pm\sqrt{1+b^4};$$

$$\therefore\ x = \pm\sqrt{b^2\pm\sqrt{1+b^4}}.$$

$$\text{But } b^2 = -\frac{1}{a^2}\{1 \mp \sqrt{1-a^4}\},$$

$$1+b^4 = 1+\frac{1}{a^4}\{2-a^4 \mp 2\sqrt{1-a^4}\} = \frac{2}{a^4}\{1 \mp \sqrt{1-a^4}\};$$

$$\therefore\ x = \pm\frac{1}{a}\{\sqrt{-1 \pm \sqrt{1-a^4} \pm \sqrt{2(1 \mp \sqrt{1-a^4})}}\},$$

or, by reduction, $x = \pm\frac{1}{a}\{(\sqrt{1+a^2}-1)(\sqrt{1-a^2}+1)\}^{\frac{1}{2}}$.

Ex. 32. $\left(x-\frac{1}{3}\right)^2-\frac{25}{9} = \dfrac{3x^2+\frac{4}{9}}{2\left(x-\frac{1}{3}\right)+\sqrt{x\left(x-\frac{8}{3}\right)}}$; find x.

$$x^2-\frac{2x}{3}-\frac{24}{9} = \frac{\left(3x^2+\frac{4}{9}\right).\left\{2\left(x-\frac{1}{3}\right)-\sqrt{x\left(x-\frac{8}{3}\right)}\right\}}{4\left(x^2-\frac{2x}{3}+\frac{1}{9}\right)-x^2+\frac{8x}{3}} = \frac{\text{Num}^{\text{r}}.}{3x^2+\frac{4}{9}},$$

$$= 2\left(x-\frac{1}{3}\right)-\sqrt{x\left(x-\frac{8}{3}\right)},$$

$$x^2-\frac{8x}{3}+\sqrt{x^2-\frac{8x}{3}} = 2,$$

$$x^2-\frac{8x}{3}+\sqrt{x^2-\frac{8x}{3}}+\frac{1}{4} = 2+\frac{1}{4} = \frac{9}{4},$$

$$\sqrt{x^2-\frac{8x}{3}} = \pm\frac{3}{2}-\frac{1}{2} = 1, \text{ or } -2.$$

$$x^2-\frac{8x}{3}+\left(\frac{4}{3}\right)^2 = \frac{16}{9}+1,$$

$$x-\frac{4}{3} = \pm\frac{5}{3};$$

$$\therefore\ x = \frac{4\pm5}{3} = 3, \text{ or } -\frac{1}{3}.$$

$$\text{or } x^2-\frac{8x}{3}+\left(\frac{4}{3}\right)^2 = \frac{16}{9}+4 = \frac{52}{9},$$

$$x-\frac{4}{3} = \frac{\pm2\sqrt{13}}{3};$$

$$\therefore\ x = \frac{4\pm2\sqrt{13}}{3}.$$

Ex. 33. $\dfrac{(n-1)(a^4+a^2x^2+x^4)}{(n+1)(a^4-a^2x^2+x^4)} = \left(2-\frac{1}{n}\right).\left(\frac{ax}{a^2-x^2}\right)^2$; find x.

$$\frac{a^4+a^2x^2+x^4}{a^4-a^2x^2+x^4}.\left(\frac{a^2-x^2}{ax}\right)^2 = \frac{2n-1}{n}.\frac{n+1}{n-1},$$

$$\frac{\frac{a^2}{x^2}+1+\frac{x^2}{a^2}}{\frac{a^2}{x^2}-1+\frac{x^2}{a^2}}.\left(\frac{a}{x}-\frac{x}{a}\right)^2 = \frac{2n-1}{n}.\frac{n+1}{n-1}.$$

For $\frac{a^2}{x^2}+\frac{x^2}{a^2}$ write z, then

$$\frac{z+1}{z-1}.(z-2) = \frac{2n^2+n-1}{n^2-n},$$

$$(n^2-n)(z^2-z-2) = (2n^2+n-1)(z-1),$$

$$(n^2-n)z^2-(3n^2-1)z+3n-1 = 0,$$

from which we find $z = \frac{3n-1}{n-1}$, or $\frac{1}{n}$,

$$\therefore \left(\frac{x}{a}+\frac{a}{x}\right)^2 = z+2 = \frac{5n-3}{n-1}, \text{ or } \frac{1}{n}+2,$$

and $\left(\frac{x}{a}-\frac{a}{x}\right)^2 = z-2 = \frac{n+1}{n-1}$, or $\frac{1}{n}-2$;

$$\therefore x = \frac{a}{2}\left\{\sqrt{\frac{5n-3}{n-1}}+\sqrt{\frac{n+1}{n-1}}\right\},$$

$$\text{or } \frac{a}{2}\left\{\sqrt{\frac{1}{n}+2}+\sqrt{\frac{1}{n}-2}\right\}.$$

Ex. 34. $\left(\frac{2x+3}{2x-3}\right)^{\frac{1}{3}}+\left(\frac{2x-3}{2x+3}\right)^{\frac{1}{3}} = \frac{8}{13}\cdot\frac{4x^2+9}{4x^2-9}$; find x.

Cubing, $\left(\frac{8}{13}\right)^3\left(\frac{4x^2+9}{4x^2-9}\right)^3 = \frac{2x+3}{2x-3}+\frac{2x-3}{2x+3}+3\times\frac{8}{13}\cdot\frac{4x^2+9}{4x^2-9}$,

$$= \frac{2(4x^2+9)+\frac{24}{13}(4x^2+9)}{4x^2-9},$$

$$= \frac{50}{13}\cdot\frac{4x^2+9}{4x^2-9},$$

$$\left(\frac{4x^2+9}{4x^2-9}\right)^2 = 50\times\frac{(13)^2}{8^3} = \frac{25\times(13)^2}{2^8},$$

$$\frac{4x^2+9}{4x^2-9} = \frac{5\times13}{2^4} = \frac{65}{16},$$

$$\frac{4x^2}{9} = \frac{81}{49},$$

$$\frac{2x}{3} = \pm\frac{9}{7};$$

$$\therefore x = \pm\frac{27}{14} = \pm1\tfrac{13}{14}.$$

Ex. 35. $a^2x^2-b^2-2x(a-cx)\sqrt{\frac{a^2-b^2}{1-x^2}+\left(\frac{a-cx}{1-x^2}\right)^2}+\frac{1+x^2}{1-x^2}(a-cx)^2 = 0.$

$$a^2x^2(1-x^2)-b^2(1-x^2)+(a-cx)^2(1+x^2)-2x(a-cx)\sqrt{(a^2-b^2)(1-x^2)+(a-cx)^2} = 0,$$

$$(a^2-b^2)(1-x^2)-(a^2-a^2x^2)(1-x^2)+(a-cx)^2+(a-cx)^2x^2$$
$$-2x(a-cx)\sqrt{(a^2-b^2)(1-x^2)+(a-cx)^2}=0,$$

$$(a^2-b^2)(1-x^2)+(a-cx)^2-2x(a-cx)\sqrt{(a^2-b^2)(1-x^2)+(a-cx)^2}$$
$$+x^2(a-cx)^2=a^2(1-x^2)^2,$$

$$\sqrt{(a^2-b^2)(1-x^2)+(a-cx)^2}-x(a-cx)=\pm a(1-x^2),$$

$$(a^2-b^2)(1-x^2)+(a-cx)^2=x^2(a-cx)^2+a^2(1-x^2)^2\pm 2ax(a-cx)(1-x^2),$$

$$(a^2-b^2)(1-x^2)+(a-cx)^2(1-x^2)=a^2(1-x^2)^2\pm 2ax(a-cx)(1-x^2),$$

$$a^2-b^2+(a-cx)^2=a^2(1-x^2)\pm 2ax(a-cx),$$

$$(a-cx)^2\mp 2ax(a-cx)+a^2x^2=b^2,$$

$$a-cx\mp ax=\pm b,$$

$$(c\pm a)x=a\mp b;$$

$$\therefore\ x=\frac{a\mp b}{c\pm a}.$$

Ex. 36. $(1-x)\sqrt{a\left(1+\frac{1}{x}\right)-2}=\sqrt{x+1}+\sqrt{3x-1}$; find x.

$$(1-x)\sqrt{a\left(1+\frac{1}{x}\right)-2}-\sqrt{1+x}=\sqrt{3x-1},$$

$$(1-x^2)\left\{a\left(1+\frac{1}{x}\right)-2\right\}-2(1-x)\sqrt{1+x}.\sqrt{a\left(1+\frac{1}{x}\right)-2}+1+x=3x-1,$$

$$(1-x)^2\left\{a\left(1+\frac{1}{x}\right)-2\right\}-2(1-x)\sqrt{1+x}.\sqrt{a\left(1+\frac{1}{x}\right)-2}+2(1-x)=0,$$

$$(1-x)\left\{a.\frac{1+x}{x}-2\right\}-2\sqrt{1+x}.\sqrt{a\left(1+\frac{1}{x}\right)-2}+2=0,$$

$$a.\frac{1-x^2}{x}+2x-2\sqrt{1+x}.\sqrt{a\left(1+\frac{1}{x}\right)-2}=0,$$

$$a-(a-2)x^2-2\sqrt{1+x}.\sqrt{ax+(a-2)x^2}=0,$$

$$ax+(a-2)x^2+2\sqrt{1+x}.\sqrt{ax+(a-2)x^2}=a+ax=a(1+x),$$

$$ax+(a-2)x^2+2\sqrt{1+x}.\sqrt{ax+(a-2)x^2}+1+x=(a+1)(1+x),$$

$$\sqrt{ax+(a-2)x^2}+\sqrt{1+x}=\pm\sqrt{a+1}.\sqrt{1+x},$$

$$\sqrt{ax+(a-2)x^2}=(\pm\sqrt{a+1}-1)\sqrt{1+x},$$

$$ax+(a-2)x^2=(\pm\sqrt{a+1}-1)^2+ax+2(1\mp\sqrt{a+1})x,$$

$$(a-2)x^2+2(\pm\sqrt{a+1}-1)x=(\pm\sqrt{a+1}-1)^2,$$

Ex. 37. $\left.\begin{aligned} x-y &= 8, \\ x^4-y^4 &= 14560, \end{aligned}\right\}$; find x and y.

Assume $\left.\begin{aligned} x &= z+v, \\ \text{and} \quad y &= z-v, \end{aligned}\right\}$ (Art. 213),

then $x-y = 2v = 8$; $\therefore v = 4$.

Also $x^2+y^2 = (z+4)^2+(z-4)^2 = 2z^2+32$,

and $x^2-y^2 = (z+4)^2-(z-4)^2 = 16z$,

$$\therefore (x^2+y^2)(x^2-y^2) = x^4-y^4 = (2z^2+32)16z = 14560;$$

$$\therefore z^3+16z = 455,$$

$$z^4+16z^2 = 455z = 65\times 7z,$$

$$z^4+65z^2+\left(\frac{65}{2}\right)^2 = 49z^2+65\times 7z+\left(\frac{65}{2}\right)^2,$$

$$z^2+\frac{65}{2} = 7z+\frac{65}{2},$$

$$z^2 = 7z;$$

$$\therefore z = 7.$$

Hence $\left.\begin{aligned} x &= z+4 = 11, \\ \text{and} \quad y &= z-4 = 3. \end{aligned}\right\}$

Ex. 38. $\left.\begin{aligned} x(bc-xy) &= y(xy-ac)\ldots(1) \\ xy(ay+bx-xy) &= abc(x+y-c)\ldots(2) \end{aligned}\right\}$; find x and y.

From (1) $c(bx+ay) = xy(x+y)$,

$$c(bx+ay-xy) = xy(x+y-c)\ldots\ldots(3);$$

$\therefore \dfrac{xy}{c} = \dfrac{abc}{xy}$, dividing (2) by (3),

or $(xy)^2 = abc^2$,

or $xy = c.\sqrt{ab}$;

and from (1), $$\frac{x}{y} = \frac{xy-ac}{bc-xy} = \frac{c\sqrt{ab}-ac}{bc-c\sqrt{ab}},$$
$$= \frac{\sqrt{ab}-a}{b-\sqrt{ab}} = \frac{\sqrt{a}.(\sqrt{b}-\sqrt{a})}{\sqrt{b}.(\sqrt{b}-\sqrt{a})},$$
$$= \sqrt{\frac{a}{b}};$$

$$\therefore\ x^2 = xy.\frac{x}{y} = c\sqrt{ab}.\sqrt{\frac{a}{b}} = ac,$$

$$\left.\begin{aligned}\therefore\ x &= \pm\sqrt{ac}\\ \text{and } y &= \pm\sqrt{bc}\end{aligned}\right\}.$$

Ex. 39. $$\left.\begin{aligned}x^3+y(xy-1) &= 0\ldots(1)\\ y^3-x(xy+1) &= 0\ldots(2)\end{aligned}\right\};\ \text{find } x \text{ and } y.$$

From (1), $x^4+x^2y^2-xy = 0$,

from (2), $y^4-x^2y^2-xy = 0$;

$$\therefore\ x^4-y^4+2x^2y^2 = 0,$$
$$x^4+2x^2y^2+y^4 = 2y^4,$$
$$x^2+y^2 = \sqrt{2}.y^2,$$
$$x^2 = (\sqrt{2}-1)y^2,$$
$$\frac{x^2}{y^2} = \sqrt{2}-1,\ \ldots\ldots\ (a)$$

$$\therefore\ \frac{x}{y} = \sqrt{\sqrt{2}-1}.$$

Again from (1), and (2), $\frac{x^3}{y^3} = \frac{y}{x}.\frac{1-xy}{1+xy}$;

$$\therefore\ \frac{1-xy}{1+xy} = \frac{x^4}{y^4} = 3-2\sqrt{2},\ \text{from } (a)$$

$$\therefore\ \text{by Art. 195, } xy = \frac{2\sqrt{2}-2}{4-2\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\therefore\ xy\times\frac{x}{y} = x^2 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{2}-1};$$

$$\therefore\ x = \frac{\sqrt[4]{\sqrt{2}-1}}{\sqrt[4]{2}} = \sqrt[4]{\frac{1}{2}(\sqrt{2}-1)}.$$

$$\text{And } y = \frac{1}{\sqrt{2}.x} = \frac{1}{\sqrt[4]{2(\sqrt{2}-1)}}.$$

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Ex. 40. $$\left.\begin{aligned}xy &= a(x+y)\ldots(1)\\ xz &= b(x+z)\ldots(2)\\ yz &= c(y+z)\ldots(3)\end{aligned}\right\};\ \text{find } x,\ y,\ z.$$

$$\text{From (1),}\quad \frac{x+y}{xy}=\frac{1}{a},$$

$$\text{or}\ \frac{1}{x}+\frac{1}{y}=\frac{1}{a}.$$

$$\text{From (2),}\quad \frac{1}{x}+\frac{1}{z}=\frac{1}{b};$$

$$\therefore\ \frac{1}{y}-\frac{1}{z}=\frac{1}{a}-\frac{1}{b}.$$

$$\text{From (3),}\quad \frac{1}{y}+\frac{1}{z}=\frac{1}{c};$$

$$\therefore\ \frac{2}{y}=\frac{1}{a}+\frac{1}{c}-\frac{1}{b},$$

$$\text{and}\ \frac{2}{z}=\frac{1}{b}+\frac{1}{c}-\frac{1}{a},$$

$$\text{also}\ \frac{2}{x}=\frac{1}{a}+\frac{1}{b}-\frac{1}{c};$$

$$\left.\begin{aligned}\therefore\ x&=\frac{2abc}{ac+bc-ab},\\ y&=\frac{2abc}{ab+bc-ac},\\ z&=\frac{2abc}{ab+ac-bc}.\end{aligned}\right\}$$

Ex. 41. $x(x+y+z)=a^2$, $y(x+y+z)=b^2$, $z(x+y+z)=c^2$; find x, y, z.

Adding all together, $(x+y+z)(x+y+z)=a^2+b^2+c^2$;

$$\therefore\ x+y+z=\pm\sqrt{a^2+b^2+c^2};$$

$$\therefore\ \text{from (1),}\quad x=\pm\frac{a^2}{\sqrt{a^2+b^2+c^2}},$$

$$\ldots\ldots\ \text{(2),}\quad y=\pm\frac{b^2}{\sqrt{a^2+b^2+c^2}},$$

$$\ldots\ldots\ \text{(3),}\quad z=\pm\frac{c^2}{\sqrt{a^2+b^2+c^2}},$$

Ex. 42. $x(y+z)^2=1+a^3$; $x+y=\frac{3}{2}+z$; and $yz=\frac{3}{16}$; find x, y, z.

$$\text{From (1)}\quad (y+z)^2=\frac{1+a^3}{x};$$

$$\therefore\ (y-z)^2=\frac{1+a^3}{x}-4yz=\frac{1+a^3}{x}-\frac{3}{4}.$$

But from (2), $y-z=\frac{3}{2}-x$,

$$\therefore\ \frac{1+a^3}{x}-\frac{3}{4}=\left(\frac{3}{2}-x\right)^2=\frac{9}{4}-3x+x^2;$$

$$\frac{1+a^3}{x}=x^2-3x+3,$$

$$1+a^3=x^3-3x^2+3x,$$

$$a^3=x^3-3x^2+3x-1,$$

$$a=x-1,$$

$$\therefore\ x=a+1.$$

Hence from (1), $(a+1)(y+z)^2=1+a^3$,

$$(y+z)^2=a^2-a+1,$$

$$\therefore\ y+z=\pm\sqrt{a^2-a+1}=\pm\sqrt{(a-1)^2+a},$$

also $y-z=\frac{1}{2}-a$;

$$\therefore\ 2y=\frac{1}{2}-a\pm\sqrt{(a-1)^2+a},$$

$$\therefore\ y=\frac{1}{4}-\frac{a}{2}\pm\frac{1}{2}\sqrt{(a-1)^2+a};$$

and $z=\frac{a}{2}-\frac{1}{4}\pm\frac{1}{2}\sqrt{(a-1)^2+a}$.

Ex. 43. $x(y+z)=a$, $y(x+z)=b$, $z(x+y)=c$; find x, y, z.

$$xy+xz=a,$$

$$xy+yz=b,$$

$$xz+yz=c;$$

$$\therefore\ 2xy+xz+yz=a+b;$$

$$2xy=a+b-c.$$

Similarly $2yz=b+c-a$,

and $2xz=a+c-b$;

$$\therefore\ \frac{2xy\,.\,2xz}{2yz}=2x^2=\frac{(a+b-c)\,.\,(a+c-b)}{b+c-a},$$

$$\frac{2xy\,.\,2yz}{2xz}=2y^2=\frac{(a+b-c)\,.\,(b+c-a)}{a+c-b},$$

$$\frac{2yz\,.\,2xz}{2xy}=2z^2=\frac{(b+c-a)\,.\,(a+c-b)}{a+b-c};$$

$$\therefore\ x = \pm\sqrt{\frac{(a+b-c).(a+c-b)}{2(b+c-a)}},$$

$$y = \pm\sqrt{\frac{(a+b-c).(b+c-a)}{2(a+c-b)}},$$

$$z = \pm\sqrt{\frac{(b+c-a).(a+c-b)}{2(a+b-c)}}.$$

Ex. 44. $xy+n(x+y)=a$, $xz+n(x+z)=b$, $yz+n(y+z)=c$; find x, y, and z.

From (1), $(x+n)(y+n)=a+n^2$,

…… (2), $(x+n)(z+n)=b+n^2$,

…… (3), $(y+n)(z+n)=c+n^2$.

Multiply the 1st and 2nd together, and divide by the 3rd, of these equations, then

$$\frac{(x+n)^2(y+n)(z+n)}{(y+n)(z+n)},\ \text{or}\ (x+n)^2=\frac{(a+n^2)(b+n^2)}{(c+n^2)};$$

$$\therefore\ x+n=\pm\sqrt{\frac{(a+n^2)(b+n^2)}{c+n^2}},$$

$$\text{and}\ x=-n\pm\sqrt{\frac{(a+n^2)(b+n^2)}{c+n^2}}.$$

Multiply the 1st and 3rd together, and divide by the 2nd; then we have

$$y+n=\pm\sqrt{\frac{(a+n^2)(c+n^2)}{b+n^2}},$$

$$\text{and}\ y=-n\pm\sqrt{\frac{(a+n^2)(c+n^2)}{b+n^2}}.$$

Multiply the 2nd and 3rd together, and divide by the 1st; then

$$z+n=\pm\sqrt{\frac{(b+n^2)(c+n^2)}{a+n^2}},$$

$$\text{and}\ z=-n\pm\sqrt{\frac{(b+n^2)(c+n^2)}{a+n^2}}.$$

Ex. 45. $x^a y^b = r$, and $x^c y^d = s$; find x and y.

Raising the first equation to the d^{th} power, and the second to the b^{th} power,

$$x^{ad}y^{bd} = r^d,$$

$$\text{and } x^{bc}y^{bd} = s^b;$$

$$\therefore \frac{x^{ad}}{x^{bc}} = \frac{r^d}{s^b}, \text{ or } x^{ad-bc} = \frac{r^d}{s^b};$$

$$\therefore x = \left(\frac{r^d}{s^b}\right)^{\frac{1}{ad-bc}}.$$

Similarly $y = \left(\dfrac{s^a}{r^c}\right)^{\frac{1}{ad-bc}}$.

Ex. 46. $x^{x+y} = y^{4a}$, and $y^{x+y} = x^a$; find x and y.

From first equation, $x^{\frac{x+y}{4a}} = y$,

...... second .... $x^{\frac{a}{x+y}} = y$;

$$\therefore x^{\frac{x+y}{4a}} = x^{\frac{a}{x+y}}, \text{ or } \frac{x+y}{4a} = \frac{a}{x+y};$$

$$(x+y)^2 = 4a^2, \text{ or } x+y = 2a;$$

$$\therefore x = y^2, \quad \because x^{2a} = y^{4a};$$

$$\therefore y^2+y = 2a.$$

From which it is found, that $y = \frac{1}{2}\{-1 \pm \sqrt{8a+1}\}$; and then

$$x = \frac{1}{2}\{4a+1 \mp \sqrt{8a+1}\}.$$

Ex. 47. $$\left.\begin{aligned} \frac{\sqrt{y^2+1}+1}{y} &= \frac{\sqrt{x+9}+3}{\sqrt{x}} \text{......(1)} \\ \text{and } x(y+1)^2 &= 36\left(y^3+\frac{16}{9}\right) \text{....(2)} \end{aligned}\right\}; \text{ find } x \text{ and } y.$$

From (1), multiplying the Numerator and Denominator of the first fraction by $\sqrt{y^2+1}-1$, of the second by $\sqrt{x+9}-3$, then multiplying the resulting equation by (1),

$$\frac{\sqrt{y^2+1}+1}{\sqrt{y^2+1}-1} = \frac{\sqrt{x+9}+3}{\sqrt{x+9}-3} = \frac{\sqrt{\frac{x}{9}+1}+1}{\sqrt{\frac{x}{9}+1}-1},$$

$$\therefore y^2 = \frac{x}{9},$$

$$\text{or } 9y^2 = x.$$

From (2), $9y^2(y^2+2y+1) = 36y^3+64,$

$$9y^2(y^2-2y+1) = 64,$$

$$3y(y-1) = \pm 8,$$

$$y^2-y = \pm\frac{8}{3},$$

$$y^2-y+\frac{1}{4} = \frac{1}{4}\pm\frac{8}{3} = \frac{35}{12}, \text{ or } -\frac{29}{12},$$

$$y-\frac{1}{2} = \pm\frac{1}{6}\sqrt{105},$$

$$\therefore\ y = \frac{1}{6}\{3\pm\sqrt{105}\}.$$

And $x = 9y^2 = \frac{3}{2}\{19\pm\sqrt{105}\}.$

Ex. 48.
$$\left.\begin{aligned} 2x+\sqrt{x^2-y^2} &= \frac{14}{y}\left\{\sqrt{\frac{x+y}{2}}+\sqrt{\frac{x-y}{2}}\right\} \\ \text{and } \left(\frac{x+y}{2}\right)^{\frac{3}{2}}+\left(\frac{x-y}{2}\right)^{\frac{3}{2}} &= 9 \end{aligned}\right\}\text{; find } x \text{ and } y.$$

Assume $\frac{x+y}{2} = v$, and $\frac{x-y}{2} = w$,

then $2x = 2(v+w)$,

also $x^2-y^2 = 4vw$, and $y = v-w$;

∴ by substitution in the first equation

$$2(v+w)+2\sqrt{vw} = \frac{14}{v-w}.\{v^{\frac{1}{2}}+w^{\frac{1}{2}}\};$$

$$\text{or } v+w+\sqrt{vw} = \frac{7}{v^{\frac{1}{2}}-w^{\frac{1}{2}}};$$

$$\therefore\ v^{\frac{3}{2}}-w^{\frac{3}{2}} = 7.$$

But $v^{\frac{3}{2}}+w^{\frac{3}{2}} = 9$, from the second equation;

$$\therefore\ 2v^{\frac{3}{2}} = 16, \quad\text{or } v = 4,$$

$$\text{and } 2w^{\frac{3}{2}} = 2, \quad\text{or } w = 1;$$

$$\therefore\ x = 5, \text{ and } y = 3.$$

By finding *all* the values of v and w which satisfy the equations $v^3-64 = 0$, $w^3-1 = 0$, by the method employed in Ex. 20, other values of x and y may be determined.

Ex. 49. $$\left.\begin{aligned} \frac{y}{2x}+\frac{2}{3}\cdot\frac{y-\sqrt{x-1}}{y^2-2\sqrt{x^2-1}} &= \frac{\sqrt{x+1}}{x} \\ \text{and } \frac{1}{4}y^4 &= y^2x-1 \end{aligned}\right\}\text{; find } x \text{ and } y.$$

From 2nd equation $y^4-4xy^2=-4$,

$$y^4-4x.y^2+4x^2=4x^2-4,$$

$$y^2-2x=\pm2\sqrt{x^2-1},$$

$$y^2=2x\pm2\sqrt{x^2-1};$$

$$\text{and } \therefore\ y=\pm(\sqrt{x+1}\pm\sqrt{x-1})\ldots\ldots(\text{I}).$$

Substituting for y in the first equation, and taking the upper signs in (I),

$$\frac{\sqrt{x+1}}{2x}\mp\frac{\sqrt{x-1}}{2x}+\frac{2}{3}\cdot\frac{\sqrt{x+1}}{2x}=\frac{\sqrt{x+1}}{x},$$

$$3\sqrt{x-1}=\sqrt{x+1},$$

$$9x-9=x+1,$$

$$8x=10;$$

$$\therefore\ x=\frac{10}{8}=1\tfrac{1}{4}.$$

$$\text{And } y=\sqrt{x+1}+\sqrt{x-1}=2.$$

By taking the lower signs in (I) other values of x and y may be obtained.

Ex. 50. $$\left.\begin{aligned} x^2y-4 &= 4yx^{\frac{1}{2}}-\frac{1}{4}y^3\ldots(1) \\ \text{and } x^{\frac{3}{2}}-3 &= x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}})\ldots(2) \end{aligned}\right\}\text{; find } x \text{ and } y.$$

From (1), $$y\left(x^2+\frac{y^2}{4}\right)=4yx^{\frac{1}{2}}+4,$$

$$y\left(x^2+xy+\frac{y^2}{4}\right)=xy^2+4yx^{\frac{1}{2}}+4;$$

$$\therefore\ y^{\frac{1}{2}}\left(x+\frac{y}{2}\right)=\pm(x^{\frac{1}{2}}y+2).$$

Taking the + sign, $xy^{\frac{1}{2}}-x^{\frac{1}{2}}y=2-\dfrac{y^{\frac{3}{2}}}{2}$,

$$\text{or } x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}}) = 2-\frac{y^{\frac{3}{2}}}{2};$$

$$\therefore \text{ by (2), } x^{\frac{3}{2}}-3 = 2-\frac{y^{\frac{3}{2}}}{2},$$

$$2x^{\frac{3}{2}}+y^{\frac{3}{2}} = 10 \ldots\ldots\ldots\ldots(3).$$

Also from (2), $3x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}}) = 3x^{\frac{3}{2}}-9,$

$$= x^{\frac{3}{2}}-y^{\frac{3}{2}}+1, \text{ by (3)};$$

$$\therefore x^{\frac{3}{2}}-y^{\frac{3}{2}}-3x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}}) = -1;$$

extract cube root, $x^{\frac{1}{2}}-y^{\frac{1}{2}} = -1;$

$$\therefore y^{\frac{1}{2}} = x^{\frac{1}{2}}+1 \ldots\ldots\ldots\ldots(4).$$

Substituting in (3), $2x^{\frac{3}{2}}+x^{\frac{3}{2}}+1+3x^{\frac{1}{2}}(x^{\frac{1}{2}}+1) = 10,$

$$3x^{\frac{3}{2}}+3x+3x^{\frac{1}{2}} = 9,$$

$$x^{\frac{3}{2}}+x+x^{\frac{1}{2}} = 3;$$

$$\therefore (x^{\frac{3}{2}}-1)+(x-1)+(x^{\frac{1}{2}}-1) = 0,$$

$$\text{or } (x^{\frac{1}{2}}-1)(x+2x^{\frac{1}{2}}+3) = 0;$$

$$\left.\begin{array}{r}\therefore x^{\frac{1}{2}}-1 = 0, \text{ and } x = 1,\\ \text{substituting in (4), } y^{\frac{1}{2}} = 1+1 = 2;\ \therefore y = 4.\end{array}\right\}$$

Also $x+2x^{\frac{1}{2}} = -3;$

$\therefore x+2x^{\frac{1}{2}}+1 = (x^{\frac{1}{2}}+1)^2 = -2,$ which is impossible;

$\therefore x = 1,\ y = 4,$ are the required values.

Ex. 51. $\left.\begin{array}{r} a(1-xy) = x\sqrt{1-y^2}\ldots(1)\\ \text{and } \sqrt{x}(1-xy) = y-x\ldots\ldots\ldots(2)\end{array}\right\}$; find x and y.

From (2), $x^{\frac{1}{2}}-x^{\frac{3}{2}}y = y-x;$

$$\therefore y = \frac{x^{\frac{1}{2}}+x}{1+x^{\frac{3}{2}}};$$

$$1-xy = 1-\frac{x^{\frac{3}{2}}+x^2}{1+x^{\frac{3}{2}}} = \frac{1-x^2}{1+x^{\frac{3}{2}}},$$

$$\text{and } 1-y^2 = 1-\frac{x+2x^{\frac{3}{2}}+x^2}{1+2x^{\frac{3}{2}}+x^3},$$

$$= \frac{1-x-x^2+x^3}{(1+x^{\frac{3}{2}})^2} = \frac{(1-x)^2(1+x)}{(1+x^{\frac{3}{2}})^2}.$$

Substituting in (1), $a.\dfrac{1-x^2}{1+x^{\frac{3}{2}}} = x.\dfrac{1-x}{1+x^{\frac{3}{2}}}\sqrt{1+x}$;

$$\therefore\ a(1+x) = x\sqrt{1+x};\ \text{also } 1-x = 0, \text{ or } x = 1 \text{..........(i)},$$

$$\therefore\ x = a.\sqrt{1+x};\ \text{also } 1+x = 0, \text{ or } x = -1 \text{........(ii)}.$$

$$x^2 = a^2 + a^2x,$$

$$x^2 - a^2x + \frac{a^4}{4} = \frac{a^4}{4} + a^2;$$

$$\therefore\ x = \frac{a^2 \pm a\sqrt{a^2+4}}{2} \text{......................(iii)}.$$

Also $y = 1$, or -1, or $\pm\dfrac{a^2-1}{a^2+1}$, or $\dfrac{2a}{\sqrt{(a^2+1+a\sqrt{a^2+4})^2-1-2a}}$...(iv).

Ex. 52. $y^4 = x^2(ay-bx)$, and $x^2 = ax-by$; find x and y.

From first equation, $\dfrac{y^4}{x^2} = ay-bx$,

and by second equation, $x^2 = ax-by$;

$$\therefore\ \frac{y^4}{x^4} = \frac{ay-bx}{ax-by} = \frac{a\frac{y}{x}-b}{a-b\frac{y}{x}}.$$

Assume $\dfrac{y}{x} = t$, then $t^4 = \dfrac{at-b}{a-bt}$;

$$\therefore\ at^4 - bt^5 = at - b,$$

$$\text{or } at(t^3-1) - b(t^5-1) = 0,$$

$$at(t^2+t+1) - b(t^4+t^3+t^2+t+1) = 0;$$

$$\therefore\ t-1 = 0, \text{ or } t = 1 \text{..................(1)}.$$

Also $bt^4 + (b-a)t^3 + (b-a)t^2 + (b-a)t + b = 0$,

$$b\left(t^2+\frac{1}{t^2}\right) - (a-b)\left(t+\frac{1}{t}\right) - (a-b) = 0,$$

$$b\left(t+\frac{1}{t}\right)^2 - (a-b)\left(t+\frac{1}{t}\right) - (a+b) = 0,$$

$$\left(t+\frac{1}{t}\right)^2 - \frac{a-b}{b}\left(t+\frac{1}{t}\right) = \frac{a+b}{b};$$

$$\therefore\ t+\frac{1}{t} = \frac{a-b}{2b} \pm \sqrt{\frac{\overline{a-b}|^2}{4b^2} + \frac{a+b}{b}},$$

$$= \frac{a-b \pm \sqrt{a^2+2ab+5b^2}}{2b} = m, \text{ suppose};$$

$$\text{Then } t^2-mt=-1,$$

$$\text{or } t=\frac{m}{2}\pm\sqrt{\frac{m^2}{4}-1}\text{.........(2).}$$

If from (1) $t=1$, then $x=y=a-b$..................................(i).

If $t=\frac{m}{2}\pm\sqrt{\frac{m^2}{4}-1}$, $x=a-bt=a-\frac{b}{2}(m\pm\sqrt{m^2-4})$,............(ii).

and $y=tx=\frac{1}{2}\{a-\frac{b}{2}(m\pm\sqrt{m^2-4})\}\{m\pm\sqrt{m^2-4}\}$......................(iii).

$$\text{where } m=\frac{1}{2b}\{a-b\pm\sqrt{a^2+2ab+5b^2}\}.$$

Ex. 53. $\left.\begin{array}{l} x^5+y^5=xy(x+y)^3\text{......(1)} \\ \text{and } y^2\sqrt{x}=(x+y)^{\frac{3}{2}}\text{.........(2)}\end{array}\right\}$; find x and y.

From (1), dividing by $x+y$,

$$x^4-x^3y+x^2y^2-xy^3+y^4=xy(x^2+2xy+y^2),$$

$$=x^3y+2x^2y^2+xy^3,$$

$$x^4-2x^3y-x^2y^2-2xy^3+y^4=0;$$

$$\text{dividing by } x^2y^2,\ \frac{x^2}{y^2}-2.\frac{x}{y}-1-2.\frac{y}{x}+\frac{y^2}{x^2}=0,$$

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}-2\left(\frac{x}{y}+\frac{y}{x}\right)=1,$$

$$\left(\frac{x}{y}+\frac{y}{x}\right)^2-2\left(\frac{x}{y}+\frac{y}{x}\right)+1=4,$$

$$\frac{x}{y}+\frac{y}{x}=1\pm2=3,\text{ or }-1.$$

Taking the first value and multiplying by $\frac{x}{y}$,

$$\left(\frac{x}{y}\right)^2+1=3.\frac{x}{y},$$

$$\left(\frac{x}{y}\right)^2-3.\frac{x}{y}+\frac{9}{4}=\frac{5}{4},$$

$$\therefore\ \frac{x}{y}=\frac{3\pm\sqrt{5}}{2}.$$

But from (2), $xy^4=(x+y)^3$,

$$\therefore\ \frac{3\pm\sqrt{5}}{2}.y^5=\left(\frac{x}{y}+1\right)^3.y^3=\left(\frac{5\pm\sqrt{5}}{2}\right)^3.y^3,$$

$$\therefore\ y^2 = \frac{(5\pm\sqrt{5})^3}{4(3\pm\sqrt{5})} = \frac{50\pm20\sqrt{5}}{3\pm\sqrt{5}} = \frac{50\pm10\sqrt{5}}{4};$$

$$\therefore\ y = \frac{5}{2}\sqrt{2\pm\frac{2}{5}\sqrt{5}},$$

$$\text{and } x = \sqrt{\frac{50\pm20\sqrt{5}}{3\mp\sqrt{5}}} = \frac{5}{2}\sqrt{10\pm\frac{22}{5}\sqrt{5}}.$$

Ex. 54. $\left.\begin{aligned}(x^6+1)y &= (y^2+1)x^3\ldots(1)\\ \text{and } (y^6+1)x &= 9(x^2+1)y^3\ldots(2)\end{aligned}\right\}$; find x and y.

From (1), $$\left(x^3+\frac{1}{x^3}\right)y = y^2+1,$$

$$\therefore\ x^3+\frac{1}{x^3} = y+\frac{1}{y}\ldots\ldots\ldots\ldots(3).$$

From (2), $$y^3+\frac{1}{y^3} = 9\left(x+\frac{1}{x}\right),$$

$$\text{or } \frac{1}{3}\left(y^3+\frac{1}{y^3}\right) = 3\left(x+\frac{1}{x}\right);$$

$$\therefore\ \frac{1}{3}\left(y^3+\frac{1}{y^3}\right)+y+\frac{1}{y} = x^3+3\left(x+\frac{1}{x}\right)+\frac{1}{x^3},$$

$$y^3+\frac{1}{y^3}+3\left(y+\frac{1}{y}\right) = 3\left(x+\frac{1}{x}\right)^3,$$

$$\therefore\ y+\frac{1}{y} = \left(x+\frac{1}{x}\right)\sqrt[3]{3}.$$

$$\text{And } \therefore\ x^3+\frac{1}{x^3} = \left(x+\frac{1}{x}\right)\sqrt[3]{3}, \text{ by (3)};$$

dividing by $x+\frac{1}{x}$, $x+\frac{1}{x} = 0$, i.e. $x^2 = -1$, which is impossible.

$$\text{Also } x^2-1+\frac{1}{x^2} = \sqrt[3]{3},$$

$$x^2+2+\frac{1}{x^2} = \sqrt[3]{3}+3,$$

$$\therefore\ x+\frac{1}{x} = \sqrt{\sqrt[3]{3}+3}.$$

$$\text{Similarly } x-\frac{1}{x} = \sqrt{\sqrt[3]{3}-1};$$

$$\therefore\ x = \frac{1}{2}\{\sqrt{\sqrt[3]{3}+3}+\sqrt{\sqrt[3]{3}-1}\}.$$

$$\text{And } y+\frac{1}{y} = \left(x+\frac{1}{x}\right)\sqrt[3]{3},$$

$$= \sqrt[3]{3}.\sqrt{\sqrt[3]{3}+3}\,;$$

$$\text{from which } y = \frac{1}{2}\{\sqrt[3]{3}.\sqrt{\sqrt[3]{3}+3}\pm(\sqrt{\sqrt[3]{9}-1})\}.$$

Ex. 55. $$\left.\begin{array}{l} y+3\sqrt[3]{y}\{\sqrt[3]{a+bx}-\sqrt[3]{y}\}\sqrt[3]{a+bx} = 2a\ldots(1) \\ \text{and } \dfrac{y-3\sqrt[3]{y}.\sqrt[3]{a^2-b^2x^2}}{\sqrt[3]{y}-\sqrt[3]{a+bx}} = 2a.\sqrt[n]{a-bx}\ldots\ldots.(2) \end{array}\right\};\ \text{find } x \text{ and } y.$$

From (1), $y-3y^{\frac{2}{3}}(a+bx)^{\frac{1}{3}}+3y^{\frac{1}{3}}(a+bx)^{\frac{2}{3}}-(a+bx) = a-bx$;

∴ extracting the cube root,

$$\sqrt[3]{y}-\sqrt{a+bx} = \sqrt[3]{a-bx}\ldots\ldots\ldots(3),$$

$$\text{or } \sqrt[3]{y} = \sqrt[3]{a+bx}+\sqrt[3]{a-bx}\,;$$

$$\therefore\ y = a+bx+a-bx+3\sqrt[3]{a^2-b^2x^2}.\{\sqrt[3]{a+bx}+\sqrt[3]{a-bx}\},$$

$$= 2a+3\sqrt[3]{y}.\sqrt[3]{a^2-b^2x^2}\,;$$

$$\therefore\ y-3\sqrt[3]{y}.\sqrt[3]{a^2-b^2x^2} = 2a\ldots\ldots\ldots(4).$$

Substituting (3) and (4) in (2),

$$\frac{2a}{\sqrt[3]{a-bx}} = 2a.\sqrt[n]{a-bx},$$

$$(a-bx)^{\frac{1}{n}+\frac{1}{3}} = 1,$$

$$a-bx = 1\,;$$

$$\therefore\ x = \frac{a-1}{b}.$$

$$\text{And } y = \{\sqrt[3]{2a-1}+1\}^3.$$

Ex. 56. $$\left.\begin{array}{l} (2+4xy-3x^2)^2 = 2-4x^2y^2+3x^4\ldots\ldots(1) \\ \text{and } 5y^2+\dfrac{27x^2}{32} = \dfrac{9xy}{2}+\dfrac{2xy+1}{x^2}\ldots\ldots(2) \end{array}\right\};\ \text{find } x \text{ and } y.$$

From (1), $4+16xy+16x^2y^2-12x^2-24x^3y+9x^4=2-4x^2y^2+3x^4,$

$$\text{or } 2+16xy+20x^2y^2-12x^2-24x^3y+6x^4=0,$$

$$1+8xy+10x^2y^2-6x^2-12x^3y+3x^4=0,$$

$$x^4-4x^3y+4x^2y^2=\frac{2x^2y^2}{3}+2x^2-2xy-\frac{2xy}{3}-\frac{1}{3},$$

$$(x^2-2xy)^2-2(x^2-2xy)+1=\frac{2x^2y^2}{3}+\frac{4xy}{3}+\frac{2}{3},$$

$$=\frac{2}{3}(x^2y^2+2xy+1);$$

$$\therefore\ x^2-2xy-1=\sqrt{\frac{2}{3}}.(xy+1)\ldots\ldots\ldots(3),$$

Again, from (2), $10y^2+\dfrac{27x^2}{16}=9xy+\dfrac{4xy+2}{x^2};$

$$\therefore\ \frac{9x^4}{16}-3x^3y=-\frac{10x^2y^2}{3}+\frac{4xy}{3}+\frac{2}{3},$$

$$\text{or } \left(\frac{3x^2}{4}\right)^2-4xy.\frac{3x^2}{4}+4x^2y^2=\frac{2x^2y^2}{3}+\frac{4xy}{3}+\frac{2}{3},$$

$$\therefore\ \frac{3x^2}{4}-2xy=\sqrt{\frac{2}{3}}(xy+1)\ldots\ldots\ldots\ldots\ldots(4);$$

$$\therefore\ \frac{3x^2}{4}-2xy=x^2-2xy-1, \text{ from } (3),$$

$$3x^2=4x^2-4,$$

$$\therefore\ x=2.$$

Substituting in (4), $3-4y=(2y+1)\sqrt{\dfrac{2}{3}};$

$$\therefore\ 2y=\frac{3-\sqrt{\frac{2}{3}}}{2+\sqrt{\frac{2}{3}}}=2-\frac{3}{2}\sqrt{\frac{2}{3}};$$

$$\therefore\ y=1-\frac{1}{4}\sqrt{6}.$$

Ex. 57. $\left.\begin{array}{l}\sqrt[3]{\dfrac{27y^{\frac{3}{2}}-1}{x^3+3y^2-2xy^{\frac{3}{2}}}}=3\sqrt{\dfrac{x}{y}}\ldots(1)\\ \text{and } 3x^2+42xy+16y^2=4\sqrt{xy}(5x+11y)\ldots(2)\end{array}\right\}$; find x and y.

From (2), $3x^2-20x^{\frac{3}{2}}y^{\frac{1}{2}}+42xy-44x^{\frac{1}{2}}y^{\frac{3}{2}}+16y^2=0$,

dividing by xy, $3\dfrac{x}{y}-20\sqrt{\dfrac{x}{y}}+42-44\sqrt{\dfrac{y}{x}}+16\dfrac{y}{x}=0$.

Let $\dfrac{x}{y}=t^2$, then $3t^2+20t+42-\dfrac{44}{t}+\dfrac{16}{t^2}=0$,

$$\therefore\ 3t^4-20t^3+42t^2-44t+16=0,$$

$$(3t-2)t^3-(3t-2)6t^2+(3t-2)10t-(3t-2)8=0,$$

$$\text{or}\ (3t-2)(t^3-6t^2+10t-8)=0;$$

$$3t-2=0,\ \text{or}\ t^3-6t^2+10t-8=0,$$

$$\text{i.e.}\ (t-4).(t^2-2t+2)=0,$$

$\therefore\ t=\dfrac{2}{3}$; or $t-4=0$, i.e. $t=4$. $\Big\{$ or $t^2-2t+2=0$, i.e. $(t-1)^2=1$, which is impossible.

Therefore, taking the first value of t, and substituting in (1)

$$\frac{\dfrac{729}{8}x^{\frac{3}{2}}-1}{x^3+\dfrac{243}{16}x^2-\dfrac{27}{4}x^{\frac{5}{2}}}=8;$$

$$\therefore\ 8x^3-54x^{\frac{5}{2}}+\frac{243}{2}x^2-\frac{729}{8}x^{\frac{3}{2}}=-1.$$

Extracting the cube root, $2x-\dfrac{9}{2}x^{\frac{1}{2}}=-1$;

$$\therefore\ x-\frac{9}{4}x^{\frac{1}{2}}+\frac{81}{64}=\frac{49}{64},$$

$$x^{\frac{1}{2}}=\frac{9\pm 7}{8}=2,\ \text{or}\ \frac{1}{4},$$

$$\left.\begin{aligned}&\therefore\ x=4,\ \text{or}\ \frac{1}{16},\\ &\text{and}\ y=\frac{9}{4}x=9,\ \text{or}\ \frac{9}{64}.\end{aligned}\right\}$$

Taking the second value of t, other values of x and y remain to be determined.

Ex. 58. $\left\{\dfrac{ab}{2}(b+c)^2+(b^2+c^2)cx\right\}\times\sqrt{\dfrac{a^2}{4}(b-c)^4+(b^2+c^2)^2x^2}$

$=\left\{\dfrac{ab}{2}(b-c)^2-(b^2+c^2)cx\right\}\times\sqrt{\dfrac{a^2}{4}(b+c)^4+(b^2+c^2)^2x^2}$; find x.

Let $b+c = m$, and $b-c = n$, so that $b^2+c^2 = \frac{1}{2}(m^2+n^2)$,

$$\text{then } \frac{m^2ab+(m^2+n^2)cx}{n^2ab-(m^2+n^2)cx} = \sqrt{\frac{m^4a^2+(m^2+n^2)^2x^2}{n^4a^2+(m^2+n^2)^2x^2}},$$

$$\frac{m^4a^2b^2+2m^2(m^2+n^2)abcx+(m^2+n^2)^2c^2x^2}{n^4a^2b^2-2n^2(m^2+n^2)abcx+(m^2+n^2)^2c^2x^2} = \frac{m^4a^2+(m^2+n^2)^2x^2}{n^4a^2+(m^2+n^2)^2x^2},$$

$$\frac{(m^4-n^4)a^2b^2+2(m^2+n^2)^2abcx}{n^4a^2b^2-2n^2(m^2+n^2)abcx+(m^2+n^2)^2c^2x^2} = \frac{(m^4-n^4)a^2}{n^4a^2+(m^2+n^2)^2x^2},$$

$$\frac{(m^2-n^2)ab^2+2(m^2+n^2)bcx}{n^4a^2b^2-2n^2(m^2+n^2)abcx+(m^2+n^2)^2c^2x^2} = \frac{(m^2-n^2)a}{n^4a^2+(m^2+n^2)^2x^2},$$

$$(m^2-n^2)n^4a^3b^2+(m^4-n^4)(m^2+n^2)ab^2x^2+2(m^2+n^2)n^4a^2bcx+2(m^2+n^2)^3bcx^3$$
$$= (m^2-n^2)n^4a^3b^2-2(m^4-n^4)n^2a^2bcx+(m^4-n^4)(m^2+n^2)ac^2x^2,$$

$$(m^4-n^4)ab^2x+2n^4a^2bc+2(m^2+n^2)^2bcx^2 = (m^4-n^4)ac^2x-2(m^2-n^2)n^2a^2bc,$$

$$2(m^2+n^2)^2bcx^2+(m^4-n^4)a(b^2-c^2)x+2m^2n^2a^2bc = 0,$$

$$\left.\begin{aligned}(m^2+n^2)^2x^2+2(m^2+n^2)mnax+m^2n^2a^2 = 0,\\ (m^2+n^2)x+mna = 0,\end{aligned}\right\}\begin{aligned}\because\ mn = b^2-c^2,\\ \text{and } bc = \frac{1}{4}(m^2-n^2);\end{aligned}$$

$$\therefore\ x = -\frac{mna}{m^2+n^2} = \frac{a}{2}\cdot\frac{c^2-b^2}{c^2+b^2}.$$

Ex. 59. $\left.\begin{aligned}\frac{3+2x^2-4x^4}{x^2-1} = y^2(1-2y^2)\ldots\ldots\ldots(1)\\ \text{and}\quad (2x^2-1)(2y^2-1) = 3\ldots\ldots\ldots(2)\end{aligned}\right\}$; find x and y.

Write u for x^2 and v for y^2, then from (2), $4uv-2(u+v) = 2$,
or $2uv-(u+v) = 1\ldots\ldots\ldots(a)$.

From (1),
$$\begin{aligned}4u^2-2u-3 &= v(2v-1)(u-1),\\ &= v(2uv-u-2v+1),\\ &= v(2-v), \text{ by } (a),\\ &= 2v-v^2;\\ \therefore\ 4u^2-3 &= 2(u+v)-v^2,\\ &= 4uv-2-v^2, \text{ by } (a),\end{aligned}$$

$$4u^2-4uv+v^2=1,$$

$$2u-v=\pm 1,$$

$$\therefore v=2u\mp 1\ldots\ldots(\beta).$$

Taking the upper sign, and substituting in (α),

$$4u^2-2u-u-2u+1=1,$$

$$\text{or } 4u^2=5u, \quad \therefore u=\frac{5}{4}, \text{ or } 0\ldots\ldots\text{(i)}.$$

Taking the lower sign, and substituting in (α),

$$4u^2+2u-u-2u-1=1,$$

$$4u^2-u=2,$$

$$4u^2-u+\left(\frac{1}{4}\right)^2=\frac{1}{16}+2=\frac{33}{16},$$

$$2u-\frac{1}{4}=\pm\frac{\sqrt{33}}{4}, \quad \therefore u=\frac{1\pm\sqrt{33}}{8}\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Hence from (i), $x=\pm\frac{1}{2}\sqrt{5}$, or 0; and $y=\pm\frac{1}{2}\sqrt{6}$; or $\pm\sqrt{-1}$,

from (ii), $x=\pm\frac{1}{2}\sqrt{\frac{1\pm\sqrt{33}}{2}}$; and $y=\pm\frac{1}{2}\sqrt{5\pm\sqrt{33}}$.

PROBLEMS.

In reducing a Problem to an Equation, the course to be pursued is stated in Art. 199; but much depends here, as in the solution of equations, upon a practical acquaintance with particular artifices, by which the most *convenient* unknown quantities are assumed, and the problem most easily translated into algebraical language.

The general question always is, having certain known quantities, represented by given symbols, and one or more other unknown quantities, represented by one or more of the letters x, y, z, &c., to connect the known and unknown symbols together by the conditions of the problem, so as to produce as many *independent* equations as there are unknown quantities.

There is also one general property of a large class of such problems, viz. that the increase or decrease, the selling or buying, &c., is after a *uniform rate*. Thus, if A is said to perform a piece of work in a days, he is supposed to work equally every day. If A is said to travel p miles in q days, he is supposed to travel one uniform distance each day. And so on, unless the contrary be expressed. So that the following Rule is of constant application, seeing that uniform increase or decrease of every sort may be represented by uniform motion:—

Rule. If v represent the space described by a body moving *uniformly* in 1 unit of time, (whether it be 1 second, 1 hour, or any other known unit), and s the space described by the same body in t such units of time, then $s = tv$.

Also $v = \frac{s}{t}$, and $t = \frac{s}{v}$; both of which forms are frequently required.

Thus, if A travels p miles in q days, then the distance (v) travelled in 1 day $= \frac{\text{whole distance }(s)}{\text{number of days }(t)} = \frac{p}{q}$; and the number of days $= \frac{\text{whole distance}}{\text{distance per day}}$.

The following Problems are added here as differing in some material respect from those in the text:—

PROB. 1. In the year 1830 A's age was 50 and B's 35. Required the year in which A is twice as old as B.

Let $1830 \pm x$ be the year required.

$$\begin{aligned} \text{Then}\quad 50 \pm x &= 2(35 \pm x), \\ \text{or}\quad 50 \pm x &= 70 \pm 2x; \\ \therefore \pm x &= -20; \end{aligned}$$

$\therefore$ the year required is 1810.

PROB. 2. In what proportions must substances of "specific gravities" a and b be mixed, so that the "specific gravity" of the mixture may be c?

[DEF. By the "specific gravity" of a body is meant the number of times which its weight is of the weight of an equal bulk of water.]

To 1 cubic foot of the first substance let x cubic feet of the second be added.

Then, since 1 cubic foot of the first weighs a cubic feet of water,

and x ...... feet ...... second ...... bx ....

$\therefore$ the whole $1 + x$ cubic feet of mixture weighs $a + bx$ cubic feet of water.

But since c is the specific gravity of the mixture, the weight of $1 + x$ cubic feet is $c(1 + x)$ cubic feet of water,

$$\begin{aligned} \therefore a + bx &= c(1 + x), \\ &= c + cx; \\ \therefore x &= \frac{a - c}{c - b}; \end{aligned}$$

that is, for every cubic foot of the substance whose specific gravity is a there must be $\frac{a-c}{c-b}$ cubic feet of the substance whose specific gravity is b.

PROB. 3. From a vessel of wine containing a gallons b gallons are drawn off, and the vessel is filled up with water. Find the quantity of wine remaining in the vessel, when this has been repeated n times.

Let $x_1, x_2, x_3, \ldots x_n$ be the number of gallons of wine remaining in the vessel after 1, 2, 3,...n drawings off respectively.

(1) Then $x_1 = a-b$.

(2) $x_2 = a-b-$quantity of wine in b gallons of first mixture.

Now, a gallons contain $a-b$ of wine;

$$\therefore \text{ 1 gallon contains } \frac{a-b}{a} \quad \ldots\ldots.$$

$$\text{and } b \text{ gallons contain } b.\frac{a-b}{a} \quad \ldots\ldots.$$

$$\therefore x_2 = a-b-b.\frac{a-b}{a} = \frac{(a-b)^2}{a}.$$

(3) $x_3 = \dfrac{(a-b)^2}{a} -$quantity of wine in b gallons of second mixture.

$$\text{But } a \text{ gallons contain } \frac{(a-b)^2}{a} \text{ of wine;}$$

$$\therefore \text{ 1 gallon contains } \frac{(a-b)^2}{a^2} \quad \ldots\ldots.$$

$$\text{and } b \text{ gallons contain } b.\frac{(a-b)^2}{a^2} \quad \ldots\ldots.$$

$$\therefore x_3 = \frac{(a-b)^2}{a} - \frac{b.(a-b)^2}{a^2} = \frac{(a-b)^3}{a^2}.$$

And so on, for each succeeding mixture; so that, generally,

$$x_n = \frac{(a-b)^n}{a^{n-1}}.$$

PROB. 4. The advance of the hour-hand of a watch before the minute-hand is measured by $15\frac{2}{3}$ of the minute divisions; and it is between 9 and 10 o'clock. Find the exact time indicated by the watch.

Let x = number of minutes past 9 o'clock; then since the minute-hand goes 12 times as fast as the hour-hand,

$$45+\frac{x}{12} = \text{number of minute divisions the hour-hand is past 9,}$$

$$\text{and } 45+\frac{x}{12}-x = \text{distance in minutes between hour-hand and minute-hand,}$$

$$= 15\tfrac{2}{3}, \text{ by the question,}$$

$$\therefore \frac{11x}{12} = 29\tfrac{1}{3} = \frac{88}{3},$$

$$\therefore x = 32.$$

Hence the time required is 28 minutes before 10 o'clock.

PROB. 5. In comparing the rates of a watch and a clock, it was observed on one morning, when it was 12^h by the clock, that the watch was at $11^h.\ 59^m.\ 49^s$; and two mornings after, when it was 9^h by the clock, the watch was at $8^h.\ 59^m.\ 58^s$. The clock is known to gain $0{\cdot}1^s$ in 24 hours, find the gaining rate of the watch.

On the 1st morning the watch is behind the clock 11^{s}; and after 45 hours it is only 2^{s} behind; $\therefore$ the watch gains *upon the clock* to the amount of 9^{s} in 45 hours, or $\frac{1}{5}$ of a second per hour.

These hours are those indicated by the clock: and since the clock gains $0{\cdot}1^{s}$ in 24 hours, $24\times60\times60^{s}+0{\cdot}1^{s}$ of clock time $= 24\times60\times60^{s}$ of real time.

$$\therefore \text{ 1 hour of real time} = \frac{24\times60\times60+0{\cdot}1}{24\times60\times50} \text{ hours of clock time;}$$

$\therefore$ the watch gains upon the clock to the amount of $\dfrac{24\times60\times60+0{\cdot}1}{24\times60\times60}\times\dfrac{1^{s}}{5}$ per hour of real time.

Let x be the gaining rate of the watch per hour: then since $\frac{0{\cdot}1}{24}$ is the gaining rate of the clock, $x-\frac{0{\cdot}1}{24}$ is the gain of the watch upon the clock *per hour.*

$$\therefore x-\frac{0{\cdot}1}{24}=\frac{24\times60\times60+0{\cdot}1}{24\times60\times60}\times\frac{1}{5},$$

$$\text{or } x=\frac{1}{240}+\frac{1}{5}+\frac{0{\cdot}1}{5\times24\times60\times60};$$

$$\therefore \text{ the diurnal gain of the watch is } 24x, \text{ or } \frac{1}{10}+4{\cdot}8+\frac{0{\cdot}0002}{36},$$

i.e. $4{\cdot}9000055\ldots$seconds.

Prob. 6. If A and B together can perform a piece of work in a days, A and C together the same in b days, and B and C together in c days; find the time in which each can perform the work separately.

Let w represent the work, and x, y, z, the times in which A, B, C, can separately do it.

$$\text{Then } \frac{w}{a} = A\text{'s daily work}+B\text{'s daily work}\ldots\ldots\ldots\ldots(1).$$

$$\frac{w}{b} = A\text{'s}\ldots\ldots\ldots\ldots\ldots+C\text{'s}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\frac{w}{c} = B\text{'s}\ldots\ldots\ldots\ldots\ldots+C\text{'s}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\therefore \frac{w}{a}-\frac{w}{b} = B\text{'s}\ldots\ldots\ldots\ldots\ldots-C\text{'s}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

and adding (3) and (4), $\frac{w}{a}+\frac{w}{c}-\frac{w}{b} = 2.B\text{'s daily work}, = 2.\frac{w}{y}$;

$$\therefore y=\frac{2abc}{ab+bc-ac}.$$

$$\text{Similarly, } x=\frac{2abc}{ac+bc-ab}; \text{ and } z=\frac{2abc}{ab+ac-bc}.$$

To shew that the denominators of these fractions are *necessarily* positive:

By the Prob. B alone could not perform w in a days,

$$\therefore B \text{ alone} \ldots\ldots \frac{w}{a} \text{ in 1 day.}$$

$$C \text{ alone} \ldots\ldots w \text{ in } b \text{ days,}$$

$$\therefore C \text{ alone} \ldots\ldots \frac{w}{b} \text{ in 1 day;}$$

$$\therefore B \text{ and } C \text{ together} \ldots\ldots \frac{w}{a}+\frac{w}{b} \text{ in 1 day,}$$

$$\text{and } \therefore B \text{ and } C \text{ together} \ldots\ldots c.\left(\frac{w}{a}+\frac{w}{b}\right) \text{ in } c \text{ days.}$$

But B and C together can perform w in c days;

$$\therefore\ c\left(\frac{w}{a}+\frac{w}{b}\right) > w, \quad \text{or } \frac{ac+bc}{ab} > 1,$$

$$\text{and } \therefore ac+bc > ab.$$

Similarly it may be shewn that $ab+bc > ac$, and $ab+ac > bc$.

PROB. 7. The fore-wheel of a coach makes 6 revolutions more than the hind wheel in going 120 yards; but if the circumference of each wheel be increased 1 yard, the fore-wheel will make only 4 revolutions more than the hind-wheel in the same distance. Required the circumference of each wheel.

Let x = circumf. of the hind-wheel, in yards;

y = ... fore-wheel ....

then $\frac{120}{x}$ = number of revolutions by former in 120 yards;

$\frac{120}{y}$ = ... latter ....

$$\therefore \frac{120}{x}+6=\frac{120}{y},$$

$$\text{or } 20x-20y = xy \ldots\ldots(1).$$

Again, on the 2nd supposition, $\frac{120}{x+1}+4=\frac{120}{y+1}$,

$$\text{or } 30(y+1)+(x+1)(y+1) = 30(x+1);$$

$$\therefore 29x-31y = xy+1 \ldots\ldots(2).$$

Subtr. (1) from (2), $9x-11y=1$,

$$\therefore 9x = 11y+1 \dots\dots\dots\dots (3).$$

But from (1), $20\times 9x - 20\times 9y = 9xy$,

$$\therefore 20(11y+1)-180y = 11y^2+y,$$

$$\text{or } 11y^2-39y = 20,$$

$$y^2-\frac{39}{11}y+\overline{\frac{39}{22}}\Big|^2 = \frac{1521}{(22)^2}+\frac{20}{11} = \frac{2401}{(22)^2},$$

$$\therefore y = \frac{39\pm 49}{22} = 4, \text{ or } -\frac{5}{11},$$

$$\text{and } x = \frac{11y+1}{9} = 5, \text{ or } -\frac{4}{9}:$$

∴ the circumference of the wheels are 4 and 5 yards respectively.

Prob. 8. Find two numbers in the ratio of m to n, whose sum is equal to their product.

Let mx, and nx, be the two numbers,

$$\text{then } mx+nx = mx.nx,$$

$$m+n = mnx,$$

$$\left.\begin{aligned} \therefore mx &= \frac{m}{n}+1 \\ \text{and } nx &= \frac{n}{m}+1 \end{aligned}\right\}, \text{ the two numbers.}$$

Prob. 9. The product of two numbers is p, and the difference of their cubes is equal to m times the cube of their difference. Find the numbers.

Let $x+y$, and $x-y$, be the two numbers,

$$\text{then } \left.\begin{aligned} (x+y)(x-y), \text{ or } x^2-y^2 &= p, \\ (x+y)^3-(x-y)^3 &= m(2y)^3 \end{aligned}\right\}.$$

$$\left.\begin{aligned} \text{From 2nd equation, } 2y^3+6x^2y &= 8my^3 \\ \dots\dots \text{ 1st } \dots\dots\dots\dots 6x^2y-6y^3 &= 6py \end{aligned}\right\};$$

$$\therefore 8y^3 = 8my^3-6py,$$

$$4(m-1)y^2 = 3p,$$

$$\therefore y = \frac{1}{2}\sqrt{\frac{3p}{m-1}}.$$

$$\text{Also } x^2 = y^2+p = \frac{1}{4}\cdot\frac{3p}{m-1}+p = \frac{(4m-1)p}{4(m-1)};$$

$$\therefore\ x=\frac{1}{2}\sqrt{\frac{(4m-1)p}{m-1}}.$$

Hence the required numbers are

$$\frac{1}{2}\cdot\frac{\sqrt{(4m-1)p}+\sqrt{3p}}{\sqrt{m-1}},\ \text{and}\ \frac{1}{2}\cdot\frac{\sqrt{(4m-1)p}-\sqrt{3p}}{\sqrt{m-1}}.$$

PROB. 10. Find two numbers whose product is equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

Let x, and xy, be the two numbers,

$$\left.\begin{aligned}\text{then}\ x^2y &= x^2y^2-x^2\ldots\ldots(1)\\ \text{and}\ x^2y^2+x^2 &= x^3y^3-x^3\ldots\ldots(2)\end{aligned}\right\}.$$

$$\text{From (1)},\quad y=y^2-1,$$

$$\therefore\ y^2-y+\frac{1}{4}=\frac{5}{4},$$

$$\text{and}\ y=\frac{1}{2}(\sqrt{5}+1).$$

$$\text{From (2)},\ y^2+1=xy^3-x,$$

$$\therefore\ x=\frac{y^2+1}{y^3-1}=\frac{y+2}{2y}=\frac{1}{2}+\frac{1}{y},$$

$$=\frac{1}{2}+\frac{2}{\sqrt{5}+1}=\frac{1}{2}+\frac{\sqrt{5}-1}{2}=\frac{1}{2}\sqrt{5}.$$

$$\therefore\ \text{the required numbers are}\ \frac{1}{2}\sqrt{5},\ \text{and}\ \frac{1}{4}(5+\sqrt{5}).$$

PROB. 11. There are four numbers in Arithmetical Progression. The sum of the two extremes is 8, and the product of the means is 15. What are the numbers?

Let $x-3y$, $x-y$, $x+y$, $x+3y$, be the numbers;

$$\text{then, by the question,}\ x-3y+x+3y=8,$$

$$\therefore\ x=4.$$

$$\text{Also}\ (x-y)(x+y)=15,$$

$$\text{or}\ x^2-y^2=15,$$

$$\therefore\ 16-y^2=15,\ \text{or}\ y=1;$$

$\therefore$ the numbers are 1, 3, 5, 7.

Prob. 12. There are three numbers in Geometrical Progression, whose product is 64, and sum 14. What are the numbers?

Let $\frac{x}{y}$, x, xy, be the numbers; then, by the question,

$$\frac{x}{y}.x.xy \text{ or } x^3 = 64, \therefore\ x = 4.$$

Also $\frac{x}{y}+x+xy = 14$, or $\frac{1}{y}+1+y = \frac{14}{x} = \frac{7}{2}$,

$$\therefore\ y^2 - \frac{5}{2}y + \left(\frac{5}{4}\right)^2 = \frac{25}{16} - 1 = \frac{9}{16},$$

$$y = \frac{5 \pm 3}{4} = 2, \text{ or } \frac{1}{2};$$

∴ the numbers are 2, 4, 8, or 8, 4, 2.

Prob. 13. Two labourers A and B, whose rates of working are as 3 to 5, were employed to dig a ditch; A worked 12 hours and B 10 hours a day: B being called away, A worked one day alone in order to complete the work: when they were paid, B received as many pence more than A as the number of days they worked together. Now, had B been called away a day sooner, A would have received 3*s.* 11*d.* more than B at the conclusion of the work. Required their respective daily wages, on supposition that the payment to each was in proportion to the work performed.

Let x be the number of days they worked together;

$3w$, and $5w$, the work per hour of A and B respectively;

$\therefore\ 36w = A$'s daily work,

$50w = B$'s ....

let then $36y = A$'s daily wages, in pence,

$50y = B$'s ....

and we have $50y \times x - 36y(x+1) = x$,

or $14xy - 36y = x$..............(1).

Again, on the second supposition, A and B work $x-1$ days together,

∴ the work done in that time $= 86w(x-1)$,

but the whole work $= 86wx + 36w$;

∴ work left to be done by $A = 122w$;

$$\therefore\ A \text{ works } x-1+\frac{122w}{36w} \text{ days,}$$

$$\text{or } x+2+\frac{7}{18} \text{ days.}$$

$$\text{Hence } \left(x+2+\frac{7}{18}\right)36y-(x-1)50y=47,$$

$$\text{or } 136y-14xy=47\,\ldots\ldots\ldots\ldots\ldots(2).$$

$$\text{From (1), } x=\frac{36y}{14y-1};$$

$$\therefore\ 136y-\frac{14\times 36y^2}{14y-1}=47,$$

$$14\times 136y^2-136y-14\times 36y^2=47\times 14y-47,$$

$$1400y^2-794y=-47,$$

$$y^2-\frac{794}{1400}y+\left.\overline{\frac{397}{1400}}\right|^2=\frac{91809}{(1400)^2};$$

$$\therefore\ y=\frac{397\pm 303}{1400}=\frac{1}{2}, \text{ or } \frac{47}{700};$$

$\therefore$ *A*'s daily wages $= 36\times\frac{1}{2} = 18$*d.* $=$ 1*s.* 6*d.*

and *B*'s...............$= 50\times\frac{1}{2} = 25$*d.* $=$ 2*s.* 1*d.*

PROB. 14. It was calculated that, if the gross revenue of a state were increased in the proportion of $2\frac{1}{4} : 1$, after deducting the interest of the national debt and the cost of collection (the latter of which varies as the square root of the sum collected), the available income would be increased in the proportion of $3\frac{16}{31} : 1$. If, on the other hand, the gross revenue were diminished in the proportion of $1\frac{7}{9} : 1$, the available income would be reduced in the proportion of $7\frac{3}{4} : 1$, and would in fact amount only to 4 millions. Find the amount of the revenue, and the interest of the debt.

$$\left.\begin{aligned} \text{Let } x &= \text{the gross revenue,}\\ y &= \text{the interest on the debt,}\\ z &= \text{the expense of collection,}\end{aligned}\right\}\ \textit{in millions of pounds sterling.}$$

Then $\frac{9x}{4}$ = increased revenue, and expense of collecting : $z :: \sqrt{x} : \sqrt{\frac{9x}{4}}$;

$$\therefore\ \text{expense of collecting} = \frac{3z}{2};$$

$$\text{Diminished revenue, and expense of collecting} = \frac{3z}{4};$$

$$\therefore \frac{9x}{16}-y-\frac{3z}{4} = 4 \ldots\ldots\ldots\ldots(1).$$

$$\text{Also } x-y-z : \frac{9x}{16}-y-\frac{3z}{4} :: 7\tfrac{3}{4} : 1 :: 31 : 4,$$

$$\therefore \text{ by (1), } x-y-z = 31 \ldots\ldots\ldots\ldots\ldots(2).$$

$$\text{Again, } x-y-z : \frac{9x}{4}-y-\frac{3z}{2} :: 1 : 3\tfrac{16}{31} :: 31 : 109;$$

$$\therefore \text{ by (2), } \frac{9x}{4}-y-\frac{3z}{2} = 109 \ldots\ldots\ldots(3).$$

$$\left.\begin{array}{r} \text{Subtracting (1) from (2) } \dfrac{7x}{16}-\dfrac{z}{4} = 27, \\ \ldots\ldots\ldots\ldots\ldots \text{(2) from (3) } \dfrac{5x}{4}-\dfrac{z}{2} = 78, \end{array}\right\};$$

$$\therefore \frac{5x}{4}-\frac{7x}{8} = 78-54 = 24,$$

$$\frac{3x}{8} = 24, \text{ and } x = 64.$$

$$\text{Hence } \frac{z}{4} = \frac{7x}{16}-27 = 28-27 = 1, \text{ and } \therefore z = 4.$$

$$\text{Also } y = x-z-31 = 60-31 = 29.$$

PROB. 15. A steam-boat sets out from London 3 miles behind a wherry, and having got to the same distance a-head it overtakes a barge *floating* down the stream, and reaches Gravesend $1\frac{1}{2}$ hours afterwards. Having waited to land the passengers $\frac{1}{6}$th of the time of coming down, it starts to return, and meets the wherry in $\frac{3}{4}$ of an hour, the barge being then $5\frac{1}{4}$ miles a-head of the steam-boat, and arrives at London in the same time that the wherry was in coming down. Find the distance between London and Gravesend, and the rate of each vessel.

Let x = rate of the boat, y = rate of the wherry, t = rate of the tide, that is, of the barge; then $\because$ boat's speed against the tide = wherry's speed with it, $x-t = y+t$, $\therefore t = \dfrac{x-y}{2}$.

$$\text{Hence } x+\frac{x-y}{2}, \text{ or } \frac{3x-y}{2} = \text{boat's speed down,}$$

$$x-\frac{x-y}{2}, \text{ or } \frac{x+y}{2} = \ldots\ldots\ldots\ldots\ldots \text{ up,}$$

$$\frac{6}{x-y} = \text{time before the boat overtakes the barge;}$$

$$\therefore \frac{6}{x-y}+\frac{3}{2} = \text{whole time of boat down,}$$

$$\frac{6}{5(x-y)}+\frac{3}{10} = \text{time for landing the passengers;}$$

$\therefore \dfrac{6}{5(x-y)}+\dfrac{3}{10}+\dfrac{3}{4}+\dfrac{3}{2}$, or $\dfrac{6}{5(x-y)}+\dfrac{51}{20}$ = interval of time between the boat passing the barge, and meeting the wherry in returning, in which time the barge moves over $\dfrac{x-y}{2}\left\{\dfrac{6}{5(x-y)}+\dfrac{51}{20}\right\}$, or $\dfrac{3}{5}+\dfrac{51}{40}(x-y)$ miles, and the boat has come up $\dfrac{3}{4}\cdot\dfrac{x+y}{2}$ miles,

$\therefore \dfrac{3}{4}\cdot\dfrac{x+y}{2}+\dfrac{3}{5}+\dfrac{51}{40}(x-y)+\dfrac{21}{4}$ (= distance from Gravesend when boat passed the barge down) $= \dfrac{3}{2}\cdot\dfrac{3x-y}{2}$, from which equation we get

$$4x+y = 39 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again $\dfrac{8\frac{1}{4}}{y}$ = time wherry takes to get from 3 miles behind to $5\frac{1}{4}$ miles a-head of the barge,

$$\therefore \frac{33}{4y} = \frac{6}{5(x-y)}+\frac{51}{20} \ldots\ldots\ldots\ldots (2),$$

from which two equations (1) and (2), $x = 9$, $y = 3$.

Hence also distance from London to Gravesend = time down × speed,

$$= \frac{5}{2}\times 12 = 30 \text{ miles.}$$

Prob. 16. *A* and *B* travelled on the same road and at the same rate to London. At the 50th mile-stone from London *A* overtook a flock of geese, which travelled at the rate of 3 miles in 2 hours; and 2 hours afterwards he met a stage-waggon which travelled at the rate of 9 miles in 4 hours. *B* overtook the flock of geese at the 45th mile-stone from London, and met the stage-waggon 40 minutes before he came to the 31st mile-stone. Where was *B* when *A* reached London?

Since *A* and *B* travel in the same direction on the same road and at the same rate, the distance between them is always the same.

Let x = the number of miles per hour of *A*'s and *B*'s travelling.

Then, since the places at which *A* and *B* overtake the geese are 5 miles apart, which the geese travel over in $5\div\dfrac{3}{2}$, or $\dfrac{10}{3}$ hours; therefore in that time *A* has moved forward $\dfrac{10x}{3}$ miles;

$$\therefore \frac{10x}{3}-5 = \text{distance in miles between } A \text{ and } B \ldots\ldots\ldots\ldots (1).$$

Again, A met the waggon $50-2x$ miles from London,

$$B \ldots\ldots\ldots\ldots\ldots 31+\frac{2x}{3} \ldots\ldots\ldots\ldots\ldots$$

$\therefore$ distance the waggon travelled between the meetings is $\frac{8x}{3}-19$ miles; and the time elapsed between A and B meeting the waggon

$$= \left(\frac{8x}{3}-19\right)\div\frac{9}{4}, \text{ or } \frac{4}{9}\left(\frac{8x}{3}-19\right) \text{ hours.}$$

During this time A has moved forward $\frac{4}{9}\left(\frac{8x}{3}-19\right)x$ miles;

$$\therefore \text{distance between } A \text{ and } B = \frac{8x}{3}-19+\frac{4x}{9}\left(\frac{8x}{3}-19\right) \text{ miles} \ldots\ldots\ldots (2);$$

$$\text{equating (1) and (2), } \frac{8x}{3}-19+\frac{4x}{9}\left(\frac{8x}{3}-19\right) = \frac{10x}{3}-5,$$

$$\frac{4x}{9}\left(\frac{8x}{3}-19\right) = \frac{2x}{3}+14,$$

$$16x^2-114x = 9x+189,$$

$$16x^2-123x = 189,$$

$$\therefore 16x^2-\frac{123}{4}.4x+\left(\frac{123}{8}\right)^2 = \frac{15129}{64}+189 = \frac{27225}{64},$$

$$\therefore 4x-\frac{123}{8} = \frac{165}{8};$$

$$4x = \frac{288}{8} = 36;$$

$$\therefore x = 9,$$

and $\frac{10x}{3}-5 = 25$, the required distance in miles of B from London.

PROB. 17. Fine gold chains are manufactured at Venice, and are sold at so much per braccio, a braccio being a measure containing about two feet English. When there are 90 links in an inch, the value of the workmanship of a braccio is equal to the whole value of a braccio when there are but 30 links in an inch; and the whole value of the braccio in the former case is equal to three-times the difference between the cost of the material and workmanship of a braccio in the latter, together with $4\frac{4}{5}$ francs. Supposing that the workmanship in each braccio varies as the number of links in an inch, and the weight of metal varies inversely as the square of that number, find the values of the material and workmanship in a braccio of each of the chains.

$$\text{Let } x = \left\{\begin{array}{l}\text{the value of the gold in a braccio 30 links to an inch, in} \\ \text{francs,}\end{array}\right.$$

$$y = \text{the value of the workmanship} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots :$$

$$\text{Then } \frac{1}{30^2} : \frac{1}{90^2} = x : \frac{x}{9}\left\{\begin{array}{l}\text{the value of gold in a braccio 90 links to} \\ \text{an inch,}\end{array}\right.$$

$$\text{and } 30 : 90 = y : 3y, \text{ the value of the workmanship;}$$

$$\therefore\ x+y = 3y, \quad \text{or } x = 2y \ldots\ldots\ldots\ldots (1),$$

$$\text{and } \frac{x}{9}+3y = 3(x-y)+4\tfrac{4}{9};$$

$$\therefore\ x+27y = 27x-27y+40,$$

$$54y = 26x+40,$$

$$27y = 13x+20,$$

$$= 26y+20, \text{ from (1);}$$

$$\therefore\ y = 20 \text{ francs, workmanship of 30 link-chain per braccio,}$$

$$x = 40 \ \ldots\ldots \quad \text{gold} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{x}{9} = 4\tfrac{4}{9} \text{ francs,} \quad \text{gold in the other chain, per braccio,}$$

$$3y = 60 \ \ldots\ldots \quad \text{workmanship} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Prob. 18. A pack of np cards is dealt regularly round to p persons with their faces uppermost, every card dealt to each person being placed upon that previously dealt to him; the hands are then taken up, turned so as to have their backs uppermost, and placed upon one another; that hand which contains a particular card (A) being always placed below r other hands. The cards are then dealt again, the hands taken up, turned, and placed upon one another as before; and so on:—Shew that, if m and q be the whole numbers next greater than $\frac{\log 2np}{\log p}$ and $\frac{rnp}{p-1}$ respectively, the card A will, at the end of the m^{th} and every succeeding operation, occupy the q^{th} place, or be restricted to the q^{th} and $\overline{q-1}^{\text{th}}$ places from the top, according as rn is indivisible or divisible by $p-1$. (*Senate-House Prob.* 1839. *by Mr. Gaskin.*)

Let the 1st operation be performed, and let a be the number of cards which stand before A in its own hand; then $a<n$, and

$$rn+a = \text{N}^{\circ} \text{ of cards before } A \text{ in the pack,}$$

$$\left.\begin{array}{l}\text{whole N}^{\circ}\text{ equal to,} \\ \text{or next less than} \ldots\end{array}\right\} \ \frac{rn+a}{p} = \left\{\begin{array}{l}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ own hand,} \\ \text{after 2nd operation,}\end{array}\right.$$

$$\left.\begin{array}{l}\text{whole N° equal to,}\\ \text{or next less than...}\end{array}\right\} rn+\frac{rn+a}{p}=\left\{\begin{array}{l}\text{N° of cards before } A \text{ in the pack,}\\ \quad\text{after 2nd operation,}\end{array}\right.$$

$$\text{..................}\quad \frac{rn}{p}+\frac{rn+a}{p^2}=\left\{\begin{array}{l}\text{.......................... own hand,}\\ \quad\text{after 3rd operation,}\end{array}\right.$$

$$\text{..................}\ rn+\frac{rn}{p}+\frac{rn+a}{p^2}=\left\{\begin{array}{l}\text{.......................... the pack,}\\ \quad\text{after 3rd operation,}\end{array}\right.$$

$$\text{\&c.} \quad = \quad \text{\&c.}$$

$\therefore$ whole number equal to, or next less than, $rn+\frac{rn}{p}+\frac{rn}{p^2}+\ldots+\frac{rn+a}{p^{m-1}}$,

that is, the whole number equal to, or next less than, $\frac{rn}{p^{m-1}}\cdot\frac{p^m-1}{p-1}+\frac{a}{p^{m-1}}$;

or $\frac{rnp}{p-1}+\frac{a}{p^{m-1}}-\frac{rn}{p^{m-1}(p-1)}$ = number of cards before A in the pack, after the m^{th} operation.

Hence, if q be the whole number next greater than $\frac{rnp}{p-1}$, it is evident, when $\frac{rn}{p-1}$ is not an integer, that the card A will be restricted to the q^{th} place, provided $\frac{a}{p^{m-1}}-\frac{rn}{p^{m-1}(p-1)}$ is a fraction so small that it cannot make a difference of 1 in the value of

$$\frac{rnp}{p-1}+\frac{a}{p^{m-1}}-\frac{rn}{p^{m-1}(p-1)}\ldots\ldots(1).$$

$$\text{Now } a<n,\ \therefore \frac{a}{p^{m-1}}<\frac{np}{p^m}.$$

If therefore $\frac{np}{p^m}=\text{or}<\frac{1}{2}$, $\frac{a}{p^{m-1}}<\frac{1}{2}$, and $\frac{a}{p^{m-1}}-\frac{rn}{p^{m-1}(p-1)}$ is a small fraction,

$$\left(\text{for, since } r<p,\ \frac{rn}{p^{m-1}(p-1)}<\frac{1}{2}\right)$$

and $\therefore \frac{a}{p^{m-1}}-\frac{rn}{p^{m-1}(p-1)}$ cannot make a difference of 1 in the above expression (1).

Now if $\frac{np}{p^m}=\frac{1}{2}$, or $p^m=2np$, $m=\frac{\log 2np}{\log p}$.

Hence at the m^{th} and every succeeding operation the card A is restricted to the q^{th} place from the top.

If $\frac{rnp}{p-1}$ be an integer, that is, rn be divisible by $p-1$, then it is evident that the number of cards before A may be either $q-1$, or $q-2$; and therefore A must be restricted to the q^{th} or $q-1^{\text{th}}$ place.

SINGLE AND DOUBLE POSITION.

MANY problems are readily solved by means of the Arithmetical Rules called "Single Position," and "Double Position," without the application of Algebra; but the Rules themselves require algebraic proof.

(1) The Rule of 'SINGLE POSITION' is applied to those cases only, in which the required quantity is some multiple, part, or parts, of some other *given* quantity; that is, if x represent the required value, a and b known quantities, the cases for 'Single Position' are such as produce an equation of the form

$$ax = b.$$

Thus, if it be required to find such a value x that $a \times x = b$, suppose s to be the value, and instead of b, we find $a \times s = b'$, then we have

$$\frac{ax}{as} = \frac{b}{b'},$$

$$\therefore\ x = \frac{b}{b'} . s,$$

which points out the Rule:—namely, *Suppose some value* (s) *to be the one sought, and having operated upon it as the question directs, let the result* (b′) *be noted. Then the true value is equal to the true result* (b) *divided by the false one* (b′) *and multiplied by the supposed value* (s).

Ex. Find the number which being added to the half and fourth of itself will produce 14.

Suppose 12 the number, then

$$12 + \tfrac{1}{2} \text{ of } 12 + \tfrac{1}{4} \text{ of } 12 \text{ is } 21;$$

$$\therefore \text{ number required} = \frac{14}{21} \times 12 = 8.$$

(2) The rule for 'DOUBLE POSITION' is applied to those cases in which the required quantity is not a multiple, part, or parts, of a given quantity, but furnishes an equation of the form

$$ax + b = cx + d.$$

By transposition this equation becomes

$$(a-c)x + b - d = 0 \ldots\ldots\ldots (1).$$

Now suppose s to be the value of x, which by substitution does not satisfy the equation, but gives

$$(a-c)s + b - d = e \ldots\ldots\ldots\ldots (2),$$

then subtracting (1) from (2),

$$(a-c)(s-x) = e.$$

Again, suppose s' to be the value of x, which by substitution and subtraction, as before, gives

$$(a-c)(s'-x) = e'.$$

$$\text{Then}\quad \frac{(a-c)(s-x)}{(a-c)(s'-x)} = \frac{e}{e'},$$

$$\text{or}\ \frac{s-x}{s'-x} = \frac{e}{e'},$$

$$\text{or}\quad e's-e'x = es'-ex,$$

$$\therefore\ x = \frac{es'-e's}{e-e'},$$

which proves the common Rule, namely, *Make two suppositions* (s *and* s′) *for the required quantity; treat each of them in the manner pointed out by the question; and note the errors* (e *and* e′); *then the required quantity will be found by dividing the difference of the products* es′, e′s *by the difference of the errors* e, e′.

Ex. . What number is that which, upon being increased by 10, becomes three times as great as it was before?

1st. Suppose the number to be 20, (s)

$$\text{then } 20+10 = 30,$$

$$\text{but } 3\times20 = 60,\ \therefore\ e = -30.$$

2nd. Suppose the number to be (30), (s')

$$\text{then } 30+10 = 40,$$

$$\text{but } 3\times30 = 90,\ \therefore\ e' = -50.$$

$$\text{Hence the true number} = \frac{-30\times30+50\times20}{-30+50} = 5.$$

PILES OF SHOT OR CANNON BALLS.

Ex. 1. In a pyramidal pile of shot of which the base is a square, the number in one side of the base is given, find the whole number in the pile.

Let n be the given number in one side of the base, then n^2 is the number in the base;

$n-1$ is the number in a side of the next superincumbent square,

$\therefore\ (n-1)^2$ .... the whole square.

Similarly $(n-2)^2$ is the number in the next square; and so on, until the squares are diminished to a single shot.

Hence the whole number in the pile is equal to the sum of the series

$$n^2+(n-1)^2+(n-2)^2+\&c.\ldots+1^2,$$

or $1^2+2^2+3^2+\ldots+n^2$, which by Art. 295, Ex. 4, is equal to

$$\frac{1}{6}n(n+1)(2n+1).$$

Ex. 2. In a pyramidal pile of shot of which the base is an equilateral triangle, the number in a side of the base is given, find the whole number in the pile.

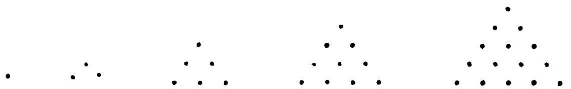
Let n be the given number in a side of the base,

then $n-1$ is the number in a side of the next superincumbent triangle,

$n-2$ .... 3rd ....

$n-3$ .... 4th ....

and so on, until we get to a single ball; that is, there are n triangles, which, beginning at the other end of the series, may be represented thus,



&c.

so that the whole number of shot in the pile, being the sum of the shot in these n triangles, will be the sum of the series

$$1+3+6+10+15+21+\&c. \text{ to } n \text{ terms.}$$

1st. Let n be even; then the series, taking pairs of terms together

$$=4+16+36+\&c. \text{ to } \frac{n}{2} \text{ terms,}$$

$$=4\{1^2+2^2+3^2+\&c. \text{ to } \frac{n}{2} \text{ terms}\},$$

$$=4\cdot\frac{1}{6}\cdot\frac{n}{2}\left(\frac{n}{2}+1\right)(n+1)=\frac{1}{6}n(n+1)(n+2). \quad \text{(Art. 295.)}$$

2nd. Let n be odd; then the series may be written

$$1+9+25+49+\&c. \text{ to } \frac{n+1}{2} \text{ terms,}$$

or $1^2+3^2+5^2+7^2+\&c.$ to $\frac{n+1}{2}$ terms; which is equal to

$$\frac{1}{3}\cdot\frac{n+1}{2}\left(4\cdot\left.\overline{\frac{n+1}{2}}\right|^2-1\right), \text{ (Art. 295, Ex. 4),}$$

$$=\frac{1}{6}(n+1)(n^2+2n),$$

$$=\frac{1}{6}n(n+1)(n+2), \text{ as before.}$$

Ex. 3. To find the number of shot in a pile of which the base is a rectangular parallelogram, whose sides contain a given number.

Let l and b represent the given number of balls in the length and breadth respectively of the base;

then bl = number of balls in the base. The next layer will be a parallelogram whose sides are $b-1$, and $l-1$; and $\therefore$ $(b-1)(l-1)$ is the number of balls in this layer. In the next the number is $(b-2)(l-2)$: and so on, until we come to $(b-\overline{b-1})(l-\overline{b-1})$, or $1\times(l-b+1)$; the last parallelogram being reduced to a straight line of $l-b+1$ balls.

Hence the whole number of balls in the pile

$$\begin{aligned}
&= bl+(b-1)(l-1)+(b-2)(l-2)+\ldots\ldots \text{to } b \text{ terms},\\
&= bl+bl+bl+\&\text{c. to } b \text{ terms}-(1+2+3+\&\text{c.}+\overline{b-1})(b+l)\\
&\qquad\qquad\qquad +1^2+2^2+3^2+\&\text{c.}+\overline{b-1}|^2,\\
&= b^2l-b.\frac{b-1}{2}.(b+l)+\frac{1}{6}(b-1)b(2b-1),\\
&= \left(b^2-b.\frac{b-1}{2}\right)l-b.\frac{b-1}{2}.\left(b-\frac{1}{3}.\overline{2b-1}\right),\\
&= \frac{b(b+1)}{2}.l-b.\frac{b-1}{2}.\frac{b+1}{3},\\
&= \frac{1}{6}b(b+1)(3l-b+1).
\end{aligned}$$

Obs. To find the number of balls in an *incomplete* pile it is only necessary to find the number in the pile which is wanting to complete the given one, and subtract that number from the number in the given pile supposed complete.

Ex. 4. To find the sum of n terms of any order of *Figurate Numbers.*

Def. *Figurate Numbers* are formed by making the n^{th} term of each order the sum of n terms of the preceding order. Thus,

Figurate Numbers of 1st order are 1, 1, 1, 1, 1, &c.
.... 2nd .... 1, 2, 3, 4, 5, &c.
.... 3rd .... 1, 3, 6, 10, 15, &c.
.... 4th .... 1, 4, 10, 20, 35, &c. and so on.

Hence, for 1st order, sum of n terms $= n$,

... 2nd ......, ... $= \frac{1}{2}n(n+1)$,

... 3rd ......, ... $= \frac{1}{6}n(n+1)(n+2)$; &c.

and for m^{th} ......, ... $= \dfrac{n(n+1)\ldots(n+m-1)}{1.2\ldots m}$.

THE

GENERAL THEORY

OF

EQUATIONS.

NATURE OF EQUATIONS.

ART. 483. ANY equation, involving the powers of one unknown quantity, may be reduced to the form

$$x^n + px^{n-1} + qx^{n-2} + \&c. = 0;$$

where the whole expression is made equal to nothing, the terms are arranged according to the dimensions of the unknown quantity, the coefficient of the highest dimension is unity, and the coefficients, p, q, r, &c. are affected with their proper signs.

An equation, in which the index of the highest power of the unknown quantity is n, is said to be of n *dimensions;* and in speaking simply of an equation of n dimensions, we understand one reduced to the above form, unless the contrary be expressed.

484. Any quantity of the form

$$x^n + px^{n-1} + qx^{n-2} \ldots + Px + Q,$$

may be supposed to arise from the multiplication of

$$(x-a)(x-b)(x-c).\&c.$$

continued to n factors.

For, by actually multiplying the factors together, we obtain a quantity of n dimensions, similar to the proposed quantity,

$$x^n + px^{n-1} + qx^{n-2} + \&c.;$$

and if a, b, c, &c. can be so assumed that the coefficients of the corresponding terms in the two quantities become equal, the whole expressions coincide. And these coefficients may be made equal, because we shall have n equations to determine the n quantities a, b, c, d, &c. (See Art. 198.) If then the quantities, a, b, c, d, &c. be properly assumed, the equation

$$x^n + px^{n-1} + qx^{n-2} + \&c. = 0,$$

is the same with $(x-a)(x-b)(x-c).\&c. = 0$*.

We cannot suppose $x^n + px^{n-1} + qx^{n-2} + \&c.$ to be made up of more, or fewer, than n simple factors; because, on either supposition, the result would not be of the same number of *dimensions* with the proposed quantity.

485. DEF. The quantities a, b, c, d, &c. are called *roots* of the equation, or *values* of x; because, if any one of them be substituted for x, the whole expression becomes nothing, which is the only condition proposed by the equation.

486. *If the signs of the terms of an equation be all positive, it cannot have a positive root; and if the signs be alternately positive and negative, it cannot have a negative root.*

If $x^n + px^{n-1} + qx^{n-2} + \&c. = 0$, where p, q, &c. are all positive, and any positive quantity, a, be substituted for x, the result is positive; consequently a is not a root of the equation.

If $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, and a negative quantity, $-a$, be substituted for x, when n is an odd number the result is negative, and when n is an even number the result is positive; therefore $-a$ cannot, in either case, be a root of the equation.

487. *Every equation has as many roots as it has dimensions, and no more.*

If $x^n + px^{n-1} + qx^{n-2} + \&c. = 0$,

or $(x-a)(x-b)(x-c).\&c.$ to n factors $= 0$;

there are n quantities, a, b, c, &c. each of which, when substituted for x, makes the whole $= 0$, because in each case one of the factors becomes $= 0$; but any quantity different from these, as e, when substituted for x, gives the product $(e-a)(e-b)(e-c)$.&c. which does not vanish, because none of the factors vanish; that is, e will not answer the condition which the equation requires, and is therefore not a *root.*

---

* This proof, which is usually given, is imperfect; for if the n equations be reduced to one, containing only one of the quantities, a, this equation is $a^n + pa^{n-1} + qa^{n-2} + \&c. = 0$, which exactly coincides with the proposed equation; in supposing therefore that a can be found, we take for granted that the equation can be solved.

488. When one of the roots, a, is obtained, the equation

$$(x-a)(x-b)(x-c).\&c. = 0,$$

$$\text{or } x^n + px^{n-1} + qx^{n-2} + \&c. = 0,$$

is divisible by $x-a$, without a remainder, and is thus reducible to $(x-b)(x-c).\&c. = 0$, an equation one dimension lower, whose roots are b, c, &c.

Ex. One root of the equation $y^3+1=0$ is -1, therefore $y+1=0$, and the equation may be depressed to a quadratic, by division. The quotient of y^3+1 by $y+1$ is y^2-y+1.

Hence the other two roots are the roots of the quadratic

$$y^2-y+1=0.$$

If *two* roots, a and b, be obtained, the equation is divisible by $(x-a)(x-b)$; and thus it may be reduced two dimensions lower.

Ex. Two roots of the equation $x^6-1=0$, are $+1$ and -1, therefore $x-1=0$, and $x+1=0$; and it may be depressed to a biquadratic by dividing x^6-1 by $(x-1)(x+1)$, or by x^2-1. The quotient is x^4+x^2+1.

Hence the equation $x^4+x^2+1=0$ contains the other *four* roots of the proposed equation $x^6-1=0$.

489. Conversely, if the equation be divisible by $x-a$, without remainder, a is a root; if by $(x-a)(x-b)$, a and b are both roots; &c. In the latter case, let Q be the quotient arising from the division, then the equation is

$$(x-a)(x-b)Q=0,$$

in which if a or b be substituted for x, the whole vanishes.

490. Cor. 1. If a, b, c, &c. be the roots of an equation, that equation is

$$(x-a)(x-b)(x-c).\&c. = 0.$$

Thus, the equation whose roots are 1, 2, 3, 4, is

$$(x-1)(x-2)(x-3)(x-4)=0;$$

$$\text{or } x^4-10x^3+35x^2-50x+24=0.$$

The equation whose roots are 1, 2, and -3, is

$$(x-1)(x-2)(x+3)=0, \quad \text{or } x^3-7x+6=0.$$

491. COR. 2. If the last term of an equation vanish, the equation is of the form $x^n+px^{n-1}+qx^{n-2}\ldots\ldots+Px=0$, which is divisible by x, or $x-0$, without remainder; therefore 0 is one of its roots; if the two last terms vanish, it is divisible by x^2, without remainder, or by $(x-0)(x-0)$, that is, two of its roots are 0; &c.

492. *The coefficient of the second term of an equation is the sum of the roots, with their signs changed; the coefficient of the third term is the sum of the products of every two roots, with their signs changed; the coefficient of the fourth term is the sum of the products of every three roots, with their signs changed, &c.; and the last term is the product of all the roots, with their signs changed.*

Let a, b, c, &c. be the roots of the equation; then $(x-a)(x-b)(x-c)$.&c. $=0$, is that equation; and by Art. 308, it appears, that when these factors are multiplied together, the coefficient of the second term is the sum of the quantities $-a$, $-b$, $-c$, &c.; the coefficient of the third term, the sum of the products of every two, &c.; and the last term, which does not contain x, is the product of all those quantities.

493. COR. 1. If the roots be all positive, the signs of the terms will be alternately + and −. For the product of an odd number of negative quantities is negative, and of an even number positive. But if the roots be all negative, the signs of all the terms will be positive, because the equation arises from the continued product $(x+a)(x+b)(x+c)$.&c, in which every sign is positive.

494. Any equation, it has been observed, may be conceived to arise from the multiplication of the simple factors $(x-a)(x-b)(x-c)$.&c. or by taking two or more of these together, it

may be supposed to arise from the multiplication of *quadratic*, *cubic*, &c. factors, if the dimensions of these factors together make up the dimensions of the proposed equation.

Thus a cubic equation may be supposed to be the product of *three* simple factors, as

$$(x-a)(x-b)(x-c)=0;$$

or of a quadratic and a simple factor, as

$$(x^2-px+q)\times(x-c)=0.$$

495. *Impossible roots enter equations by pairs.*

If $a+\sqrt{-b^2}$ be a root of the equation $x^n+px^{n-1}+\&c.=0$, then $a-\sqrt{-b^2}$ is also a root.

In the proposed equation for x substitute $a+\sqrt{-b^2}$, and the result will consist of two parts, possible quantities, which involve the powers of a and the even powers of $\sqrt{-b^2}$, and impossible quantities which involve the odd powers of $\sqrt{-b^2}$; call the sum of the possible quantities A, and of the impossible B, then $A+B$ is the whole result. Let now $a-\sqrt{-b^2}$ be substituted for x, and the possible part of the result will be the same as before, and the impossible part, which arises from the odd powers of $-\sqrt{-b^2}$, will only differ from the former impossible part in its sign; therefore the result is $A-B$; and since by the supposition $a+\sqrt{-b^2}$ is a root of the equation $A+B=0$; in which, as no part of A can be destroyed by B, $A=0$, and $B=0$; therefore $A-B=0$, that is, the result, arising from the substitution of $a-\sqrt{-b^2}$ for x, is nothing; or $a-\sqrt{-b^2}$ is a root of the equation.

496. COR. 1. Hence it follows, that an equation of an *odd* number of dimensions must have, at least, one possible root, unless some of the coefficients are impossible, in which case the equation may have an odd number of impossible roots.

497. COR. 2. By the same mode of reasoning it appears that, *when the coefficients are rational*, surd roots of the form $\pm\sqrt{b}$, or $a\pm\sqrt{b}$, enter equations by pairs.

TRANSFORMATION OF EQUATIONS.

498. *If the signs of all the terms in an equation be changed, its roots are not altered.*

Let $x^n + px^{n-1} + qx^{n-2} + \&c. = 0$, of which the roots are a, b, c, &c., then

$$(x-a)(x-b)(x-c).\&c. = 0;$$

and if the signs of all the terms be changed, the equation becomes

$$-(x-a)(x-b)(x-c).\&c. = 0;$$

which is satisfied, if a, b, c, &c. are substituted for x.

499. *If the signs of the alternate terms, beginning with the second, be changed*, the signs of all the roots are changed.*

Let $x^n + px^{n-1} + qx^{n-2} + \&c. = 0$, be an equation whose roots are a, b, $-c$, &c.; for x substitute $-y$, and when n is an even number, the equation becomes $y^n - py^{n-1} + qy^{n-2} - \&c. = 0$; but when n is an odd number, $-y^n + py^{n-1} - qy^{n-2} + \&c. = 0$, or changing all the signs (Art. 498), $y^n - py^{n-1} + qy^{n-2} - \&c. = 0$, as before; and since $x = -y$, or $y = -x$, the values of y are $-a$, $-b$, $+c$, &c.

Ex. Let it be required to change the signs of the roots of the equation $x^3 - qx + r = 0$.

This equation with all its terms is $x^3 + 0 - qx + r = 0$; and changing the signs of the alternate terms, beginning with the second, we have

$$x^3 - 0 - qx - r = 0, \text{ or } x^3 - qx - r = 0,$$

an equation whose roots differ from the roots of the given one only in their signs.

500. *To transform an equation into one whose roots are greater or less than the corresponding roots of the original equation by any given quantity.*

I. Let the roots of the equation $x^3 + px^2 + qx + r = 0$ be a, b, c; to transform it into one whose roots are $a+e$, $b+e$, $c+e$.

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* That is, supposing either the equation to be complete in all its terms, or the terms hat are wanting to be replaced by ciphers and considered as terms for this purpose.—Ed.

Assume $x+e=y$, or $x=y-e$; then

$$\begin{array}{rl} x^3 = & y^3 - 3ey^2 + 3e^2y - e^3 \\ +px^2 = & \quad + py^2 - 2pey + pe^2 \\ +qx = & \qquad + qy - qe \\ +r = & \qquad\qquad + r \end{array}$$

$$\therefore\ y^3 - (3e-p)y^2 + (3e^2 - 2pe + q)y - (e^3 - pe^2 + qe - r) = 0.$$

In this last equation, since $y = x + e$, the values of y are

$$a+e,\quad b+e,\quad c+e.$$

If $y+e$ be substituted for x, the values of y in the resulting equation will be $a-e$, $b-e$, $c-e$.

II. In general, let the roots of the equation

$$x^n + px^{n-1} + qx^{n-2} + \text{\&c.} = 0, \text{ be } a,\ b,\ c,\ \text{\&c.}$$

Assume $y = x - e$, or $x = y + e$; then, by substitution,

$$\left.\begin{array}{rl} x^n = & y^n + ney^{n-1} + n.\frac{n-1}{2}e^2y^{n-2} \ldots + \qquad ne^{n-1}y + e^n \\ +px^{n-1} = & \quad + py^{n-1} + (n-1)pey^{n-2} \ldots + (n-1)pe^{n-2}y + pe^{n-1} \\ +qx^{n-2} = & \qquad + \qquad qy^{n-2} \ldots + (n-2)qe^{n-3}y + qe^{n-2} \\ \text{\&c.} = & \qquad\qquad \text{\&c.} \end{array}\right\} = 0,$$

and since $y = x - e$, the values of y, in this equation, are

$$a-e,\quad b-e,\quad c-e,\quad \text{\&c.}$$

501. One use of this transformation is to *take away* any term out of an equation. Thus, to transform an equation into one which shall want the *second* term, e must be so assumed that $ne+p=0$, or $e = -\frac{p}{n}$ (where p is the coefficient of the second term, and n the index of the highest power of the unknown quantity); and if the roots of the transformed equation can be found, the roots of the original equation may also be found, because $x = y - \frac{p}{n}$.

Ex. To transform the equation $x^3 - 9x^2 + 7x + 12 = 0$ into one which shall want the second term.

Assume $x = y + 3$, then

$$\left.\begin{array}{rl} x^3 = & y^3 + 9y^2 + 27y + 27 \\ -9x^2 = & \quad\;\; -9y^2 - 54y - 81 \\ +7x = & \qquad\qquad\;\; +\; 7y + 21 \\ +12 = & \qquad\qquad\qquad\quad\; +12 \end{array}\right\} = 0,$$

that is, $y^3 - 20y - 21 = 0$; and if the values of y be a, b, c, the values of x are $a + 3$, $b + 3$, and $c + 3$.

502. To take away the *third* term of the equation, e must be so assumed, that

$$n \cdot \frac{n-1}{2} e^2 + (n-1)pe + q = 0.$$

In this case we shall have a quadratic to solve; and in general, to take out the m^{th} term, by this method, it will be necessary to solve an equation of $\overline{m-1}$ dimensions.

Ex. To transform the equation $x^3 - 6x^2 + 9x - 1 = 0$ into one which shall want the third term.

Here $n = 3$, $p = -6$, and $q = 9$; therefore

$$n \cdot \frac{n-1}{2} e^2 + (n-1)pe + q = 0 \text{ becomes}$$

$$3e^2 - 12e + 9 = 0, \text{ or } e^2 - 4e + 3 = 0,$$

in which the values of e are 1 and 3. Let $x = y + 3$, then

$$\left.\begin{array}{rl} x^3 = & y^3 + 9y^2 + 27y + 27 \\ -6x^2 = & \quad\;\; -6y^2 - 36y - 54 \\ +9x = & \qquad\qquad\;\; +\; 9y + 27 \\ -1 = & \qquad\qquad\qquad\quad\; -\; 1 \end{array}\right\} = 0,$$

that is, $y^3 + 3y^2 - 1 = 0$.

In the same manner, if $x = y + 1$, the transformed equation will want the third term, and will be

$$y^3 - 3y^2 + 3 = 0.$$

503. *To transform an equation into one whose roots are the* reciprocals *of the roots of the given equation.*

Let the roots of the equation

$$x^n + px^{n-1} + qx^{n-2} \ldots + Px + Q = 0$$

be a, b, c, &c.; to transform it into one whose roots are $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c.

Assume $y=\frac{1}{x}$, or $x=\frac{1}{y}$; then, by substitution,

$$\frac{1}{y^n}+\frac{p}{y^{n-1}}+\frac{q}{y^{n-2}}\dots+\frac{P}{y}+Q=0,$$

and multiplying by y^n,

$$1+py+qy^2\dots+Py^{n-1}+Qy^n=0;$$

that is, $Qy^n+Py^{n-1}\dots+qy^2+py+1=0$;

$$\text{or } y^n+\frac{P}{Q}y^{n-1}+\dots+\frac{p}{Q}y+\frac{1}{Q}=0;$$

and since $y=\frac{1}{x}$, the values of y are $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c.

504. Cor. 1. If any term in the given equation be wanting, the corresponding term will be wanting in the transformed equation; thus, if the original equation want the second term, the transformed equation will want the last term but one, &c. because the coefficients in the transformed equation are the coefficients of the original equation in an inverted order.

505. Cor. 2. If the coefficients of the terms, taken from the beginning of an equation, be the same with the coefficients of the corresponding terms, taken from the end, with the same signs, the transformed will coincide with the original equation, and their *roots* will therefore be the same.

Let a, b, c be roots of the equation

$$x^n+px^{n-1}+qx^{n-2}\dots\dots+qx^2+px+1=0;$$

the transformed equation will be

$$y^n+py^{n-1}+qy^{n-2}\dots\dots+qy^2+py+1=0,$$

and a, b, c must also be roots of this equation; but the roots of this equation are the reciprocals of the roots of the original equation, therefore $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are also roots of the original equation; that is, the roots of either equation are

$$a,\ \frac{1}{a},\ b,\ \frac{1}{b},\ c,\ \frac{1}{c},\ \text{\&c.}$$

Ex. The roots of the equations

$$x^4-px^3+qx^2-px+1=0,\ \ x^4+qx^2+1=0,$$
$$\text{and } x^4+1=0,$$

are of the form a, b, $\frac{1}{a}$, $\frac{1}{b}$.

506. Cor. 3. If the equation be of an odd number of dimensions, or if the middle term of an equation of an even number of dimensions be wanting, the same thing will hold when the signs of the corresponding terms, taken from the beginning and end, are different.

Ex. The roots of the equation $x^3-px^2+px-1=0$ are of the form 1, a, $\frac{1}{a}$. For, in this case, if the signs of all the terms of the transformed equation be changed, it will coincide with the original equation; and by changing the signs of all the terms, we do not alter the roots. (Art. 498.)

Def. The equations described in the last two corollaries are called *recurring* equations.

507. Cor. 4. One root of a recurring equation of an odd number of dimensions will be $+1$, or -1, according as the sign of the last term is $-$ or $+$; and the rest will be of the form a, $\frac{1}{a}$, b, $\frac{1}{b}$, &c.

For if $+1$, in the former case, and -1, in the latter, be substituted for the unknown quantity, the whole vanishes; thus, if

$$x^5-px^4+qx^3-qx^2+px-1=0,$$

and for x we substitute $+1$, it becomes

$$1-p+q-q+p-1=0;$$

and it appears from Art. 503, that if a, b, c, &c. be roots of the equation, $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c. are also roots.

LIMITS OF THE ROOTS OF EQUATIONS.

508. If a, b, c, $-d$, &c. be the roots of an equation, taken in order of magnitude, that is, a greater than b, b greater than c, &c.* the equation is

$$(x-a)(x-b)(x-c)(x+d).\&c.=0;$$

in which, if a quantity greater than a be substituted for x, as every factor is, on this supposition, positive, the result will be positive; if a quantity less than a, but greater than b, be substituted, the result will be negative, because the first factor will be negative and the rest positive; if a quantity between b and c be substituted, the result will again be positive, because the first two factors are negative and the rest positive: and so on. Thus quantities which are *limits* to the roots of an equation, or between which the roots lie, if substituted successively for the unknown quantity, give results alternately positive and negative.

509. Conversely, if two magnitudes, when substituted for the unknown quantity, give results affected with different signs, an odd number of roots must lie between them; and if a series of magnitudes, taken in order, can be found, which give as many results, alternately positive and negative, as the equation has dimensions, these must be limits to the roots of the equation; because an odd number of roots lies between each two succeeding terms of the series, and there are as many terms as the equation has dimensions; therefore this odd number cannot exceed 1.

510. If the results arising from the substitution of two magnitudes for the unknown quantity be both positive or both negative, either no root of the equation, or an even number of roots, lies between them.

511. COR. If m, and every quantity greater than m, when substituted for the unknown quantity, give positive results, m is greater than the greatest root of the equation.

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* In this series, the greater d is, the less is $-d$. And whenever a, b, c, $-d$, &c. are said to be the roots of an equation taken in order, a is supposed to be the greatest. Also, in speaking of the limits of the roots of an equation, we understand the limits of the *possible* roots.

512. *To find a limit greater than the greatest root of an equation.*

Let the roots of the equation be a, b, c, &c.; transform it into one whose roots are $a-e$, $b-e$, $c-e$, &c. and if, by trial, such a value of e be found, that every term of the transformed equation is positive, all its roots are negative (Art. 486), and consequently e is greater than the greatest root of the proposed equation.

Ex. 1. To find a number greater than the greatest root of the equation $x^3 - 5x^2 + 7x - 1 = 0$.

Assume $x = y + e$, and we have

$$\left.\begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ - 5y^2 - 10ey - 5e^2 \\ + 7y + 7e \\ - 1 \end{array}\right\} = 0,$$

in which equation, if 3 be substituted for e, each of the quantities

$$e^3 - 5e^2 + 7e - 1, \quad 3e^2 - 10e + 7, \quad 3e - 5,$$

is positive, or all the values of y are negative; therefore 3 is greater than the greatest value of x.

Ex. 2. In any cubic equation of this form,

$$x^3 - qx + r = 0,$$

$\sqrt{q}$ is greater than the greatest root.

By transforming the equation, as before,

$$\left.\begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ - qy - qe \\ + r \end{array}\right\} = 0,$$

and substituting $\sqrt{q}$ for e,

$$y^3 + 3\sqrt{q}.y^2 + 2qy + r = 0,$$

every term of which is positive; therefore $\sqrt{q}$ is greater than the greatest value of x.

513. Cor. If the signs of the roots be changed, a limit greater than the greatest root of the resulting equation, with its sign changed, is less than the least root of the proposed equation.

Ex. Required a limit less than the least root of the equation $y^3 - 3y + 72 = 0$.

When the signs of the roots are changed, this equation becomes $y^3 - 3y - 72 = 0$. (Art. 499.)

Assume $y = x + e$; then

$$\left.\begin{aligned} x^3 + 3ex^2 + 3e^2x + e^3 \\ -\ \ 3x - 3e \\ -72 \end{aligned}\right\} = 0;$$

and, if 5 be substituted for *e*, every term becomes positive, consequently 5 is greater than the greatest root of the equation $y^3 - 3y - 72 = 0$; and -5 less than the least root of the equation $y^3 - 3y + 72 = 0$.

514. *The greatest negative coefficient increased by unity is greater than the greatest root of an equation.*

Let $$x^n - px^{n-1} - qx^{n-2} - \&c. = 0,$$
and if the coefficients be equal to each other,

$$x^n - px^{n-1} - px^{n-2} - \&c. = 0,$$

$$\text{or } x^n - p\times(x^{n-1} + x^{n-2} \ldots\ldots + x + 1) = 0,$$

$$\text{that is, } x^n - p\times\frac{x^n - 1}{x - 1} = 0. \quad \text{(Art. 290.)}$$

In this equation substitute $1 + p$ for x, and the result is

$$(1+p)^n - p\times\frac{(1+p)^n - 1}{p}, \text{ or } +1;$$

and if any of the coefficients in the given equation be positive, or less than p, the sum of the series to be taken from x^n will be diminished, and the result greater than before. Also, if for x, any quantity still greater be substituted, as $p + m + 1$, the result is

$$(p + m + 1)^n - \frac{p}{p + m}\times(p + m + 1)^n + \frac{p}{p + m};$$

$$\text{or } \frac{m}{p + m}\times(p + m + 1)^n + \frac{p}{p + m},$$

a positive quantity; therefore $1 + p$ is greater than the greatest root (Art. 512).

515. ***The roots of the equation***

$$nx^{n-1}+(n-1)px^{n-2}+(n-2)qx^{n-3}+\&c.=0$$

are limits between the roots of the equation

$$x^n+px^{n-1}+qx^{n-2}+\&c.=0,$$

when the roots of the latter equation are possible.

Let the roots of this equation, taken in order, be a, b, c, $-d$, &c. and in it, for x, substitute $y+e$, then by Art. 500,

$$\left.\begin{array}{rr} y^n+ncy^{n-1}\ldots\ldots+ & ne^{n-1}y+e^n \\ +py^{n-1}\ldots\ldots+ & (n-1)pe^{n-2}y+pe^{n-1} \\ \ldots\ldots+ & (n-2)qe^{n-3}y+qe^{n-2} \\ +\&c. & \end{array}\right\}=0,$$

the roots of which equation are

$$a-e,\quad b-e,\quad c-e,\quad -d-e,\ \&c.,$$

and the coefficient of the last term but one of any equation of n dimensions is the sum of the products of every $n-1$ roots, with their signs changed (Art. 492); therefore

$$\left.\begin{array}{r} ne^{n-1}+(n-1)pe^{n-2} \\ +(n-2)qe^{n-3}+\&c. \end{array}\right\}=\left\{\begin{array}{l} (e-a)(e-b)(e-c).\ \&c. \\ +(e-a)(e-b)(e+d).\&c. \\ +(e-a)(e-c)(e+d).\&c. \\ +(e-b)(e-c)(e+d).\&c. \end{array}\right.$$

in which, if a, b, c, $-d$, &c. be successively substituted for e, the results are

$(a-b)(a-c)(a+d)$.&c. which is positive,
$(b-a)(b-c)(b+d)$.&c. negative,
$(c-a)(c-b)(c+d)$.&c. positive,
$(-d-a)(-d-b)(-d-c)$.&c. negative, &c.

therefore a, b, c, $-d$ are limits to the roots of the equation

$$ne^{n-1}+(n-1)pe^{n-2}+\&c.=0$$

(Art. 510); or, substituting x for e, to the roots of the equation

$$nx^{n-1}+(n-1)px^{n-2}+\&c.=0.$$

Let α, β, γ, &c. be the roots of this equation, taken in order, then a, α, b, β, c, γ, $-d$, &c. are arranged according to their magnitudes, that is, α, β, γ, &c. lie between the roots of the equation

$$x^n+px^{n-1}+qx^{n-2}+\&c.=0.$$

516. *Every equation whose roots are possible has as many changes of signs from + to −, and from − to +, as it has positive roots; and as many continuations of the same sign, from + to +, and from − to −, as it has negative roots.*

$$\text{Let } x^n - px^{n-1} \ldots\ldots \pm Sx^2 \pm Px \pm Q = 0;$$

the equation of limits is

$$nx^{n-1} - (n-1)px^{n-2} \ldots\ldots \pm 2Sx \pm P = 0,$$

which, as far as it goes, has the same signs with the former; and therefore the original equation will have one more change of signs, or one more continuation of the same sign, than the limiting equation, according as the signs of P and Q are different, or the same.

Suppose α, β, γ, &c. to be the roots of the limiting equation; then the roots of the original equation are, by Art. 515, of this form, a, b, c, $\pm d$, &c.; therefore, with its proper sign,

$$P = n \times - \alpha \times - \beta \times - \gamma \times \&\text{c.}$$

$$\text{and } Q = -a \times - b \times - c \times \mp d \times \&\text{c.} \quad \text{(Art. 492)},$$

which products will have the same sign when the multiplier d is positive, or the root $(-d)$ negative, and different signs when that root is positive. It appears then, that if the original equation have one more change of signs than the limiting equation, it has one more positive root; and, if it have one more continuation of the same sign, it has one more negative root; therefore if it can be shewn that every equation of $n-1$ dimensions, and consequently the equation

$$nx^{n-1} + (n-1)px^{n-2} + (n-2)qx^{n-3} + \&\text{c.} = 0,$$

which is the limiting equation to

$$x^n + px^{n-1} + qx^{n-2} + \&\text{c.} = 0,$$

has as many changes of signs as it has positive roots, and as many continuations of the same sign as it has negative roots, the same rule will be true in the equation

$$x^n + px^{n-1} + qx^{n-2} + \&\text{c.} = 0;$$

or, in other words, if the rule be true of every equation of one order, it is true of every equation of the next superior order.

Now in every simple equation $x-a=0$, or $x+a=0$, the rule is true, therefore it is true in every quadratic

$$x^2 \pm px \pm q = 0;$$

and if it be true in every quadratic, it is true in every cubic; and so on; that is, the rule is true in all cases.

In the demonstration each root, $\pm d$, is supposed to be distinct from the rest, and a possible quantity.

Hence, when all the roots are possible, the number of positive roots is exactly known.

Ex. The equation $x^3+x^2-14x+8=0$ has two positive roots and one negative; because the signs are +, +, −, +, in which there are two changes, one from + to −, and the other from − to +, and one continuation of the sign +.

517. *When any coefficient vanishes, it may be considered either as positive or negative, because the value of the whole expression is the same on either supposition.*

Ex. If the roots of the equation $x^3-qx+r=0$ be possible, two of them are positive and the third is negative; for there are two changes of signs in the equation

$$x^3 \pm 0 - qx + r = 0,$$

and one continuation of the same sign.

518. *To find between which of the roots of a proposed equation any given number lies.*

Let the roots of the proposed equation be diminished by the given number, and the number of negative roots in the transformed equation will shew its place among the roots of the original equation.

Ex. To find between which of the roots of the equation $x^3-9x^2+23x-15=0$ the number 2 lies.

Assume $x=y+2$; then

$$\left.\begin{array}{r} x^3 \\ -\ 9x^2 \\ +\,23x \\ -\,15 \end{array}\right\} = \left.\begin{array}{r} y^3+6y^2+12y+\ \ 8 \\ -\,9y^2-36y-36 \\ +\,23y+46 \\ -\,15 \end{array}\right\} = 0,$$

or $y^3-3y^2-y+3=0$,

which has one negative root; and the roots of the proposed equation are all positive; therefore two of them are greater, and one is less, than 2.

SOLUTION OF RECURRING EQUATIONS.

519. *The roots of a recurring equation of an even number of dimensions, exceeding a quadratic, may be found by the solution of an equation of half the number of dimensions.*

Let $x^n + px^{n-1} + \ldots. + px + 1 = 0$ be the given equation in which n is even; its roots are of the form $a, \frac{1}{a}, b, \frac{1}{b}$, &c. (Art. 505); or it may be conceived to be made up of quadratic factors,

$$(x-a)\left(x-\frac{1}{a}\right);\quad (x-b)\left(x-\frac{1}{b}\right);\ \text{\&c.}$$

$$\text{that is, if } \alpha = a + \frac{1}{a},\ \beta = b + \frac{1}{b},\ \text{\&c.}$$

of the quadratic factors

$$x^2 + \alpha x + 1,\quad x^2 + \beta x + 1,\ \text{\&c.}$$

Then, by multiplying these together, and equating the coefficients with those of the proposed equation, the values of α, β, &c. may be found. Moreover, for every value of each of the quantities α, β, &c. there are two values of x; therefore the equation for determining the value of α will rise only to half as many dimensions as x rises to in the original equation.

520. If the recurring equation be of an *odd* number of sions, +1 or −1 is a root (Art. 507); and the equation therefore be reduced to one of the same kind, of an even number of dimensions, by division.

Ex. 1. Let $x^3 - 1 = 0$. One root of this equation is 1, and by dividing $x^3 - 1$ by $x - 1$, the equation

$$x^2 + x + 1 = 0$$

is obtained, which contains the other two roots, viz.

$$\frac{-1+\sqrt{-3}}{2},\ \text{and}\ \frac{-1-\sqrt{-3}}{2}.$$

In the same manner, the roots of the equation $x^3+1=0$ are found to be

$$-1,\ \frac{1+\sqrt{-3}}{2},\ \text{and}\ \frac{1-\sqrt{-3}}{2}.$$

This also follows from Art. 499.

Ex. 2. Let $x^4-1=0$. Two roots of this equation are $+1$, -1; and by division, $(x^4-1)\div(x^2-1)=x^2+1=0$, an equation which contains the other two roots, viz.

$$+\sqrt{-1},\ \text{and}\ -\sqrt{-1}.$$

SOLUTION OF A CUBIC EQUATION BY CARDAN'S RULE.

521. If the Cubic Equation required to be solved contain the 2nd power of the unknown quantity, that term must be taken away by transformation, as in Art. 501, so that the equation is reduced to the form $x^3+qx+r=0$, where q and r may be positive or negative.

Then assume $x=a+b$, and the equation becomes

$$(a+b)^3+q\times(a+b)+r=0;$$

$$\text{or}\ a^3+b^3+3ab\times(a+b)+q\times(a+b)+r=0;$$

and since we have two unknown quantities, a and b, and have made only one supposition respecting them, viz. that $a+b=x$, we are at liberty to make another; let $3ab+q=0$, then the equation becomes $a^3+b^3+r=0$; also, since $3ab+q=0$, $b=-\frac{q}{3a}$; and, by substitution,

$$a^3+\frac{q^3}{27a^3}+r=0,\ \text{or}\ a^6+ra^3+\frac{q^3}{27}=0,$$

an equation of a quadratic form; and by completing the square,

$$a^6+ra^3+\frac{r^2}{4}=\frac{r^2}{4}-\frac{q^3}{27},\ \text{and}\ a^3+\frac{r}{2}=\pm\sqrt{\frac{r^2}{4}-\frac{q^3}{27}};$$

$$\therefore\ a^3=-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}-\frac{q^3}{27}},\ \text{and}\ a=\sqrt[3]{-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}.$$

Also, since $a^3+b^3+r=0$,

$$b^3=-\frac{r}{2}\mp\sqrt{\frac{r^2}{4}-\frac{q^3}{27}},\ \text{and}\ b=\sqrt[3]{-\frac{r}{2}\mp\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}};$$

$$\therefore\ x=\sqrt[3]{-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}+\sqrt[3]{-\frac{r}{2}\mp\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}.$$

We may observe, that when the sign of $\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}$, in one part of the expression, is positive, it is negative in the other, that is,

$$x=\sqrt[3]{-\frac{r}{2}+\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}+\sqrt[3]{-\frac{r}{2}-\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}.$$

Ex. Let $x^3+6x-20=0$; here $q=-6$, $r=-20$,

$$\therefore\ x=\sqrt[3]{10+\sqrt{108}}+\sqrt[3]{10-\sqrt{108}}$$
$$=2{\cdot}732-0{\cdot}732=2. \quad \text{(Art. 332.)}$$

Having obtained one value of x, the equation may be depressed to a quadratic, and the other roots found (Art. 488).

522. Cor. 1. Since there are 3 cube roots of a^3, (See Art. 520, Ex. 1,) viz.

$$a,\ \tfrac{1}{2}(-1+\sqrt{-3})a, \text{ and } \tfrac{1}{2}(-1-\sqrt{-3})a;$$

and also 3 cube roots of b^3, viz.

$$b,\ \tfrac{1}{2}(-1+\sqrt{-3})b, \text{ and } \tfrac{1}{2}(-1-\sqrt{-3})b,$$

there would appear to be 9 values of $a+b$, and therefore of x, in the cubic equation, which we know cannot be. The fact is, the other condition, viz. $3ab=-q$, shewing that the *product* of a and b must be a *possible* quantity, excludes 6 of the 9, and the remaining *three* are the roots required, viz.

$$a+b,\ \tfrac{1}{2}(-1+\sqrt{-3})a+\tfrac{1}{2}(-1-\sqrt{-3})b, \text{ and}$$
$$\tfrac{1}{2}(-1-\sqrt{-3})a+\tfrac{1}{2}(-1+\sqrt{-3})b.$$

523. Cor. 2. If $\frac{r^2}{4}=\frac{q^3}{27}$, then $x=2\sqrt[3]{-\frac{r}{2}}$, the 3 values of which are $2a$, $-a$, and $-a$, all the roots being *possible*, and two of them *equal*. *Cardan's Rule* is easy of application to a case like this. Thus

Ex. $x^3-3x+2=0$; required x.

Here $r=2$, $q=3$, and $\frac{r^2}{4}=1=\frac{q^3}{27}$, $\therefore \sqrt[3]{-\frac{r}{2}}=\sqrt[3]{-1}$, the 3 values of which are found by solving $x^3+1=0$ (as in p. 322), and are

$$-1,\ \tfrac{1}{2}(1+\sqrt{-3}), \text{ and } \tfrac{1}{2}(1-\sqrt{-3}).$$

$\therefore$ the 3 values of x are -2, 1, and 1.

Verification. $(x+2)(x-1)(x-1)=(x+2)(x^2-2x+1)=x^3-3x+2.$

In every other case, which can be solved by *Cardan's Rule*, two of the three roots must be *impossible*, as is obvious from Cor. 1

SOLUTION OF A BIQUADRATIC BY DES CARTES'S METHOD.

524. Any biquadratic may be reduced to the form

$$x^4 + qx^2 + rx + s = 0,$$

by taking away the second term (Art. 501). Suppose this to be made up of the two quadratics

$$x^2 + ex + f = 0, \text{ and } x^2 - ex + g = 0,$$

where $+e$ and $-e$ are made the coefficients of the second terms, because the second term of the biquadratic is wanting, that is, the sum of its roots is 0. By multiplying these quadratics together, we have

$$x^4 + (g + f - e^2).x^2 + (eg - ef).x + fg = 0,$$

which equation is made to coincide with the former by equating their coefficients, or making

$$g + f - e^2 = q, \quad eg - ef = r, \text{ and } fg = s;$$

hence $g + f = q + e^2$, also $g - f = \frac{r}{e}$,

and by taking the sum and difference of these equals

$$2g = q + e^2 + \frac{r}{e}, \text{ and } 2f = q + e^2 - \frac{r}{e};$$

$$\therefore\ 4fg = q^2 + 2qe^2 + e^4 - \frac{r^2}{e^2} = 4s,$$

and multiplying by e^2, and arranging the terms according to the dimensions of e,

$$e^6 + 2qe^4 + (q^2 - 4s).e^2 - r^2 = 0;$$

or, making $y = e^2$,

$$y^3 + 2qy^2 + (q^2 - 4s).y - r^2 = 0.$$

By the solution of this cubic a value of y, and therefore of $\sqrt{y}$, or e, is obtained; also f and g, which are respectively equal to

$$\tfrac{1}{2}\left(q + e^2 - \frac{r}{e}\right), \text{ and } \tfrac{1}{2}\left(q + e^2 + \frac{r}{e}\right),$$

are known. The biquadratic is thus resolved into two quadratics, whose roots may be found.

It may be observed that, whichever value of y is used, the same values of x are obtained.

This method of solving a Biquadratic, being dependent on Cardan's Rule for the solution of a Cubic, can obviously be applied to those cases only, where two of the roots are either equal or impossible.

WARING'S SOLUTION OF A BIQUADRATIC.

525. Let the proposed biquadratic be

$$x^4 + 2px^3 = qx^2 + rx + s;$$

now $(x^2 + px + n)^2 = x^4 + 2px^3 + (p^2 + 2n)x^2 + 2pnx + n^2$,

if therefore $(p^2 + 2n)x^2 + 2pnx + n^2$ be added to both sides of the proposed biquadratic, the first part is a complete square, $(x^2 + px + n)^2$, and the latter part,

$$(p^2 + 2n + q)x^2 + (2pn + r)x + n^2 + s,$$

is a complete square, if

$$4(p^2 + 2n + q)(n^2 + s) = (2pn + r)^2, \qquad \text{(Art. 152)},$$

that is, multiplying and arranging the terms according to the dimensions of n, if

$$8n^3 + 4qn^2 + (8s - 4rp)n + 4qs + 4p^2s - r^2 = 0.$$

From this equation let a value of n be obtained and substituted in the equation

$$(x^2 + px + n)^2 = (p^2 + 2n + q)x^2 + (2pn + r)x + n^2 + s;$$

then extracting the square root on both sides,

$$x^2 + px + n = \pm (\sqrt{p^2 + 2n + q}.x + \sqrt{n^2 + s})$$

when $2pn + r$ is positive; or

$$x^2 + px + n = \pm (\sqrt{p^2 + 2n + q}.x - \sqrt{n^2 + s}),$$

when $2pn + r$ is negative; and from these two quadratics the four roots of the given biquadratic may be determined.

Ex. Let $x^4 - 6x^3 + 5x^2 + 2x - 10 = 0$ be the proposed equation.

By comparing this with the equation

$$x^4 + 2px^3 - qx^2 - rx - s = 0,$$

we have $2p = -6$, or $p = -3$, $q = -5$, $r = -2$, $s = 10$;

and $8n^3 + 4qn^2 + (8s + 4rp)n + 4qs + 4p^2s - r^2 = 0$, becomes

$$8n^3 - 20n^2 + 56n + 156 = 0,$$

or $2n^3 - 5n^2 + 14n + 39 = 0$, one of whose roots is $-\frac{3}{2}$;

$$\text{hence } \left(x^2 - 3x - \frac{3}{2}\right)^2 = x^2 + 7x + \frac{49}{4},$$

$$\therefore\ x^2 - 3x - \frac{3}{2} = \pm\left(x + \frac{7}{2}\right);$$

$$\text{or } x^2 - 4x - 5 = 0, \text{ and } x^2 - 2x + 2 = 0;$$

the roots of these quadratics, namely,

$$-1,\ 5,\ 1 + \sqrt{-1},\ 1 - \sqrt{-1},$$

are the roots of the proposed biquadratic.

This method has the merit of not requiring the term involving x^3 to be taken away, but, depending like the former method on Cardan's Rule, it can only be applied to those equations, which have two roots either equal or impossible.

METHODS OF APPROXIMATION.

526. The most useful and general method of discovering the possible roots of numeral equations is approximation. Find by trial two numbers, which substituted for the unknown quantity give, one a positive, and the other a negative, result; and an odd number of roots lies between these two quantities, that is, one possible root at least lies between them; then, by increasing one of the limits, and diminishing the other, an approximation may be made to the root; substitute this approximate value, increased or diminished by v, for the unknown quantity in the equation; neglect all the powers of v above the first, as being small when compared with the other terms, and a simple equation is obtained for determining v nearly; thus a nearer approximation is made to the root, and by repeating the operation the approximation may be made to any required degree of exactness.

Ex. Let the roots of the equation $y^3 - 3y + 1 = 0$ be required.

When 1 is substituted for x the result is -1, and when 2 is substituted, the result is $+3$, therefore one possible root lies between 1 and 2; try 1·5, and the result is $-0·125$, or the root lies between 1·5 and 2.

Let $1·5+v=y$; then

$$\left.\begin{aligned} y^3 &= 3·375+6·75v+4·5v^2+v^3 \\ -3y &= -4·5 \quad -3v \\ +\ 1 &= +1 \end{aligned}\right\}=0,$$

$$\text{that is, } -0·125+3·75v+4·5v^2+v^3=0,$$

and, neglecting the two last terms,

$$-0·125+3·75v=0, \text{ or } v=\frac{0·125}{3·75}=0·033 \text{ nearly,}$$

$$\therefore\ y=1·5+v=1·533 \text{ nearly.}$$

Again, suppose $1·533+v=y$; by proceeding as before, we find

$$0·003686437+4·050267v=0,$$

$$\therefore\ v=\frac{-0·003686437}{4·050267}=-0·0009101 \text{ \&c.}$$

$$\therefore\ y=1·532089 \text{ nearly.}$$

The other roots may be found by the solution of a quadratic. (Art. 488.)

527. If we have two equations, containing two unknown quantities, we may discover the values of these quantities nearly in the same manner.

Ex. Let $\left\{\begin{aligned} x^2y &= 405 \\ xy-y^2 &= 20 \end{aligned}\right\}$, to find x and y.

Find, by trial, approximate values of x and y; such are 20 and 1; and let $x=20+v$, $y=1+z$;

$$\text{then } x^2y=400+40v+400z+v^2+40vz+v^2z=405,$$

$$\text{and } xy-y^2=19+v+18z+vz-z^2=20,$$

and neglecting those terms in which z or v is of more than one dimension, or in which their product is found, as being small when compared with the rest,

$$\left.\begin{aligned} 400 + 40v + 400z &= 405 \\ \text{and}\quad 19 + v + 18z &= 20 \end{aligned}\right\},$$

$$\text{or}\quad 40v + 400z = 5\,;$$

$$\therefore\ v + 10z = 0{\cdot}125.$$

$$\text{And}\quad v + 18z + 19 = 20,$$

$$\text{or}\quad v + 18z = 1,$$

$$\therefore\ 8z = 0{\cdot}875\,;$$

$$\text{and}\ z = 0{\cdot}109375.$$

$$\text{But}\ v = 0{\cdot}125 - 10z = -0{\cdot}96875\,;$$

$$\left.\begin{aligned} \therefore\ x &= 19{\cdot}03, \\ \text{and}\ y &= 1{\cdot}109. \end{aligned}\right\}$$

By making use of the values thus obtained nearer approximations may be made to x and y.

EXAMPLES.

[N.B. Where *App.* is subjoined to an Ex. it signifies that the *Solution* may be found in the Appendix. Also *Comp.* signifies that the *Solution* may be found in the "*Companion to Wood's Algebra.*" The *Solutions* of all the rest are given in the Author's KEY *for Schoolmasters.*]

INTRODUCTION.

VULGAR FRACTIONS.

(1) REDUCE to mixed numbers $\frac{602}{11}$, $\frac{4139}{15}$, $\frac{12332}{1111}$, $\frac{45739}{60}$.

Ans. $54\frac{8}{11}$, $275\frac{14}{15}$, $11\frac{111}{1111}$, $762\frac{19}{60}$.

(2) What is $\frac{2}{5}$ of $9\frac{1}{6}$? also $\frac{5}{9}$ of $\frac{12}{25}$ of $\frac{1}{3}$ of $\frac{1}{2}$ of $41\frac{1}{5}$?

(1) Ans. $3\frac{2}{3}$. (2) Ans. $1\frac{187}{225}$.

(3) Reduce to lowest terms $\frac{70}{462}$, $\frac{242}{1111}$, $\frac{799}{2961}$, $\frac{109375}{10000000}$.

Ans. $\frac{5}{33}$, $\frac{22}{101}$, $\frac{17}{63}$, $\frac{7}{640}$.

(4) Reduce to the *least* common denominator $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$.

Ans. $\frac{210}{420}$, $\frac{140}{420}$, $\frac{105}{420}$, $\frac{84}{420}$, $\frac{70}{420}$, $\frac{60}{420}$.

(5) Find the least common multiple of $2\frac{1}{2}$, $6\frac{1}{3}$, and 5. Ans. 95.

(6) Reduce to the least common *numerator* $\frac{4}{13}$, $\frac{3}{10}$, $\frac{11}{36}$; and find which is the greatest.

(1) Ans. $\frac{132}{429}$, $\frac{132}{440}$, $\frac{132}{432}$. (2) Ans. The 1st.

(7) Which is the greatest $\frac{2}{3}$, $\frac{3}{4}$, or $\frac{4}{5}$? Ans. $\frac{4}{5}$.

(8) Which is greater $7\frac{15}{16}$, or $7\frac{21}{23}$; and how much? Ans. $7\frac{15}{16}$ by $\frac{9}{368}$.

(9) What is the difference between $7\frac{4}{5}$ and $7\times\frac{4}{5}$? Ans. $2\frac{1}{5}$.

(10) What fraction of £1 is 19*s.* $10\frac{3}{4}$*d.*? Ans. $\frac{191}{192}$.

(11) What fraction of *£5* is *£3. 6s. 8d.*? Ans. $\frac{2}{3}$.

(12) Which is greater, $\sqrt{\frac{2}{3}}$, or $\sqrt[3]{\frac{2}{3}}$? (*Comp.* p. 1.) Ans. The latter.

(13) Reduce 5 yards 2 feet to the fraction of a mile. Ans. $\frac{17}{5280}$.

(14) Which is greater, $\frac{1}{19}$ of £1, or $\frac{1}{20}$ of a guinea? Ans. $\frac{1}{19}$ of £1.

(15) What is the difference between $\frac{1}{12}$ of £1, and $\frac{1}{14}$ of a guinea? Ans. 2*d.*

(16) Add together $100\frac{5}{6}$, $1\frac{3}{4}$, $\frac{1}{3}$, and $7\frac{2}{5}$. Ans. $110\frac{19}{60}$.

(17) Find the number which exceeds by $148\frac{2}{9}$ the difference between $195\frac{1}{3}$ and $95\frac{11}{12}$. Ans. $247\frac{23}{36}$.

(18) Multiply $45\frac{3}{4}$ by $17\frac{2}{3}$, and divide the product by $4\frac{1}{4}$. Ans. $190\frac{3}{17}$.

(19) Divide $2\frac{3}{4}$ of $7\frac{7}{11}$ by $\frac{1}{2}$ of $\frac{3}{4}$ of $18\frac{2}{3}$. Ans. 3.

(20) Find the value of $7\frac{1}{3}\times\frac{9}{11}\times17\times\frac{1}{2}\times8\frac{4}{5}\times\frac{1}{8}$. Ans. $53\frac{5}{6}$.

(21) Reduce $\frac{12\times11\times10\times9\times8}{1\times2\times3\times4\times5}$. Ans. 792.

(22) Find the value of $\frac{3}{7}\times1\frac{2}{5}\times12\frac{1}{2}\div6\frac{2}{3}$. Ans. $1\frac{1}{8}$.

(23) Reduce $\frac{4\frac{1}{3}\times4\frac{1}{3}\times4\frac{1}{3}-1}{4\frac{1}{3}\times4\frac{1}{3}-1}$. (*Comp.* p. 1.) Ans. $4\frac{25}{48}$.

(24) Reduce to simplest form $1\frac{1}{3}+\frac{8}{3}$ of $\frac{41}{34}+\frac{4}{5\frac{1}{10}}$. Ans. $5\frac{1}{3}$.

(25) Reduce $\frac{5}{7}\times\frac{2}{9}\times13\frac{1}{2}\div\left(\frac{1}{9}\times\frac{3}{7}+54\right)$. Ans. $\frac{9}{227}$.

(26) Reduce $\left(\frac{2}{19}+\frac{1}{3}\right)\div\left(3-\frac{1}{3}\right)\times\left(\frac{1}{3}+\frac{1}{5}\right)$. Ans. $\frac{5}{57}$.

(27) Reduce $\frac{18}{17}\times\left\{1-\frac{64}{81}\right\}+\frac{8}{11}\times\frac{1}{6}\times\left(\frac{1}{2}+\frac{5}{12}\right)$. Ans. $\frac{1}{3}$.

(28) Simplify the following peculiar fractional forms:

$$\frac{1}{\frac{1}{3}},\quad \frac{\frac{2}{3}}{\frac{5}{11}},\quad \frac{1}{1+\frac{1}{2}},\quad 2\tfrac{1}{2}+\frac{1}{3\frac{1}{3}+\frac{1}{4\frac{1}{4}}}.$$

Ans. 3, $1\frac{7}{15}$, $\frac{2}{3}$, $2\frac{71}{91}$.

(29) Simplify $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{4+\frac{1}{5}}}}$; and $\dfrac{2}{3+\dfrac{4}{5+\frac{6}{7}}}$.

(1) Ans. $\frac{68}{157}$.
(2) Ans. $\frac{82}{151}$.

(30) $\frac{2}{3}$ multiplied by 3 signifies $\frac{2}{3}$ *taken three times*, that is, $\frac{2}{3}+\frac{2}{3}+\frac{2}{3}$; what does $\frac{2}{3}$ multiplied by $\frac{3}{4}$ signify? (Art. 129, and *Comp.* p. 1.)

(31) If two-thirds of an estate be worth £220, what is the value of $\frac{3}{11}$ of the same? Ans. £90.

(32) An article which cost 3*s.* 6*d.* is sold for 3*s.* 10½*d.*; what is that per cent. profit? (*Comp.* p. 2.) Ans. $10\frac{5}{7}$.

(33) How much per cent. is 14*s.* 6*d.* of £3. 10*s.*? Ans. $20\frac{5}{7}$.

(34) How much per cent. is 27½ parts out of 36? Ans. $76\frac{7}{18}$.

(35) A shilling weighs 3 dwts. 15 grs. of which 3 parts out of 40 are alloy, and the rest pure silver. How much per cent. is there of alloy, and what weight of pure silver? (*Comp.* p. 2.)

(1) Ans. 7½ per cent.; (2) Ans. 3 dwts. $8\frac{19}{40}$ grs.

(36) The length of $\frac{1}{360}$ of the Earth's circumference is $69\frac{1}{22}$ miles nearly; what is the Earth's diameter, assuming that the diameter of a circle is $\frac{7}{22}$ of its circumference? Ans. $7908\frac{102}{121}$ miles.

(37) There are five numbers, of which the first two are $2\frac{1}{3}$, $3\frac{5}{24}$; and each number exceeds the preceding one by the same fraction; find the numbers, and the sum of them.

(1) Ans. $2\frac{1}{3}$, $3\frac{5}{24}$, $4\frac{1}{12}$, $4\frac{23}{24}$, $5\frac{5}{6}$. (2) Ans. $20\frac{5}{12}$.

(38) What is the sum $\frac{7}{11}$ of which is 5*s.* 3*d.*? Ans. 8*s.* 3*d.*

(39) Divide $\frac{3}{5}$ into two parts, so that one is greater than the other by $\frac{4}{13}$. (*Comp.* p. 3.) Ans. $\frac{59}{130}$, $\frac{19}{130}$.

DECIMAL FRACTIONS.

(1) Reduce to *vulgar* fractions 0·375, 0·8125, 4·075, 0·0064.

Ans. $\frac{3}{8}$, $\frac{13}{16}$, $4\frac{3}{40}$, $\frac{4}{625}$.

(2) What is 0·003 of 0·27 of 90? Ans. 0·0729.

(3) Divide 0·27 by 0·003; 0·06 by 60; 600 by 0·06; and 0·006 by 600.

(1) Ans. 90. (2) Ans. 0·001. (3) Ans. 10000. (4) Ans. 0·00001.

(4) *Reduce to their equivalent decimals* $\frac{2}{25}$, $\frac{1}{800}$, $\frac{1}{128}$, $\frac{4000}{256}$.

Ans. 0·08, 0·00125, 0·0078125, 15·625.

(5) Find the value in shillings and pence of £0·97216. Ans. 19*s.* $5\frac{1}{4}$*d.*

(6) Reduce to vulgar fractions 0·8333..., 4·041666..., 0·09009009...

(1) Ans. $\frac{5}{6}$; (2) Ans. $4\frac{1}{24}$; (3) Ans. $\frac{10}{111}$.

(7) Add together $2\frac{1}{3}$, $72\frac{5}{8}$, $316\frac{1}{6}$, and 2·875. Ans. 394.

(8) Add together $1\frac{17}{51}$, $\frac{8}{3}$ of $\frac{41}{34}$, $\frac{4}{5\frac{1}{10}}$, and 0·6666... Ans. 6.

(9) What decimal of a square mile is one acre? Ans. 0·0015625.

(10) What decimal of a year is 1 second? Ans. 0·0000000317.

(11) Divide 0·454545... by 0·121212... Ans. 3·75.

(12) Find the decimal which does not differ from $\frac{333}{106}$ by the ten-thousandth part of an unit. Ans. 3·1415.

(13) Shew that $3+\cfrac{1}{7+\cfrac{1}{16}} = 3{\cdot}14159$ nearly.

(14) Divide $2\frac{1}{2}+\frac{1}{6}$ by $3\frac{1}{2}-\frac{1}{8}$; and express the result in a decimal form. Ans. 0·790123.

(15) Extract the square root of $1\frac{3}{8}$ to 4 places of decimals; and the cube-root of $3\frac{4}{5}$ to two places. (1) Ans. 1·1726. (2) Ans. 1·56.

(16) Find $\frac{1}{\sqrt{3}}$ correct to 7 places of decimals; and $\sqrt{\frac{5{\cdot}04}{0{\cdot}012}}$ to two places. (*Comp.* p. 3.) (1) Ans. 0·5773503. (2) Ans. 20·49.

(17) Express in a decimal form $2+\frac{3}{5}+\frac{5}{1000}+\frac{14}{20000}$. Ans. 2·6057.

(18) Reduce the following expressions to simple decimals having 7 decimal places:

(1) $\frac{1}{2}\times\{6\frac{1}{2}+2\frac{2}{3}-3\}$. (1) Ans. 3·0833333.

(2) $2\times\left\{\frac{1}{3}+\frac{1}{3}\times\frac{1}{3^3}+\frac{1}{5}\times\frac{1}{3^5}+\frac{1}{7}\times\frac{1}{3^7}+\ldots\right\}$. (2) Ans. 0·6931472.

(3) $2\times\left\{\frac{1}{5}+\frac{1}{3}\times\frac{1}{5^3}+\frac{1}{5}\times\frac{1}{5^5}+\frac{1}{7}\times\frac{1}{5^7}+\ldots\right\}$. (3) Ans. 0·4054651.

(19) Express the following in decimals of 4 places:

(1) $\frac{1}{2}+\frac{1}{3\times2^3}+\frac{1}{5\times2^5}+\frac{1}{7\times2^7}$. (1) Ans. 0·5490.

(2) $16\times\left\{\frac{1}{5}-\frac{1}{3\times5^3}+\frac{1}{5\times5^5}-\frac{1}{7\times5^7}+\&\text{c.}\right\}-\frac{4}{239}$. (*Comp.* p. 4.)

(2) Ans. 3·1416.

(20) Express a degree (69½ miles) in metres, 32 metres being equal to 35 yards nearly. (*Comp.* p. 4.) Ans. 111835·42857.

(21) The true length of a year is 365·24224 days. Find what the error amounts to by the common reckoning in 4 centuries. (*Comp.* p. 4.) Ans. 0·104 days.

(22) A square inch plate of metal of 0·05 inches thickness is drawn into a wire of uniform thickness 50 feet long: find the thickness of the wire. (*Comp.* p. 5.) Ans. Section of wire = 0·0000833... of a square inch.

(23) A quadrant of the meridian in French metres is 10000565·278, and 1 metre = 39·37079 English inches: required the length of the quadrant in English feet. Ans. 32810846·2868.

(24) The number of degrees in an arc of a circle which is equal to the radius is 57·29578: required the number of seconds in the same. Ans. 206264·8.

(25) Given that $\frac{1\text{ French foot}}{1\text{ English foot}}=1\cdot0657654$, and that the equatoreal and polar radii of the Earth are respectively 3271953·854 and 3261072·9 *toises*, each *toise* being 6 French feet: find the Earth's equatoreal and polar radii in English feet. Ans. 20922811, and 20853232.

(26) The length of the pendulum which vibrates seconds in the latitude of Greenwich is 39·1393 inches, and the acceleration of gravity is measured by the product of (length of seconds pendulum)×$(3\cdot1415927)^2$: required the expression for the acceleration of gravity in feet. Ans. 32·1908.

(27) Two distances are measured in inches, and are known to be correct within a quarter of a hundredth of an inch each way, being 11·87 and 9·95. How far can their *product* be depended upon for accuracy? (*Comp.* p. 5.) Ans. Only in its integral part.

ALGEBRA.

NOTATION, &c.

(1) **What** is the difference between $3+a$, and $3a$, when $a=5$?
Ans. 7.

(2) What is the difference between $3a+x$, and $3ax$, when $a=2$, and $x=3$?
Ans. 9.

(3) How many *terms* are there in $3a$: and in $3+a$?
(1) Ans. 1. (2) Ans. 2.

(4) How many *terms* are there in each of the following quantities?
(1) $a+bx-cy$, (2) $abcxy$, (3) $2a-3b+4ax\times mnp$.
(1) Ans. 3. (2) Ans. 1. (3) Ans. 3.

(5) What are the *coefficients* of a and x in $na+x$?
(1) Ans. n. (2) Ans. 1.

(6) What is the *coefficient* of x in xy, and of a in $2ax$?
(1) Ans. y. (2) Ans. $2x$.

(7) What is the difference between $3a$, and a^3, when $a=3$? Ans. 18.

(8) What are the *simple factors* of $2ab(a+b)$? Ans. $2, a, b, a+b$.

(9) What are the *simple factors* of $m(a+b)(c-d)$?
Ans. $m, a+b, c-d$.

(10) Shew that $\dfrac{x+1}{\frac{1}{x}+1}=x$, when $x=1$, or 2, or 3, or *any* number.

(11) Find the value of $\dfrac{a+b}{a-b}$, when $a=\dfrac{1}{2}$, and $b=\dfrac{2}{5}$. Ans. 9.

(12) Find $\dfrac{ax+by}{b+x}$, when $a=5$, $b=3$, $x=7$, $y=5$. Ans. 5.

(13) Find the value of $\dfrac{3}{1+x}+\dfrac{3}{1-x}$, when $x=\dfrac{1}{2}$. Ans. 8.

(14) Find $\dfrac{ax^2+b^2}{bx-a^2-c}$, when $a=3$, $b=5$, $c=2$, $x=6$. Ans. 7.

(15) Find x^3-2xy^2+y-13, when $x=2$, and $y=-3$. Ans. -44.

(16) Find the value of $\dfrac{x-1}{x+1}+\dfrac{x+3}{x-3}-2.\dfrac{x+2}{x-2}$, when $x=5$. Ans. 0.

(17) Shew that $\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}=23\frac{1}{3}$, if $x=19$.

(18) Find the value of $\frac{a^2+b^2}{a^3-b^3}$, when $a=3$, and $b=-2$. Ans. $\frac{13}{35}$.

(19) Find the value of $\frac{x}{y}-\sqrt{\frac{1+x}{1-y}}$, when $x=\frac{1}{4}$, and $y=\frac{1}{5}$. Ans. 0.

(20) Find the value of $5\sqrt{62+3x}-\frac{1}{2}\sqrt{95\frac{2}{3}-5x}$, when $x=6\frac{1}{3}$. Ans. 41.

(21) Find the value of $\sqrt{\frac{3}{4}-x}+\sqrt{2x}-\frac{3}{2}\sqrt{1-4x}$, when $x=\frac{1}{12}$.

Ans. 0.

(22) Find the value of $\frac{a+\sqrt{a^2+b^2}}{a^3-2b(a^2-b^2)}$, when $a=-4$, and $b=-3$.

Ans. $-\frac{1}{22}$.

(23) Find the value of $3a^2b+\frac{a^2+b^2}{c}+abc+\frac{2ab}{3}+\frac{a^2}{b^3}+\frac{a-b}{a^3+b^3+c^4}$, when $a=4$, $b=3$, $c=2$. Ans. $189\frac{589}{5778}$.

ADDITION AND SUBTRACTION.

(1) ADD together $3x^2-5x+1$, $7x^2+2x-4$, and $-x^2-4x+13$.

Ans. $9x^2-7x+10$.

(2) Add together $a-3b+3c-d$, and $a+3b+3c+d$. Ans. $2a+6c$.

(3) Add together $a+b+c-d$, $a+b+d-c$, $a+c+d-b$, and $b+c+d-a$.

Ans. $2a+2b+2c+2d$.

(4) Add together $x^3-2ax^2+a^2x$, x^3+3ax^2, and $2a^3-ax^2-a^2x$.

Ans. $2x^3+2a^3$.

(5) Add together $5ax-7by+cz$, and $ax+2by-cz$. Ans. $6ax-5by$.

(6) Add together $8a^2-2a+2$, $6b^2-5ab+5c^2-3bc$, a^2+2b^2+a+2, and $2ab+3bc+3c^2$. Ans. $9a^2-a+8b^2-3ab+8c^2+4$.

(7) Add together $4a^2y-4aby-2ab^2+2b^3$, and $a^2y+aby+ab^2-b^3$.

Ans. $5a^2y-3aby-ab^2+b^3$.

(8) Add together $a^2+b^2+c^2+d^2$, $ab-2a^2+ac-2c^2+ad-2d^2$, $a^3-3ab+b^3-3ac+c^3-3ad$, and $2ab-a+2ac-b+2ad-c$.

Ans. $a^3+b^3+c^3-a^2+b^2-c^2-d^2-a-b-c$.

(9) Add together $5ab^2+\frac{3}{5}a^2b$, and $\frac{2}{3}ab^2-4a^2b$. Ans. $\frac{17}{3}ab^2-\frac{17}{5}a^2b$.

(10) Add together $\frac{1}{2}a-\frac{1}{3}b+\frac{1}{5}c$, $\frac{1}{4}a-\frac{1}{5}b-\frac{1}{3}c$, and $\frac{1}{3}a+\frac{1}{4}b+\frac{1}{2}c$.
Ans. $\frac{13}{12}a-\frac{17}{60}b+\frac{11}{30}c$.

(11) Add together $a^2-3ab-\frac{14}{21}b^2$, $2b^2-\frac{2}{3}b^3+c^2$, $ab-\frac{1}{3}b^2+b^3$, and $2ab-\frac{1}{3}b^3$. Ans. $a^2+b^2+c^2$.

(12) Add together $a^m-b^n+3x^p$, $2a^m-3b^n-x^p$, and $a^m+4b^n-x^q$.
Ans. $4a^m+2x^p-x^q$.

(13) Add together $2(a+b)+c$, $2(a+b+c)+d$, and $2(a+b+c+d)+e$.
Ans. $6(a+b)+5c+3d+e$.

(14) Add together $(a+7)x^2-by^2$, and $(b-7)x^2-7y^2$.
Ans. $(a+b)x^2-(b+7)y^2$.

(15) Add together $ax-by$, $x+y$, and $(a-1)x-(b+1)y$.
Ans. $2ax-2by$.

(16) Add together $2(a+b)xy^2$, $3(a-2b)xy^2$, and $4(2a-b)xy^2$.
Ans. $(13a-8b)xy^2$.

(17) Add together $at^n+3bt^n-9at^n+7bt^n$. Ans. $(10b-8a)t^n$.

(18) Add together $1-(1-\overline{1-x})$, $2x-(3-5x)$, and $2-(-4+5x)$.
Ans. $4+x$.

(19) From $6a-b-c$ take $a-b+2c$. Ans. $5a-3c$.

(20) From $8a+x-5b-5c$ take $x+2b-5c$. Ans. $8a-7b$.

(21) From $7x^2-2x+4$ take $2x^2+3x-1$. Ans. $5x^2-5x+5$.

(22) From $8xy-7x^2-x-y$ take $7xy+x^2+x+y$.
Ans. $xy-8x^2-2x-2y$.

(23) From $4a^m+2x^p-x^q$ take $a^m-b^n+3x^p$ and $2a^m-3b^n-x^p$.
Ans. $a^m+4b^n-x^q$.

(24) From $a+b$ take $\frac{a}{2}-\frac{b}{2}$. Ans. $\frac{a}{2}+\frac{3b}{2}$.

(25) From $\frac{a+b}{2}$ take $\frac{a-b}{2}$; and $\frac{1}{2}(a+b)$ from $a-b$.

(1) Ans. b. (2) Ans. $\frac{a}{2}-\frac{3b}{2}$.

(26) From $2(a+b)-3(c-d)$ take $a+b-4(c-d)$. Ans. $a+b+c-d$.

(27) From $(a+b)x+(b+c)y$ take $(a-b)x-(b-c)y$. Ans. $2b(x+y)$.

(28) From $(a^2+bc)x^2-(a^2-c^2)bx$ take $bcx^2-(a^2-b^2)bx$.
Ans. $a^2x^2-(b^2-c^2)bx$.

(29) From x^3-ax^2+bx-c take x^3-px^2+qx-r.
Ans. $(p-a)x^2-(q-b)x+r-c$.

(30) From $a-x-(x-2a)+2a-x$ take $a-2x-(2a-x)+(x-2a)$.
Ans. $8a-3x$.

(31) Simplify the following quantities:

$a-\{b-(2b+x)\}+\{b-(x-2b)\}$. Ans. $a+4b$.

$a-(b-c)-(a-c)+c-(a-b)$. Ans. $3c-a$.

$a-\{a+b-[a+b+c-(a+b+c+d)]\}$. Ans. $-b-d$.

$a+b-(2a-3b)-(5a+7b)-(-13a+2b)$. Ans. $7a-5b$.

MULTIPLICATION.

(1) Multiply $3a^2b$ by $2ac$, and the product by $-5b$. Ans. $-30a^3b^2c$.

(2) Multiply $a+2x-3x^2$ by $-m$. Ans. $3mx^2-2mx-ma$.

(3) Multiply $4a^2-3ac+2$ by $5ax$. Ans. $20a^3x-15a^2cx+10ax$.

(4) Multiply $5a-2ab+10$ by $-9ab$. Ans. $-45a^2b+18a^2b^2-90ab$.

(5) Multiply $2x+3y$ by $2x-3y$. Ans. $4x^2-9y^2$.

(6) Multiply $4a^2-6a+9$ by $2a+3$. Ans. $8a^3+27$.

(7) Multiply $a^4+a^3b+a^2b^2+ab^3+b^4$ by $a-b$. Ans. a^5-b^5.

(8) Multiply $2a+bc-2b^2$ by $2a-bc+2b^2$.
Ans. $4a^2-b^2c^2+4b^3c-4b^4$.

(9) Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$.
Ans. $a^6-3a^4b^2+3a^2b^4-b^6$.

(10) Multiply $x^4-2ax^3+4a^2x^2-8a^3x+16a^4$ by $2a+x$.
Ans. x^5+32a^5.

(11) Multiply $4ab-2ac$ by $6ab+3ac$. Ans. $24a^2b^2-6a^2c^2$.

(12) Multiply $a-b+c-d$ by $a+b-c-d$. Ans. $a^2-b^2-c^2+d^2-2ad+2bc$.

(13) Multiply $a+bx$ by $a+cx$. Ans. $a^2+(ab+ac)x+bcx^2$.

(14) Multiply x^3-ax^2+bx-c by x^2-px+q.
Ans. $x^5-(a+p)x^4+(b+ap+q)x^3-(c+bp+aq)x^2+(bq+cp)x-cq$.

(15) Find the product of $(x-a)(x-b)(x-c)$.
Ans. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.

(16) Find the product of $(x-10)(x+1)(x+4)$. Ans. $x^3-5x^2-46x-40$.

(17) Find the product of $(x-5)(x+6)(x-7)(x+8)$.
Ans. $x^4+2x^3-85x^2-86x+1680$.

(18) Find the continued product of $x+1$, $x+2$, $x+3$, and $x+4$.
Ans. $x^4+10x^3+35x^2+50x+24$.

(19) Multiply a^2+ax+x^2 by a^2-ax+x^2. Ans. $a^4+a^2x^2+x^4$.

(20) Find the continued product of $x-a$, $x+a$, x^2-ax+a^2, and x^2+ax+a^2. Ans. x^6-a^6.

(21) Multiply $1+\frac{1}{2}a+\frac{1}{3}b$ by $1-\frac{1}{2}a+\frac{1}{3}b$. Ans. $1-\frac{1}{4}a^2+\frac{2}{3}b+\frac{1}{9}b^2$.

(22) Multiply $x^2-\frac{1}{2}x+\frac{2}{3}$ by $\frac{1}{3}x+2$. Ans. $\frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{9}x+\frac{4}{3}$.

(23) Multiply x^3-x^{-3} by $x-x^{-1}$. Ans. $x^4+x^{-4}-x^2-x^{-2}$.

(24) Multiply $(1+a)a^2y+y^2+ay^2$ by a^2-y. Ans. $(1+a)y(a^4-y^2)$.

(25) Multiply $a^{\frac{4}{5}}+a^{\frac{3}{5}}x^{\frac{1}{5}}+a^{\frac{2}{5}}x^{\frac{2}{5}}+a^{\frac{1}{5}}x^{\frac{3}{5}}+x^{\frac{4}{5}}$ by $a^{\frac{1}{5}}-x^{\frac{1}{5}}$. Ans. $a-x$.

(26) Multiply $x+2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$ by $x-2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$.
Ans. $x^2-4y+6xz^{\frac{1}{3}}+9z^{\frac{2}{3}}$.

(27) Multiply $a^{\frac{5}{2}}-2a^2b^{\frac{1}{3}}+4a^{\frac{3}{2}}b^{\frac{2}{3}}-8ab+16a^{\frac{1}{2}}b^{\frac{4}{3}}-32b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}+2b^{\frac{1}{3}}$.
Ans. a^3-64b^2.

(28) Multiply $3a^{\frac{1}{2}}b^{\frac{2}{3}}+5a^{\frac{1}{3}}b^{\frac{2}{3}}$ by $3a^{\frac{1}{2}}b^{\frac{2}{3}}-5a^{\frac{1}{3}}b^{\frac{2}{3}}$. Ans. $9ab^{\frac{4}{3}}-25a^{\frac{2}{3}}b^{\frac{4}{3}}$.

(29) Multiply $a^{2-2m}b^{2p+1}c^3$ by $a^{5m-2}b^{-2}c^{-1}$. Ans. $a^{3m}b^{2p-1}c^2$.

(30) Multiply a^m-2c^n by a^m-c^n. Ans. $a^{2m}-3a^mc^n+2c^{2n}$.

(31) Multiply $a^{n-1}b-a^{n-2}b^2+ab^{n-1}$ by ab. Ans. $a^nb^2-a^{n-1}b^3+a^2b^n$.

(32) Multiply $x^{-3p}+ax^{-2p}-a^2x^{-p}$ by a^mx^p. Ans. $a^mx^{-2p}+a^{m+1}x^{-p}-a^{m+2}$.

(33) Find the continued product of $a^{n+1}b^{n-1}\times c^{2p}d^n\times a^{p-1}b^{p+1}\times c^{n-p}d^p$.
Ans. $a^{n+p}b^{n+p}c^{n+p}d^{n+p}$.

(34) Find the coefficient of x^4 in the product of $x^4-ax^3+bx^2-cx+d$ and x^2+px+q. (*Comp.* p. 6.) Ans. $b-ap+q$.

(35) Multiply $(2b-c)a^2-(4b^2-2bc+c^2)a+8b^3-4b^2c$ by $(2b+c)a$.
Ans. $(4b^2-c^2)a^3-(8b^3+c^3)a^2+(16b^4-4b^2c^2)a$.

(36) Multiply $a^{(p-1)q}-b^{(q-1)p}$ by a^q-b^p. Ans. $a^{pq}-a^qb^{(q-1)p}-b^pa^{(p-1)q}+b^{pq}$.

(37) Multiply $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\ldots\ldots$ by $1-\frac{1}{3}x+\frac{1}{5}x^2-\frac{1}{7}x^3+\ldots$

Ans. $1+\frac{1}{6}x+\frac{11}{30}x^2+\frac{121}{1260}x^3+\ldots$

(38) Simplify $\frac{1}{6}\{x(x+1)(x+2)+x(x-1)(x-2)\}+\frac{2}{3}(x-1)x(x+1)$.
(*Comp.* p. 6.) Ans. x^3.

(39) Prove that $(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)$ is equal to $(a-b)(b-c)(a-c)$. (*Comp.* p. 6.)

(40) Find the difference between $a(b+c)^2+b(a+c)^2+c(a+b)^2$ and $(a+b)(a-c)(b-c)+(a-b)(a-c)(b+c)-(a-b)(b-c)(a+c)$.
(*Comp.* p. 7.) Ans. $12abc$.

DIVISION.

(1) DIVIDE $30a^3b^2c$ by $5abc$; and $-ac^2x^4$ by $-ax^3$.
(1) Ans. $6a^2b$. (2) Ans. c^2x.

(2) Divide $2ab+6abc-8abcd$ by $2ab$; Ans. $1+3c-4cd$.
…… $5xy+20x^2y-45axy$ by $5xy$; Ans. $1+4x-9a$.
…… $-9a^2bc-12ab^2c+15abc^2$ by $-3abc$. Ans. $3a+4b-5c$.

(3) Divide $2a^2+a-6$ by $2a-3$. Ans. $a+2$.

(4) Divide a^5+b^5 by $a+b$. Ans. $a^4-a^3b+a^2b^2-ab^3+b^4$.

(5) Divide a^4+4b^4 by $a^2-2ab+2b^2$. Ans. $a^2+2ab+2b^2$.

(6) Divide $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$.
Ans. $x^4+2ax^3+3a^2x^2+2a^3x+a^4$.

(7) Divide x^6-a^6 by $x^3+2ax^2+2a^2x+a^3$. Ans. $x^3-2ax^2+2a^2x-a^3$.

(8) Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.
Ans. $a^2+b^2+c^2-ab-ac-bc$.

(9) Divide $m^2+2mp-n^2-2nq+p^2-q^2$ by $m-n+p-q$.
Ans. $m+n+p+q$.

(10) Divide $1+2x$ by $1-3x$. Ans. $1+5x+15x^2+45x^3+\ldots$

(11) Divide a^4-81 by $a-3$. Ans. $a^3+3a^2+9a+27$.

(12) Divide $64-a^6$ by $2-a$. Ans. $32+16a+8a^2+4a^3+2a^4+a^5$.

(13) Divide $552a^{11}b^3x^4y^3z$ by $184a^6b^3x^2yz$. Ans. $3a^5x^2y^2$.

(14) Divide $115a^mb^nc^{2p+q}d^{2r-1}$ by $-69a^nb^nc^{p+q}d$. Ans. $-\frac{5}{3}a^{m-n}c^pd^{2r-2}$.

(15) Divide $a+b$ by $\sqrt[3]{a}+\sqrt[3]{b}$. Ans. $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.

(16) Divide $x^{m+n}+x^ny^n+x^my^m+y^{m+n}$ by x^n+y^m. Ans. x^m+y^n.

(17) Divide $a^{2m}-3a^mc^n+2c^{2n}$ by a^m-c^n. Ans. a^m-2c^n.

(18) Divide $ax^3-(a^2+b)x^2+b^2$ by $ax-b$. Ans. x^2-ax-b.

(19) Divide $x^3-apx^2+a^2px-a^3$ by $x-a$. Ans. $x^2-apx+ax+a^2$.

(20) Divide $x^5-px^4+qx^3-qx^2+px-1$ by $x-1$.
Ans. $x^4-(p-1)x^3+(q-p+1)x^2-(p-1)x+1$.

(21) Divide $mpx^3+(mq-np)x^2-(mr+nq)x+nr$ by $mx-n$.
Ans. px^2+qx-r.

(22) Divide $x^3-2ax^2+(a^2-ab-b^2)x+a^2b+ab^2$ by $x-a-b$.
Ans. $x^2-(a-b)x-ab$.

(23) Divide $(x^3-1)a^3-(x^3+x^2-2)a^2+(4x^2+3x+2)a-3(x+1)$ by
$(x-1)a^2-(x-1)a+3$. Ans. $(x^2+x+1)a-(x+1)$.

(24) Divide $x(x-1)a^3+(x^3+2x-2)a^2+(3x^2-x^3)a-x^4$ by $a^2x+2a-x^2$.
Ans. $(x-1)a+x$

(25) Divide $x^4+x^{-4}-x^2-x^{-2}$ by $x-x^{-1}$. Ans. x^3-x^{-3}.

(26) Divide $-2a^{-8}x^5+17a^{-4}x^6-5x^7-24a^4x^8$ by $2a^{-3}x^3-3ax^4$.
Ans. $-a^{-5}x^2+7a^{-1}x^3+8a^3x^4$.

(27) Divide $(2x-y)^2a^4-(x+y)^2a^2x^2+(x+y)2ax^4-x^6$ by
$(2x-y)a^2-(x+y)ax+x^3$. Ans. $(2x-y)a^2+(x+y)ax-x^3$.

(28) Divide $(3c-6b)a^2-(c^2-4b^2)a+c^3-6bc^2+12b^2c-8b^3$ by $c-2b$.
Ans. $3a^2-(c+2b)a+c^2-4bc+4b^2$.

(29) Divide $\frac{1}{3}-6a^2+27a^4$ by $\frac{1}{3}+2a+3a^2$. Ans. $1-6a+9a^2$.

(30) Divide $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$ by $x^2-\frac{1}{2}x$. Ans. $x^2-\frac{3}{4}x+1$.

(31) Divide $\frac{3x^5}{4}-4x^4+\frac{77x^3}{8}-\frac{43x^2}{4}-\frac{33x}{4}+27$ by $\frac{x^2}{2}-x+3$.
Ans. $\frac{3x^3}{2}-5x^2+\frac{x}{4}+9$.

(32) Divide $\frac{x^4}{3}-\frac{11x^3}{12}+\frac{41x^2}{8}-\frac{23x}{4}+6$ by $\frac{2x^2}{3}-\frac{5x}{6}+1$.

Ans. $\frac{x^2}{2}-\frac{3x}{4}+6$.

(33) Divide $\frac{2}{5}a^4x-\frac{136}{75}a^3x^2+\frac{8}{5}a^2x^3+\frac{3}{10}ax^4-x^5$ by $\frac{2}{3}a^3-\frac{4}{5}a^2x+\frac{1}{2}x^3$.

Ans. $\frac{3}{5}ax-2x^2$.

(34) Divide $\frac{1}{2}a^{\frac{3}{4}}-\frac{3}{10}a^{\frac{7}{10}}-\frac{1}{3}a^{\frac{7}{12}}+\frac{1}{5}a^{\frac{8}{15}}+\frac{1}{4}a^{\frac{1}{2}}-\frac{3}{20}a^{\frac{9}{20}}$ by $\sqrt[4]{a}-\frac{3}{5}\sqrt[5]{a}$.

Ans. $\frac{1}{2}\sqrt{a}-\frac{1}{3}\sqrt[3]{a}+\frac{1}{4}\sqrt[4]{a}$.

(35) Divide $x^{\frac{3}{2}}-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. Ans. $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

(36) Divide $a-b$ by $\sqrt[4]{a}-\sqrt[4]{b}$. Ans. $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}$.

(37) Divide $a^{3m-2n}b^{2p}c-a^{2m+n-1}b^{1-p}c^n+a^{-n}b^{-1}c^m+a^{3m-n}b^{3p+2}c^n-a^{2m+2n-1}b^3c^{2n-1}+b^{p+1}c^{m+n-1}$ by $a^{-n}b^{-p-1}+bc^{n-1}$. Ans. $a^{3m-n}b^{3p+1}c-a^{2m+2n-1}b^2c^n+b^pc^m$.

(38) Divide $x^{pq}-1$ by x^q-1, and write down the last three terms of the quotient. Ans. $x^{(p-1)q}+x^{(p-2)q}+x^{(p-3)q}+\ldots x^{2q}+x^q+1$.

N.B. Since the Product of two quantities ÷ Multiplicand = Multiplier, or Product ÷ Multiplier = Multiplicand, each example in Multiplication with its Answer supplies *two others* in Division. Also, since Dividend ÷ Quotient = Divisor, or Divisor × Quotient = Dividend, each Example in Division with its Answer supplies *two others*, one in Division, the other in Multiplication.

GREATEST COMMON MEASURE.

(1) Find the Greatest Common Measure of $3a^4x^5y$ and $6a^2bx$. Ans. $3a^2x$.

(2) Find the G.C.M. of $ax+x^2$ and $abc+bcx$. Ans. $a+x$.

(3) Find the G.C.M. of $a^2+ab-12b^2$ and $a^2-5ab+6b^2$. Ans. $a-3b$.

(4) Find the G.C.M. of $6a^2+7ax-3x^2$ and $6a^2+11ax+3x^2$.

Ans. $2a+3x$.

(5) Find the G.C.M. of $x^4+a^2x^2+a^4$ and $x^4+ax^3-a^3x-a^4$.

Ans. x^2+ax+a^2.

(6) Find the G.C.M. of $3x^2+16x-35$ and $5x^2+33x-14$. Ans. $x+7$.

(7) Find the G.C.M. of $3x^4+14x^3+9x+2$ and $2x^4+9x^3+14x+3$.

Ans. x^2+5x+1.

(8) Find the G.C.M. of $20x^4+x^2-1$ and $25x^4+5x^3-x-1$. Ans. $5x^2-1$.

(9) Find the G.C.M. of $6a^3-6a^2y+2ay^2-2y^3$ and $12a^2-15ay+3y^2$.
Ans. $a-y$.

(10) Find the G.C.M. of $48x^2+16x-15$ and $24x^3-22x^2+17x-5$.
Ans. $12x-5$.

(11) Find the G.C.M. of $6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$. Ans. $2x^3-4x^2+x-1$.

(12) Find the G.C.M. of $x^4+ax^3-9a^2x^2+11a^3x-4a^4$ and $x^4-ax^3-3a^2x^2+5a^3x-2a^4$. Ans. $(x-a)^3$.

(13) Find the G.C.M. of $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$. Ans. x^2-1.

(14) Find the G.C.M. of $3x^2-(4a+2b)x+2ab+a^2$ and $x^3-(2a+b)x^2+(2ab+a^2)x-a^2b$. Ans. $x-a$.

(15) Find the G.C.M. of $2x^3+(2a+3b)x^2+(2b+3ab)x+3b^2$ and $2x^2+(2c+3b)x+3bc$. Ans. $2x+3b$.

(16) Find the G.C.M. of x^2-2x-3, $x^2-7x+12$, and x^2-x-6.
Ans. $x-3$.

(17) Find the G.C.M. of x^2+5x+4, x^2+2x-8, and $x^2+7x+12$.
Ans. $x+4$.

(18) Find the G.C.M. of $15a^4+10a^3b+4a^2b^2+6ab^3-3b^4$ and $6a^3+19a^2b+8ab^2-5b^3$. Ans. $3a^2+2ab-b^2$.

(19) Find the G.C.M. of $ab+2a^2-3b^2-4bc-ac-c^2$ and $9ac+2a^2-5ab+4c^2+8bc-12b^2$. Ans. $2a+3b+c$.

(20) Find the G.C.M. of $qnp^3+3np^2q^2-2npq^3-2nq^4$ and $2mp^2q^2-4mp^4-mp^3q+3mpq^3$. Ans. $p-q$.

(21) Find the G.C.M. of $x^6+4x^5-3x^4-16x^3+11x^2+12x-9$ and $6x^5+20x^4-12x^3-48x^2+22x+12$. Ans. x^3+x^2-5x+3.

(22) Find the G.C.M. of $a^2+2b^2+(a+2b)\sqrt{ab}$ and $a^2-b^2+(a-b)\sqrt{ab}$. (*Comp.* p. 7.) Ans. $\sqrt{a}+\sqrt{b}$.

(23) Find the G.C.M. of $\frac{x^2}{3}+\frac{11x}{6}\sqrt{x+1}-x-1$ and $x^2-\frac{x}{4}-\frac{1}{4}$. (*Comp.* p. 8.)

Ans. $x-\frac{1}{2}\sqrt{x+1}$.

(24) Find the G.C.M. of $(b-c)x^2+(2ab-2ac)x+a^2b-a^2c$ and $(ab-ac+b^2-bc)x+a^2c+ab^2-a^2b-abc$. Ans. $b-c$.

(25) Find the G.C.M. of $a^2x^3+a^5-2abx^3+b^2x^3+a^3b^2+2a^4b$ and $2a^2x^4-5a^4x^2+3a^6-2b^2x^4+5a^2b^2x^2-3a^4b^2$. (*Comp.* p. 8.) Ans. $(a-b)(x+a)$.

(26) Find the G.C.M. of $2(y^3-2y^2-y+2)x^3+3(y^2-1)x^2-(2y^3-y^2-2y+1)$ and $3(y^3-4y^2+5y-2)x^2+7(y^2-2y+1)x-(3y^3-5y^2+y+1)$. (*Comp.* p. 9.) Ans. $xy-x-y+1$.

(27) Find the value of y which will make $2(y^2+y)x^2+(11y-2)x+4$ and $2(y^3+y^2)x^3+(11y^2-2y)x^2+(y^2+5y)x+5y-1$ have a common measure. (*Comp.* p. 9.) Ans. $y=5$.

(28) Find the G.C.M. of $6ax^2-6ab^2-54a-21x^2+21b^2+189$, $6ax^5+60a-21x^5-210$, and $6ab^3-12a-21b^3+42$. (*Comp.* p. 9.) Ans. $6a-21$.

(29) Find the G.C.M. of $6a^3x^2+4ax^3-10axy-3a^2xy-2x^2y+5y^2$, $10a^2x^2-2ax^3y-6axy^2-5axy+x^2y^2+3y^3$, and $-4ax^2-6a^2bxy+2ab^2x^2y-2abx+2xy+3aby^2-b^2xy^2+by$. (*Comp.* p. 10.) Ans. $y-2ax$.

(30) If $x+a$ be the G.C.M. of x^2+px+q, and $x^2+p'x+q'$, shew that $a=\dfrac{q-q'}{p-p'}$. (*Comp.* p. 10.)

(31) Shew that x^2+qx+1, and x^3+px^2+qx+1 have a common factor of the form $x+a$, when $(p-1)^2-q(p-1)+1=0$. (*Comp.* p. 11.)

N.B. Since each proposed quantity is divisible without remainder by the G. C. M., each Example in Greatest Common Measure with its Answer supplies two or more Examples in Division.

LEAST COMMON MULTIPLE.

(1) FIND the Least Common Multiple of $8a^2$, $12a^3$, and $20a^4$. Ans. $120a^4$.

(2) Find the L.C.M. of $1-a$, $1+a$, and $1-a^2$. Ans. $1-a^2$.

(3) Find the L.C.M. of a^3-x^3 and a^2-x^2. Ans. $a^4+a^3x-ax^3-x^4$.

(4) Find the L.C.M. of $2x-1$, $4x^2-1$, and $4x^2+1$. Ans. $16x^4-1$.

(5) Find the L.C.M. of x^2+5x+4, x^2+2x-8, and $x^2+7x+12$. Ans. $x^4+6x^3+3x^2-26x-24$.

(6) Find the L.C.M. of a^2-b^2, a^2+b^2, $(a-b)^2$, $(a+b)^2$, a^3-b^3, and a^3+b^3. Ans. $a^{10}-a^6b^4-a^4b^6+b^{10}$.

(7) Find the L.C.M. of $x-1$, x^2-1, $x-2$, and x^2-4. Ans. x^4-5x^2+4.

(8) Find the L.C.M. of $4(1-x)^2$, $8(1-x)$, $8(1+x)$, and $4(1+x^2)$. Ans. $8(1-x)(1-x^4)$.

(9) Find the L.C.M. of $6x^3-11x^2+5x-3$, and $9x^3-9x^2+5x-2$. Ans. $18x^4-45x^3+37x^2-19x+6$.

(10) Find the L.C.M. of $3x^2-11x+6$, $2x^2-7x+3$, and $6x^2-7x+2$. (*Comp.* p. 11.) Ans. $6x^3-25x^2+23x-6$.

(11) Find the L.C.M. of $6x^2-13x+6$, $12x^2-5x-2$, and $15x^2+2x-8$. Ans. $120x^4-134x^3-129x^2+74x+24$.

(12) Find the L.C.M. of x^3-3x^2+3x-1, x^3-x^2-x+1, x^4-2x^3+2x-1, and $x^4-2x^3+2x^2-2x+1$. (*Comp.* p. 12.) Ans. $x^6-2x^5+x^4-x^2+2x-1$.

(13) Shew that if $x+c$ be the G.C.M. of x^2+ax+b, and $x^2+a'x+b'$, their least com. mult. will be $x^3+(a+a'-c)x^2+(aa'-c^2)x+(a-c)(a'-c)c$. (*Comp.* p. 12.)

N.B. Since the Least Com. Mult. is divisible without remainder by each of the proposed quantities, each Example in Least Common Multiple with its Answer supplies two or more Examples in Division.

ADDITION, SUBTRACTION, AND REDUCTION OF FRACTIONS.

(1) ADD together $\frac{a}{6x}$, $\frac{3a}{2x}$, and $\frac{5a}{8x}$. Ans. $\frac{55a}{24x}$.

(2) Add together $\frac{1}{x}$, $\frac{1}{2x}$, $\frac{1}{3x}$, and $\frac{1}{4x}$. Ans. $\frac{25}{12x}$.

(3) Add together $\frac{p}{ab}$, $\frac{q}{ac}$, and $\frac{r}{bc}$. Ans. $\frac{pc+qb+ra}{abc}$.

(4) Subtract $\frac{2a}{7b}$ from $\frac{9a}{7b}$; and $-\frac{3}{x}$ from $\frac{4}{x}$. Ans. $\frac{a}{b}$, $\frac{7}{x}$.

(5) Subtract $\frac{2x-15}{11a}$ from $\frac{2x-8}{11a}$. Ans. $\frac{7}{11a}$.

(6) Subtract $5\frac{1}{2}.x-\frac{a}{b}$ from $6x-\frac{2a}{b}$. Ans. $\frac{1}{2}x-\frac{a}{b}$.

(7) What is the difference between $\frac{ab}{a-b}$, and $\frac{ab}{a}-\frac{ab}{b}$? Ans. $a+\frac{b^2}{a-b}$.

(8) What is the difference between $\frac{a-b}{ab}$, and $\frac{a}{ab}-\frac{b}{ab}$? Ans. 0.

Reduce the following fractional expressions :—

(9) $\dfrac{3x-1}{24}-\dfrac{3x-5}{24}+\dfrac{5}{6}$. Ans. 1.

(10) $\dfrac{4m-3n}{3(1-n)}-\dfrac{m+3n}{3(1-n)}+\dfrac{2n}{1-n}$. Ans. $\dfrac{m}{1-n}$.

(11) $\dfrac{a-b}{ab}-\dfrac{a-c}{ac}+\dfrac{b-c}{bc}$. Ans. 0.

(12) $\dfrac{7x-10}{5}-\dfrac{3x-7}{6}-\dfrac{27x-30}{30}$. Ans. $\dfrac{1}{6}$.

(13) $\dfrac{3x-4y}{7}-\dfrac{2x-y-1}{3}+\dfrac{15x-4}{12}$. Ans. $\dfrac{85x-20y}{84}$.

(14) $\dfrac{a}{c}-\dfrac{(ad-bc)x}{c(c+dx)}-\dfrac{a}{c+dx}$. Ans. $\dfrac{bx}{c+dx}$.

(15) $\dfrac{a+\dfrac{b-a}{1+ba}}{1-a.\dfrac{b-a}{1+ba}}$; $2\times\dfrac{x^2-\dfrac{1}{4}}{2x+1}+\dfrac{1}{2}$. (1) Ans. b. (2) Ans. x.

(16) $\dfrac{\dfrac{1}{a}+\dfrac{1}{ab^3}}{b-1+\dfrac{1}{b}}$; $\dfrac{a+b+\dfrac{b^2}{a}}{a+b+\dfrac{a^2}{b}}$. (1) Ans. $\dfrac{b+1}{ab^2}$. (2) Ans. $\dfrac{b}{a}$.

(17) $\dfrac{1}{n-1-(n-1)x}-\dfrac{1}{n+1+(n+1)x}$. Ans. $\dfrac{2}{n^2-1}\cdot\dfrac{1+nx}{1-x^2}$.

(18) $\dfrac{a}{b}-\dfrac{(a^2-b^2)x}{b^2}+\dfrac{a(a^2-b^2)x^2}{b^2(b+ax)}$. Ans. $\dfrac{a+bx}{b+ax}$.

(19) $\dfrac{1}{2}\cdot\dfrac{3m+2n}{3m-2n}-\dfrac{1}{2}\cdot\dfrac{3m-2n}{3m+2n}$. Ans. $\dfrac{6mn}{9m^2-4n^2}$.

(20) $\dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^3-x^2y}{y^3-x^2y}$. Ans. $\dfrac{y}{x+y}$.

(21) $\dfrac{x^4-y^4}{x^2-2xy+y^2}\div\dfrac{x^2+xy}{x-y}$. Ans. $x+\dfrac{y^2}{x}$.

(22) $\dfrac{x(x+1)(x+2)}{3}-\dfrac{x(x+1)(2x+1)}{1\ .\ 2\ .\ 3}$. (*Comp.* p. 13.) Ans. $\dfrac{x(x+1)}{2}$.

(23) $\dfrac{1}{x-1}-\dfrac{1}{2(x+1)}-\dfrac{x+3}{2(x^2+1)}$. Ans. $\dfrac{x+3}{x^4-1}$.

(24) $\dfrac{3}{4(1-x)^2}+\dfrac{3}{8(1-x)}+\dfrac{1}{8(1+x)}-\dfrac{1-x}{4(1+x^2)}$. Ans. $\dfrac{1+x+x^2}{1-x-x^4+x^5}$.

(25) $\dfrac{3a}{(a-2x)^2}+\dfrac{2a+x}{(a+x)(a-2x)}-\dfrac{5}{a+x}$. Ans. $\dfrac{20ax-22x^2}{(a+x)(a-2x)^2}$.

(26) $\dfrac{1}{4a^3(a+x)}+\dfrac{1}{4a^3(a-x)}+\dfrac{1}{2a^2(a^2+x^2)}$. Ans. $\dfrac{1}{a^4-x^4}$.

(27) $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}$. (*Comp.* p. 13.) Ans. $x^{2n}+2$.

(28) $\dfrac{a^2+b^2+ab}{(a^3-b^3)(x^{\frac{1}{2}}-a^{\frac{1}{2}})}-\dfrac{x^{\frac{1}{2}}}{(a-b)(x-a)}$. (*Comp.* p. 13.) Ans. $\dfrac{a^{\frac{1}{2}}}{(a-b)(x-a)}$.

(29) $\dfrac{8}{5(x-2)}-\dfrac{13}{80(x+3)}-\dfrac{5}{4(x-1)^2}-\dfrac{23}{16(x-1)}$. Ans. $\dfrac{x^2+4}{(x-1)^2(x-2)(x+3)}$.

(30) $\dfrac{1}{(a-b)(a-c)(x+a)}-\dfrac{1}{(a-b)(b-c)(x+b)}+\dfrac{1}{(a-c)(b-c)(x+c)}$.

(*Comp.* p. 14.) Ans. $\dfrac{1}{(x+a)(x+b)(x+c)}$.

(31) $bc.\dfrac{a+d}{(a-b)(a-c)}+ac.\dfrac{b+d}{(b-a)(b-c)}+ab.\dfrac{c+d}{(c-a)(c-b)}$. (*Comp.* p. 14.)

Ans. d.

(32) $\dfrac{a+b}{ab}(a^2+b^2-c^2)+\dfrac{b+c}{bc}(b^2+c^2-a^2)+\dfrac{a+c}{ac}(a^2+c^2-b^2)$. (*Comp.* p. 15.)

Ans. $2(a+b+c)$.

(33) $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}+\dfrac{b^2-(c-a)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$. (*Comp.* p. 15.) Ans. 1.

(34) $\dfrac{1}{a+\dfrac{1}{b+\dfrac{1}{c}}}$, $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}$. (*Comp.* p. 15.)

(1) Ans. $\dfrac{bc+1}{abc+a+c}$. (2) Ans. $\dfrac{4}{3x}$.

(35) Reduce to lowest terms $\dfrac{ax}{3ax}$, and $\dfrac{m^2-n^2}{3m-3n}$. Ans. $\dfrac{1}{3}$, $\dfrac{m+n}{3}$.

(36) Reduce to lowest terms $\dfrac{a^{m-1}b^{2n}}{a^{2m}b^{n-1}}$, $\dfrac{ax^2y}{bx^{\frac{3}{2}}y^{\frac{3}{2}}}$, and $\dfrac{px^{-1}y^{-2}z^{-3}}{qx^{-\frac{1}{2}}y^{-\frac{3}{2}}z^{-\frac{5}{2}}}$.

(1) Ans. $\dfrac{b^{n+1}}{a^{m+1}}$. (2) Ans. $\dfrac{ax^{\frac{1}{2}}}{by^{\frac{1}{2}}}$. (3) Ans. $\dfrac{p}{qx^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}}$.

(37) Reduce to lowest terms $\dfrac{2a^3-2ab}{5a^3-5ab}$, $\dfrac{ax+x^2}{3bx-cx}$, $\dfrac{14a^2-7ab}{10ac-5bc}$, $\dfrac{12a^3x^4+2a^2x^5}{18ab^2x+3b^2x^2}$, and $\dfrac{5a^2+5ax}{a^2-x^2}$. Ans. $\dfrac{2}{5}$, $\dfrac{a+x}{3b-c}$, $\dfrac{7a}{5c}$, $\dfrac{2a^2x^3}{3b^2}$, $\dfrac{5a}{a-x}$.

Reduce to lowest terms the following fractions :—

(38) $\dfrac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}$. Ans. $\dfrac{x+a}{x+b}$.

(39) $\dfrac{ac+by+ay+bc}{af+2bx+2ax+bf}$. Ans. $\dfrac{c+y}{f+2x}$.

(40) $\dfrac{6ac+10bc+9ax+15bx}{6c^2+9cx-2c-3x}$. Ans. $\dfrac{3a+5b}{3c-1}$.

(41) $\dfrac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}$. Ans. $\dfrac{a+b+c}{a-b-c}$.

(42) $\dfrac{x^8+x^6y^2+x^2y+y^3}{x^4-y^4}$. Ans. $\dfrac{x^6+y}{x^2-y^2}$.

(43) $\dfrac{a^3+ab^2-a^2b-b^3}{4a^4-2a^2b^2-4a^3b+2ab^3}$. Ans. $\dfrac{a^2+b^2}{2a(2a^2-b^2)}$.

(44) $\dfrac{2a^2+ab-b^2}{a^3+a^2b-a-b}$. Ans. $\dfrac{2a-b}{a^2-1}$.

(45) $\dfrac{6x^5+15x^4y-4x^3z^2+10x^2yz^2}{9x^3y-27x^2yz-6xyz^2+18yz^3}$. Ans. $\dfrac{x^2}{3y}\cdot\dfrac{2x+5y}{x-3z}$.

(46) $\dfrac{acx^2+(ad+bc)x+bd}{a^2x^2-b^2}$. Ans. $\dfrac{cx+d}{ax-b}$.

(47) $\dfrac{a^3+(a+b)ax+bx^2}{a^4-b^2x^2}$. Ans. $\dfrac{a+x}{a^2-bx}$.

(48) $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}$. Ans. $\dfrac{2x-3a}{4x^2+6ax+9a^2}$.

(49) $\dfrac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}$. Ans. $\dfrac{x^{m-1}}{b(a+bx)}$.

(50) $\dfrac{30a^{3n-1}b^rc^{r+2}-6a^{2n-4}b^3c^rd^{r-1}}{20a^nb^{r-1}c^2d^2-4a^{-3}b^2d^{r+1}}$. Ans. $\dfrac{3a^{2n-1}bc^r}{2d^2}$.

(51) $\dfrac{a^2(b^2-c^2)-ab(2b^2+bc-c^2)+b^3(b+c)}{a^3(b^2+2bc+c^2)-a^2b(2b^2+3bc+c^2)+ab^3(b+c)}$. (*Comp.* p. 16.)

Ans. $\dfrac{1}{a}\cdot\dfrac{a(b-c)-b^2}{a(b+c)-b^2}$.

(52) $\dfrac{(c-d)a^2+6(bc-bd)a+9(b^2c-b^2d)}{(bc-bd+c^2-cd)a+3(b^2c+bc^2-b^2d-bcd)}$. (*Comp.* p. 16.)

Ans. $\dfrac{a+3b}{b+c}$.

(53) $\dfrac{1+x^{\frac{1}{2}}+x+x^{\frac{3}{2}}}{2x+2x^{\frac{3}{2}}+3x^2+3x^{\frac{5}{2}}}$. Ans. $\dfrac{1+x}{2x+3x^2}$.

(54) $\frac{3bcq+30mp+18bc+5mpq}{4adq-42fg+24ad-7fgq}$. Ans. $\frac{3bc+5mp}{4ad-7fg}$.

(55) $\frac{x-4-3x^{\frac{1}{2}}+4y^{\frac{1}{2}}-(xy)^{\frac{1}{2}}}{x-8-2x^{\frac{1}{2}}+12y^{\frac{1}{2}}-3(xy)^{\frac{1}{2}}}$. (*Comp.* p. 16.) Ans. $\frac{x^{\frac{1}{2}}-y^{\frac{1}{2}}+1}{x^{\frac{1}{2}}-3y^{\frac{1}{2}}+2}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

(1) MULTIPLY $a-\frac{x^2}{a}$ by $\frac{a}{x}+\frac{x}{a}$. Ans. $\frac{a^2}{x}-\frac{x^3}{a^2}$.

(2) Multiply $a+\frac{1}{a}$ by $a+\frac{1}{a}$. Ans. $a^2+\frac{1}{a^2}+2$.

(3) Multiply $\frac{b}{a}+b$ by $\frac{a}{b}+\frac{1}{b}$. Ans. $a+\frac{1}{a}+2$.

(4) Multiply $\frac{2a-b}{4a}$ by $\frac{6a-2b}{b^2-2ab}$. Ans. $\frac{b-3a}{2ab}$.

(5) Multiply x^2+x+1 by $\frac{1}{x^2}-\frac{1}{x}+1$. Ans. $x^2+1+\frac{1}{x^2}$.

(6) Multiply $x+1+\frac{1}{x}$ by $x-1+\frac{1}{x}$. Ans. $x^2+1+\frac{1}{x^2}$.

(7) Multiply $\frac{a}{a+b}+\frac{b}{a-b}$ by $\frac{a}{a+c}-\frac{b}{b+c}$. Ans. $\frac{(a^2+b^2)c}{(a+b)(a+c)(b+c)}$.

(8) Multiply $3x^{\frac{7}{4}}-\frac{5xy}{x^{\frac{1}{4}}}$ by $x^{\frac{1}{4}}-\frac{7y}{x^{\frac{3}{4}}}$. Ans. $3x^2-26xy+35y^2$.

(9) Multiply $\frac{a^2}{x^2}-\frac{ab}{2xy}+\frac{b^2}{y^2}$ by $\frac{3a^2}{x^2}-\frac{2ab}{5xy}+\frac{b^2}{y^2}$.

Ans. $\frac{3a^4}{x^4}-\frac{19a^3b}{10x^3y}+\frac{21a^2b^2}{5x^2y^2}-\frac{9ab^3}{10xy^3}+\frac{b^4}{y^4}$.

(10) Multiply $\frac{a^2}{b^3}+\frac{2c^3d^4}{b^5}-\frac{7c^2}{2a^4b^3}$ by $\frac{a^2}{b^3}-\frac{2c^3d^4}{b^5}+\frac{7c^2}{2a^4b^3}$.

Ans. $\frac{a^4}{b^6}-\frac{4c^6d^8}{b^{10}}+\frac{14c^5d^4}{a^4b^8}-\frac{49c^4}{4a^8b^6}$.

(11) Divide $\frac{3x}{2x-2}$ by $\frac{2x}{x-1}$. Ans. $\frac{3}{4}$.

(12) Divide $x+\frac{2x}{x-3}$ by $x-\frac{2x}{x-3}$. Ans. $\frac{x-1}{x-5}$.

(13) Divide $\frac{4a(a^2-x^2)}{3b(c^2-x^2)}$ by $\frac{a^2-ax}{bc+b}$. Ans. $\frac{4(a+x)}{3(c-x)}$.

(14) Divide $x^4-\frac{1}{x^4}$ by $x-\frac{1}{x}$. Ans. $x^3+\frac{1}{x^3}+x+\frac{1}{x}$.

(15) Divide $a^6+\frac{1}{a^6}+a^4+\frac{1}{a^4}+a^2+\frac{1}{a^2}+2$ by $a^3+\frac{1}{a^3}+a+\frac{1}{a}$. Ans. $a^3+\frac{1}{a^3}$.

(16) Divide $\frac{a^2}{bc}-\frac{2a}{d}+\frac{ac}{be}+\frac{bc}{d^2}-\frac{c^2}{de}$ by $\frac{a}{b}-\frac{c}{d}$. Ans. $\frac{a}{c}-\frac{b}{d}+\frac{c}{e}$.

(17) Divide $\frac{a^2x^3}{bd}+\frac{abx^2}{c^2d}-\frac{acx^2}{d^2}-\frac{b^2x}{cd^2}+\frac{a^2x}{bc}-\frac{a}{d}$ by $\frac{ax}{c}-\frac{b}{d}$; and verify the result by multiplication.

(18) Divide $a'+x$ by $a+x$ to 4 terms of quotient.

Ans. $\frac{a'}{a}+\frac{a-a'}{a^2}.x-\frac{a-a'}{a^3}.x^2+\frac{a-a'}{a^4}.x^3-\ldots$

(19) Divide a^2 by $a^2+2ax+x^2$. Ans. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\ldots$

(20) Divide $a-bx$ by $a+cx$ to 4 terms.

Ans. $1-(b+c)\frac{x}{a}+(b+c)\frac{cx^2}{a^2}-(b+c)\frac{c^2x^3}{a^3}+\ldots$

MISCELLANEOUS FRACTIONS.

(1) FIND the value of $\left(x+\frac{2x}{x-3}\right)\div\left(x-\frac{2x}{x-3}\right)$, when $x=5\frac{1}{2}$. Ans. 9.

(2) Find the value of $\frac{x}{2}-\left\{\frac{2x-3}{3}-\frac{3x-1}{4}\right\}\div\frac{x-1}{2}$, when $x=4\frac{1}{3}$.

Ans. $2\frac{5}{6}$.

(3) Find the value of $ax+by$, when $x=\frac{cq-br}{aq-bp}$, and $y=\frac{ar-cp}{aq-bp}$.

Ans. c.

(4) Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$. (*Comp.* p. 17.)

Ans. 2.

(5) Find the value of $\frac{a^n}{2na^n-2nx}+\frac{b^n}{2nb^n-2nx}$, when $x=\frac{a^n+b^n}{2}$.

(*Comp.* p. 17.) Ans. $\frac{1}{n}$.

(6) If two fractions are together equal to 1, shew that their difference is the same as the difference of their squares.

(7) If the difference of two fractions is equal to $\frac{p}{q}$, shew that p times their sum is equal to q times the difference of their squares.

(8) Two fractions are together equal to $\frac{a}{b}$, and the one exceeds the other by $\frac{c}{d}$; what are the fractions? Ans. $\frac{1}{2}\left(\frac{a}{b}+\frac{c}{d}\right)$, and $\frac{1}{2}\left(\frac{a}{b}-\frac{c}{d}\right)$.

(9) Divide $\frac{83}{89}$ into two parts differing by $\frac{14}{19}$. Ans. $\frac{2823}{3382}$, $\frac{331}{3382}$.

INVOLUTION AND EVOLUTION.*

(1) FIND the square and the cube of $3a^2bc^3$. Ans. $9a^4b^2c^6$, $27a^6b^3c^9$.

(2) Find the square and the cube of $\frac{2}{5}a^3x^{m+2}y^{p-1}$.

(1) Ans. $\frac{4}{25}a^6x^{2m+4}y^{2p-2}$. (2) Ans. $\frac{8}{125}a^9x^{3m+6}y^{3p-3}$.

(3) Find the square and cube of each of the following quantities, and verify the results by extracting their square and cube roots :—

(1) $2a+3b$, (2) $xy-3$, (3) $2ax-5mn$, (4) $2a^2-a-2$.

(4) Find the square of $\frac{2}{5}a-\frac{1}{2}b$. Ans. $\frac{4}{25}a^2-\frac{2}{5}ab+\frac{1}{4}b^2$.

(5) Find the cube of $\frac{1}{2}a-\frac{2}{3}b$. Ans. $\frac{1}{8}a^3-\frac{8}{27}b^3-\frac{1}{2}a^2b+\frac{2}{3}ab^2$.

(6) Find the cube of $1+2x+3x^2+4x^3$.
Ans. $1+6x+21x^2+56x^3+111x^4+174x^5+219x^6+204x^7+144x^8+64x^9$.

(7) Find the 4th power of $a+bx$. Ans. $a^4+4a^3bx+6a^2b^2x^2+4ab^3x^3+b^4x^4$.

(8) Find the cube of $\sqrt{a}-\sqrt{b}$. Ans. $(a+3b)\sqrt{a}-(b+3a)\sqrt{b}$.

(9) Find the cube of $\sqrt[3]{x}-\sqrt[3]{y}$. Ans. $x-y-3\sqrt[3]{xy}.(\sqrt[3]{x}-\sqrt[3]{y})$.

(10) Find the cube of $x-\frac{1}{x}$. Ans. $x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)$.

(11) Find the cube of $x-\frac{1}{x}-1$. Ans. $x^3-\frac{1}{x^3}-3x^2-3\frac{1}{x^2}+5$.

(12) Find the square of $a^{\frac{2}{3}}-a^{\frac{1}{3}}+1$. Ans. $a^{\frac{4}{3}}-2a+3a^{\frac{2}{3}}-2a^{\frac{1}{3}}+1$.

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* **Each Example in Involution with its Answer supplies also an Example in Evolution, and *vice versâ*.**

(13) Find the cube of $\frac{x}{a^2}-\frac{a^2}{x}$. Ans. $\frac{x^3}{a^6}-\frac{3x}{a^2}+\frac{3a^2}{x}-\frac{a^6}{x^3}$.

(14) Find the cube of e^x-e^{-x}. Ans. $e^{3x}-e^{-3x}-3(e^x-e^{-x})$.

(15) Find the cube of $x^{pq}-1$. Ans. $x^{3pq}-3x^{2pq}+3x^{pq}-1$.

(16) Find the square of $a^mb^n-3b^{2n-1}c^p$. Ans. $a^{2m}b^{2n}-6a^mb^{3n-1}c^p+9b^{4n-2}c^{2p}$.

(17) If $x+\frac{1}{x}=p$, prove that $x^3+\frac{1}{x^3}=p^3-3p$.

(18) Find the square of $1-\frac{1}{2}x+\frac{2}{3}x^2-\frac{3}{4}x^3+\frac{4}{5}x^4-\&c.$

Ans. $1-x+\frac{19}{12}x^2-\frac{13}{6}x^3+\ldots$

(19) Prove that $\left(x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\ldots\right)^2+\left(1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-\ldots\right)^2$ is equal to 1. (*Comp.* p. 17.)

(20) Find the square root of $9a^4b^2c^{-12}$, and of $16x^{2m+2}a^{-2n}$.

(1) Ans. $3a^2bc^{-6}$, (2) Ans. $4x^{m+1}a^{-n}$.

(21) Find the square root of $4x^4-12x^3+25x^2-24x+16$.

Ans. $2x^2-3x+4$.

(22) Find the square root of $9a^4-12a^3b+34a^2b^2-20ab^3+25b^4$.

Ans. $3a^2-2ab+5b^2$.

(23) Find the square root of $x^6+4x^5+10x^4+20x^3+25x^2+24x+16$.

Ans. x^3+2x^2+3x+4.

(24) Find the square root of $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$. Ans. $x^2-x+\frac{1}{4}$.

(25) Find the square root of $\frac{25a^2b^2}{4}+\frac{c^4}{9}-\frac{5abc^2}{3}$. Ans. $\frac{5ab}{2}-\frac{c^2}{3}$.

(26) Find the square root of $x^2+\frac{a^2}{9}+\frac{b^2}{4}+\frac{2}{3}ax-\frac{1}{3}ab-bx$.

Ans. $x+\frac{1}{3}a-\frac{1}{2}b$.

(27) Find the square root of $\frac{1051x^2}{25}-\frac{6x}{5}-\frac{14x^3}{5}+9+49x^4$.

Ans. $7x^2-\frac{x}{5}+3$.

(28) Find the square root of $a^2-\frac{3a\sqrt{a}}{2}-\frac{3\sqrt{a}}{2}+\frac{41a}{16}+1$.

Ans. $a-\frac{3\sqrt{a}}{4}+1$.

(29) Find the square root of $\frac{a^2}{b^2}+\frac{b^2}{a^2}-2$. Ans. $\frac{a}{b}-\frac{b}{a}$.

(30) Find the square root of $\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{2a}{b}+\frac{2b}{a}+3$. Ans. $\frac{a}{b}+\frac{b}{a}+1$.

(31) Find the square root of $a^{2m}-4a^{m+n}+4a^{2n}$. Ans. a^m-2a^n.

(32) Find the square root of $\frac{x^2}{y^2}\left(\frac{x^2}{4y^2}+1\right)+\frac{4y^2}{x^2}\left(\frac{y^2}{x^2}+1\right)+3$.

Ans. $\frac{x^2}{2y^2}+\frac{2y^2}{x^2}+1$.

(33) Find the square root of $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)\sqrt{2}+2\frac{1}{2}$.

Ans. $\frac{x}{y}+\frac{y}{x}-\frac{1}{\sqrt{2}}$.

(34) Find the square root of $\frac{4a^6+9a^4x^2+6a^2x^4+x^6}{4a^2+x^2}$. Ans. a^2+x^2.

(35) Find the cube root of $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$.

Ans. a^2-2a+1.

(36) Find the cube root of $\frac{a^6}{b^2}-6a^4+12a^2b^2-8b^4$. Ans. $\frac{a^2}{b^{\frac{2}{3}}}-2b^{\frac{4}{3}}$.

(37) Find the cube root of $\frac{a^3c^3}{b^3}x^6-\frac{3a^2c}{b}x^5+\frac{3ab}{c}x^4-\frac{b^3}{c^3}x^3$.

Ans. $\frac{ac}{b}x^2-\frac{b}{c}x$.

(38) Find the square root of $(x+x^{-1})^2-4(x-x^{-1})$. Ans. $x-x^{-1}-2$.

(39) Find the square root of $49a^{4m-6}-42a^{6m-2}+9a^{8m+2}$.

Ans. $7a^{2m-3}-3a^{4m+1}$.

(40) Find the square root of $9a^{2m}+6a^{3m+1}+25c^{2m-4}-30a^mc^{m-2}+a^{4m+2}-10a^{2m+1}c^{m-2}$.

Ans. $3a^m+a^{2m+1}-5c^{m-2}$.

(41) Find the cube root of $(a+1)^{6m}x^3+6ca^p(a+1)^{4m}x^2+12c^2a^{2p}(a+1)^{2m}x+8c^3a^{3p}$.

Ans. $(a+1)^{2m}x+2ca^p$.

(42) Find the 6th root of $a^6+\frac{1}{a^6}-6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)-20$.

Ans. $a-\frac{1}{a}$.

(43) Find the 4th root of $a^8-\frac{3m^2}{n}a^7+\frac{27m^4}{8n^2}a^6-\frac{27m^6}{16n^3}a^5+\frac{81m^8}{256n^4}a^4$.

Ans. $a^2-\frac{3m^2}{4n}a$.

(44) Find the square root of $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$.

Ans. a^2+b^2.

(45) Find the square root (without the aid of the common rule) of $4\{(a^2-b^2)cd+(c^2-d^2)ab\}^2+\{(a^2-b^2)(c^2-d^2)-4abcd\}^2$. (*Comp.* p. 18.)

Ans. $(a^2+b^2)(c^2+d^2)$.

(46) Find the coefficient of x^8 in the square, and in the cube, of $1-2x+3x^2+4x^4-x^7$. (1) Ans. 20. (2) Ans. 168.

(47) Find the coefficient of x^2 in the square of $(x+a)(x+b)(x+c)$. (*Comp.* p. 18.) Ans. $a^2b^2+a^2c^2+b^2c^2+4abc(a+b+c)$.

MISCELLANEOUS.

(1) If two numbers differ by a unit, prove that the difference of their squares is the sum of the two numbers.

(2) Shew that the sum of the cubes of any three consecutive integers is divisible by three times the second of them. (*Comp.* p. 18.)

(3) In the extraction of the square root according to the common rule, shew that, if the remainder is not greater than twice the root obtained, the last digit in the root is correctly found.

(4) Prove that $x^4+px^3+qx^2+rx+s$ is a perfect square, if $p^2s=r^2$ and $q=\frac{p^2}{4}+2\sqrt{s}$. (*Comp.* p. 18.)

(5) Find the relations subsisting between a, b, c, d, when ax^3+bx^2+cx+d is a complete cube. (*Comp.* p. 19.)

Ans. $ac^3=db^3$, and $b^2=3ac$.

(6) Find the relations subsisting between a, b, c, d, e, when $ax^4+bx^3+cx^2+dx+e$ is a complete 4th power. (*Comp.* p. 19.)

Ans. $bc=6ad$, $cd=6be$, and $bd=16ae$.

SURDS.

(1) **Reduce** to simplest form $\sqrt{150}$, $\sqrt{1805}$, $\sqrt[3]{320}$, $\sqrt[3]{2808}$, $\sqrt[4]{144}$.
Ans. $5\sqrt{6}$, $19\sqrt{5}$, $4\sqrt[3]{5}$, $6\sqrt[3]{13}$, $2\sqrt{3}$.

(2) Reduce to simplest form $\sqrt[4]{768}$, $\sqrt[5]{608}$, $\sqrt[7]{640}$, $\sqrt[n]{a^{m+n}b^n}$.
Ans. $4\sqrt[4]{3}$, $2\sqrt[5]{19}$, $2\sqrt[7]{5}$, $ab\sqrt[n]{a^m}$.

(3) Reduce to simplest form $\sqrt{a^4b^3c}$, $\sqrt[3]{a^5b^3}$, $\sqrt{a^{4n}b^{2p}c^3}$, $\sqrt{a^3-a^2b}$.
Ans. $a^2b\sqrt{bc}$, $ab\sqrt[3]{a^2}$, $a^{2n}b^pc\sqrt{c}$, $a\sqrt{a-b}$.

(4) Reduce to simplest form $\sqrt{ax^2-6ax+9a}$, $\sqrt{(x^2-y^2)(x+y)}$.
Ans. $(x-3)\sqrt{a}$, $(x+y)\sqrt{x-y}$.

Simplify the following surds :—

(5) $\frac{11}{7}\sqrt{72}-\frac{17}{9}\sqrt{50}$. Ans. $\frac{1}{63}\sqrt{2}$.

(6) $\sqrt{48ab^2}+b\sqrt{75a}+\sqrt{3a(a-9b)^2}$. Ans. $a\sqrt{3a}$.

(7) $6\sqrt[6]{4a^2}+2\sqrt[3]{2a}+\sqrt[9]{8a^3}$. Ans. $9\sqrt[3]{2a}$.

(8) $2\sqrt{8}-7\sqrt{18}+5\sqrt{72}-\sqrt{50}$. Ans. $8\sqrt{2}$.

(9) $2\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}+\sqrt{\frac{3}{5}}$. Ans. $\frac{28}{15}\sqrt{15}$.

(10) $\frac{8\sqrt{3}}{4}-\frac{1}{2}\sqrt{12}+4\sqrt{27}-2\sqrt{\frac{3}{16}}$. Ans. $\frac{25}{2}\sqrt{3}$.

(11) $7\sqrt[3]{54}+3\sqrt[3]{16}+\sqrt[3]{2}-5\sqrt[3]{128}$. Ans. $8\sqrt[3]{2}$.

(12) $\sqrt[4]{16}+\sqrt[3]{81}-\sqrt[3]{-512}+\sqrt[3]{192}-7\sqrt[6]{9}$. Ans. 10.

(13) $\sqrt[3]{(54}+\sqrt[3]{250}+\sqrt[3]{128})\times(\sqrt[3]{54}+\sqrt[3]{250}-\sqrt[3]{128})$. Ans. $48\sqrt[3]{4}$.

(14) $\sqrt{18a^5b^3}+\sqrt{50a^3b^3}$. Ans. $(3a^2b+5ab)\sqrt{2ab}$.

(15) $\sqrt{25a^2b}+\sqrt{144a^2b}-\sqrt{289a^2b}$. Ans. 0.

(16) $3b\sqrt[3]{2a^5b^2}-7\sqrt[3]{2a^5b^5}+8a\sqrt[3]{2a^2b^5}$. Ans. $4ab\sqrt[3]{2a^2b^2}$.

(17) $\sqrt[3]{\frac{27a^5x}{2b}}-\sqrt[3]{\frac{a^2x}{2b}}$. Ans. $(3a-1)\sqrt[3]{\frac{a^2x}{2b}}$.

(18) $c\sqrt{ab}.\sqrt{\frac{a}{b}}$, and $\frac{c\sqrt{ab}-ac}{bc-c\sqrt{ab}}$. (1) Ans. ac. (2) Ans. $\sqrt{\frac{a}{b}}$.

(19) $\frac{ab}{b-c} \pm \sqrt{\frac{a^2b^2}{(b-c)^2} - \frac{a^2b}{b-c}}$. (*Comp.* p. 19.) Ans. $\frac{a\sqrt{b}}{\sqrt{b}\mp\sqrt{c}}$.

(20) $\sqrt{\frac{ab^3}{c^2}} + \frac{1}{2c}\sqrt{a^3b-4a^2b^2+4ab^3}$. Ans. $\frac{a}{2c}\sqrt{ab}$.

(21) $5\sqrt{3}\times 7\sqrt{\frac{8}{3}}\times\sqrt{2}$. Ans. 140.

(22) $\sqrt[3]{4}\times 7\sqrt[3]{6}\times\frac{1}{2}\sqrt[3]{5}$. Ans. $7\sqrt[3]{15}$.

(23) $(3+\sqrt{5})\times(2-\sqrt{5})$. Ans. $1-\sqrt{5}$.

(24) $\left(-5-\sqrt{\frac{3}{4}}\right)\times\left(-5+\sqrt{\frac{3}{4}}\right)$. Ans. $24\frac{1}{4}$.

(25) $\left(4\sqrt{\frac{7}{3}}+5\sqrt{\frac{1}{2}}\right)\times\left(\sqrt{\frac{7}{3}}+2\sqrt{\frac{1}{2}}\right)$. Ans. $14\frac{1}{3}+13\sqrt{\frac{7}{6}}$.

(26) $5\sqrt{2}\times3\sqrt{4+6\sqrt{2}}$. Ans. $30\sqrt{2+3\sqrt{2}}$.

(27) $(\sqrt[3]{4}+\sqrt[3]{9}+\sqrt[3]{48})\times(\sqrt[3]{2}+\sqrt[3]{3})$. Ans. $5+3\sqrt[3]{18}+3\sqrt[3]{12}$.

(28) $(\sqrt[3]{12+\sqrt{19}})\times(\sqrt[3]{12-\sqrt{19}})$. Ans. 5.

(29) $\sqrt[3]{40}-\frac{1}{2}\sqrt[3]{320}+\sqrt[3]{135}$. Ans. $3\sqrt[3]{5}$.

(30) $2\sqrt[4]{24\sqrt[3]{18}}\div\sqrt{2\sqrt{12}}$. Ans. $2\sqrt[6]{\frac{3}{2}}$.

(31) $\frac{\sqrt{3}+1}{2-\sqrt{3}}$, $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$. (1) Ans. $5+3\sqrt{3}$, (2) Ans. $4+\sqrt{2}$.

(32) $\frac{\frac{1}{2}\cdot\sqrt{\frac{1}{2}}}{\sqrt{2}+3\sqrt{\frac{1}{2}}}$, $(\sqrt{2}+\sqrt{3})^3$. (1) Ans. $\frac{1}{10}$, (2) Ans. $11\sqrt{2}+9\sqrt{3}$.

(33) $\frac{\frac{1}{4}(\sqrt{5}\mp1)}{\frac{1}{2\sqrt{2}}\sqrt{5\pm\sqrt{5}}}$. (*Comp.* p. 20.) Ans. $\sqrt{1\mp\frac{2}{5}\sqrt{5}}$.

(34) $\frac{\frac{1}{2\sqrt{2}}\sqrt{5\mp\sqrt{5}}}{\frac{1}{4}(\sqrt{5}\pm1)}$. Ans. $\sqrt{5\mp2\sqrt{5}}$.

(35) $\sqrt[3]{\sqrt[4]{8a^3}}$, $\sqrt[2]{\sqrt[5]{9a^2}}$, $(\sqrt[5]{a^4b})^2$. Ans. $\sqrt[4]{2a}$, $\sqrt[5]{3a}$, $a\sqrt[5]{a^3b^2}$.

(36) $\sqrt[5]{\sqrt[3]{a^2}}.\sqrt[6]{\sqrt[4]{a^9}}$, $\dfrac{b}{\sqrt{a}}\sqrt[3]{ac}.\dfrac{\sqrt[4]{c^3}}{\sqrt{b}}$. (1) Ans. $\sqrt{a}.\sqrt[120]{a}$, (2) Ans. $c\sqrt[12]{\dfrac{b^6c}{a^2}}$.

(37) $\sqrt[m]{a^{2m-n}b^{5m+1}c^{3p}}\times\sqrt[m]{a^nb^{m-1}c^{m-3p}}$. Ans. a^2b^6c.

(38) $\sqrt[2n]{x}.\sqrt[n]{x^2}.\sqrt[3n]{x^3}$. Ans. $\sqrt[2n]{x^7}$.

(39) $2\sqrt[n]{5x}.6\sqrt[n]{2xy}.\sqrt[n]{x^{n-2}y^{n-1}}$. Ans. $12\sqrt[n]{10}.xy$.

(40) $\sqrt[4]{\dfrac{a}{b}}\times\sqrt{\dfrac{b}{a}}$, $\sqrt[6]{\dfrac{1}{a^3b^2}}\times\sqrt[5]{a^2}\times\sqrt[3]{b}$. (1) Ans. $\sqrt[4]{\dfrac{b}{a}}$, (2) Ans. $\dfrac{1}{\sqrt[10]{a}}$.

(41) $\dfrac{\sqrt{1-x}+\dfrac{1}{\sqrt{1+x}}}{1+\dfrac{1}{\sqrt{1-x^2}}}$. Ans. $\sqrt{1-x}$.

(42) $a\sqrt{\dfrac{a^3b}{3a^2+6ac+3c^2}}-\dfrac{bc\sqrt{ab}}{a+c}$. Ans. $\left(\dfrac{a^2\sqrt{3}}{3}-bc\right)\dfrac{\sqrt{ab}}{a+c}$.

(43) $\dfrac{2x^2}{(1-x^2)^{\frac{3}{2}}}-\dfrac{1}{(1-x^2)^{\frac{1}{2}}}$. Ans. $\dfrac{3x^2-1}{(1-x^2)^{\frac{3}{2}}}$.

(44) $\dfrac{1}{\sqrt{2}}.\dfrac{2b^2-2c^2-bc\sqrt{2}}{2b^2-c^2}$. Ans. $\dfrac{c\sqrt{2}-b}{c-b\sqrt{2}}$.

(45) $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}$. Ans. $2x$.

(46) $\dfrac{\sqrt{18a^5b^3}+\sqrt{50a^3b^5}}{\sqrt{8ab^3}+\sqrt{2a^3b-8a^2b^2+8ab^3}}$. Ans. $3ab+5b^2$.

(47) $\dfrac{\sqrt[3]{x^4}+\sqrt[3]{x^2y^2}-\sqrt[3]{x^3y}}{\sqrt[3]{x^4}+\sqrt[3]{xy^3}-\sqrt[3]{x^3y}-\sqrt[3]{y^4}}$. (*Comp.* p. 20.) Ans. $\dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}-\sqrt[3]{y^2}}$.

(48) $\dfrac{x}{x-7}(\sqrt[18]{\{3a^2x^3-63a^2x^2+441a^2x-1029a^2\}})^6$. Ans. $x\sqrt[3]{3a^2}$.

(49) $\sqrt{a^2+\sqrt[3]{a^4b^2}}+\sqrt{b^2+\sqrt[3]{a^2b^4}}$. (*Comp.* p. 20.) Ans. $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$.

(50) $\sqrt[4]{\left(\dfrac{a\sqrt{b}}{\sqrt[3]{ab}}\right)^3}$, $\left.\sqrt{\left\{\dfrac{a^{-2m}}{b^{-2n}}\right\}^{\frac{p}{m}}}\,\right|^{\frac{q}{n}}$. (1) Ans. $\sqrt[8]{a^4b}$, (2) Ans. $\left\{\dfrac{b^{\frac{1}{m}}}{a^{\frac{1}{n}}}\right\}^{pq}$.

(51) $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$. Ans. $\dfrac{a}{x}+\sqrt{\dfrac{a^2}{x^2}-1}$.

(52) $\sqrt{\frac{ay}{x}}.\sqrt[3]{\frac{bx}{y^2}}.\sqrt[6]{\frac{y^2}{b^2a^3}}.$ Ans. $\sqrt[6]{\frac{y}{x}}.$

(53) Multiply $\sqrt[4]{29+4\sqrt{30}}$ by $\sqrt[4]{29-4\sqrt{30}}.$ Ans. $\sqrt{19}.$

(54) Multiply $x+\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}$ by $x+\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}.$ Ans. $x^2+px-q.$

(55) Multiply $\frac{x}{b}\sqrt{\frac{a}{b}}+\sqrt{\frac{c}{d}}$ by $\frac{x}{b}\sqrt{\frac{a}{b}}-\sqrt{\frac{c}{d}}.$ Ans. $\frac{ax^2}{b^3}-\frac{c}{d}.$

(56) Multiply $\sqrt{x^3}-1+\frac{1}{\sqrt{x^3}}$ by $\sqrt{x^3}+1+\frac{1}{\sqrt{x^3}}.$ Ans. $x^3+\frac{1}{x^3}+1.$

(57) Multiply $\sqrt[3]{a^{-\frac{1}{2}}}+\sqrt{\sqrt[3]{a^{\frac{1}{2}}b}}$ by $\sqrt[3]{a^{-\frac{1}{2}}}-\sqrt[3]{\sqrt{a^{\frac{1}{2}}b}}.$ Ans. $\sqrt[3]{a^{-1}}-\sqrt[3]{a^{\frac{1}{2}}b}.$

(58) Divide $2x^3+2x^2-3x-3$ by $\sqrt{2}.x-\sqrt{3}.$

Ans. $\sqrt{2}.x^2+(\sqrt{2}+\sqrt{3})x+\sqrt{3}.$

(59) Divide $4a+8\sqrt{ax}-9b+12\sqrt{bx}$ by $2\sqrt{a}-3\sqrt{b}+4\sqrt{x}.$

Ans. $2\sqrt{a}+3\sqrt{b}.$

(60) Divide $\sqrt[4]{a^3}-\sqrt[4]{b^3}$ by $\sqrt[4]{a}-\sqrt[4]{b}.$ Ans. $\sqrt{a}+\sqrt[4]{ab}+\sqrt{b}.$

(61) Divide $\sqrt[5]{a^3}-\sqrt[5]{b^3}$ by $\sqrt[5]{a}-\sqrt[5]{b}.$ Ans. $\sqrt[5]{a^2}+\sqrt[5]{ab}+\sqrt[5]{b^2}.$

(62) Divide $a^3-2\sqrt[4]{a^2b^3}-a^2\sqrt[6]{a^3b^2}+2b\sqrt[12]{b}$ by $\sqrt{a}-\sqrt[3]{b}.$

Ans. $a^2\sqrt{a}-2\sqrt[4]{b^3}.$

(63) Divide $\sqrt[12]{a^{10}b^9}-c\sqrt[10]{a^7}.\sqrt[6]{b^5}-\frac{3}{2}a\sqrt[4]{b^3}+\frac{3abc}{2}\sqrt[30]{\frac{1}{a^4b^5}}$ by

$\sqrt{ab}-\frac{3}{2}\sqrt[6]{a^4b^3}.$ Ans. $\sqrt[3]{a}.\sqrt[4]{b}-c\sqrt[5]{a}.\sqrt[3]{b}.$

(64) Divide $x-a$ by $x^{\frac{1}{n}}-a^{\frac{1}{n}}.$

Ans. $x^{1-\frac{1}{n}}+a^{\frac{1}{n}}x^{1-\frac{2}{n}}+a^{\frac{2}{n}}x^{1-\frac{3}{n}}+\ldots\ldots+a^{1-\frac{2}{n}}x^{\frac{1}{n}}+a^{1-\frac{1}{n}}.$

(65) Square $\frac{c}{2}+\frac{1}{2}\sqrt{a^2-c^2}.$ Ans. $\frac{a^2}{4}+\frac{c}{2}\sqrt{a^2-c^2}.$

(66) Square the expression $\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$

Ans. $a+\sqrt{b}.$

(67) Find the 5th root of $\dfrac{3^{10}a^{-5}(a+b)^5.(2+x)^{-10}}{32c^5d^{-15}}$. Ans. $\dfrac{9d^3(a+b)}{2ac(2+x)^2}$.

(68) Find the 9th root of $2^{36}a^{45}b^9.\dfrac{(a+b)^{27}}{a^9}$. Ans. $16a^4b(a+b)^3$.

(69) Find the square root of $-2+a^{2\sqrt{2}}+a^{-2\sqrt{2}}$. Ans. $a^{\sqrt{2}}-a^{-\sqrt{2}}$.

(70) Find the square root of $\dfrac{a^2c}{b}-cf+2ac\sqrt{-\dfrac{f}{b}}$.

Ans. $a\sqrt{\dfrac{c}{b}}+\sqrt{-cf}$.

(71) Find the square root of

$x^m+\dfrac{1}{4}\sqrt[\frac{np}{2}]{b^px^{2n}}-\sqrt{b}.\sqrt[2p]{x^{mp+4}}$. Ans. $x^{\frac{m}{2}}-\dfrac{1}{2}\,b^{\frac{1}{n}}x^{\frac{2}{p}}$.

(72) Find the square root of $\sqrt[x]{a^{-\frac{2}{y}}}+\sqrt[r]{a^{\frac{1}{y}}b}+2\sqrt{\sqrt[r]{b}}.\sqrt{\sqrt[x]{\sqrt[y]{\sqrt[r]{a^{x-2r}}}}}$.

(*Comp.* p. 20.) Ans. $\sqrt[x]{a^{-\frac{1}{y}}}+\sqrt{\sqrt[r]{a^{\frac{1}{y}}b}}$.

Find the square root of each of the following quantities :

(73) $18+2\sqrt{77}$. Ans. $\sqrt{11}+\sqrt{7}$.

(74) $94-42\sqrt{5}$. Ans. $7-3\sqrt{5}$.

(75) $28+10\sqrt{3}$. Ans. $5+\sqrt{3}$.

(76) $13+2\sqrt{30}$. Ans. $\sqrt{10}+\sqrt{3}$.

(77) $10\frac{1}{2}+2\sqrt{5}$. Ans. $\sqrt{10}+\dfrac{1}{2}\sqrt{2}$.

(78) $a+x+\sqrt{2ax+x^2}$. Ans. $\sqrt{a+\dfrac{x}{2}}+\sqrt{\dfrac{x}{2}}$.

(79) Find the square root of $\dfrac{a^2}{4}+\dfrac{c}{2}\sqrt{a^2-c^2}$. Ans. $\dfrac{c}{2}+\dfrac{1}{2}\sqrt{a^2-c^2}$.

(80) ... ... $x+y+z+2\sqrt{xz+yz}$. Ans. $\sqrt{x+y}+\sqrt{z}$.

(81) $M\times(\sqrt[3]{7}-\sqrt[3]{5})$ is a rational quantity; find M.

Ans. $\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}$.

(82) $M\times(5^{\frac{1}{2}}+2^{\frac{2}{3}})$ is a rational quantity; find M.

Ans. $5^{\frac{5}{2}}-25\times2^{\frac{2}{3}}+5^{\frac{3}{2}}\times2^{\frac{4}{3}}-20+5^{\frac{1}{2}}\times2^{\frac{8}{3}}-2^{\frac{10}{3}}$.

(83) $M\times(\sqrt{2}+\sqrt[3]{3})$ is rational; find M.

Ans. $2^{\frac{5}{2}}-4\times 3^{\frac{1}{3}}+2^{\frac{3}{2}}\times 3^{\frac{2}{3}}-6+2^{\frac{1}{2}}\times 3^{\frac{4}{3}}-3^{\frac{5}{3}}$.

(84) $M\times(\sqrt[3]{9}+\sqrt{5})$ is rational; find M.

Ans. $3^{\frac{10}{3}}-3^{\frac{8}{3}}\times 5^{\frac{1}{2}}+45-3^{\frac{4}{3}}\times 5^{\frac{3}{2}}+3^{\frac{2}{3}}\times 25-5^{\frac{5}{2}}$.

(85) Reduce $\dfrac{\sqrt[3]{5\cdot 12}+\sqrt[3]{0\cdot 03375}}{\sqrt[3]{80}-\sqrt[3]{0\cdot 01}}$ to its equivalent simple decimal.

(*Comp.* p. 21.) Ans. 0·5.

(86) Reduce the following fractions to equivalent fractions with rational denominators :—

(1) $\dfrac{1+\sqrt{3}}{2\sqrt{2}-3\sqrt{3}}$, (2) $\dfrac{1+\sqrt{2}}{1+\sqrt{2}+\sqrt{3}}$, (3) $\dfrac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}}$,

(4) $\dfrac{\sqrt{10}}{\sqrt{2}-\sqrt[3]{3}}$, (5) $\dfrac{a-b}{\sqrt{a}-\sqrt[4]{b}}$, (6) $\dfrac{n}{a+\sqrt[5]{b}}$.

(1) Ans. $-\dfrac{1}{19}\{3\sqrt{3}+2\sqrt{2}+2\sqrt{6}+9\}$.

(2) Ans. $\dfrac{1}{4}\{4+3\sqrt{2}-2\sqrt{3}-\sqrt{6}\}$. (3) Ans. $\sqrt[3]{40}+\sqrt[3]{20}+\sqrt[3]{10}$.

(4) Ans. $-\sqrt{10}.\{4\sqrt{2}+4\sqrt[3]{3}+2\sqrt{2}.\sqrt[3]{9}+6+3\sqrt{2}.\sqrt[3]{3}+3\sqrt[3]{9}\}$.

(5) Ans. $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}$.

(6) Ans. $\dfrac{n(a^4-a^3b^{\frac{1}{5}}+a^2b^{\frac{2}{5}}-ab^{\frac{3}{5}}+b^{\frac{4}{5}})}{a^5+b}$.

(87) Prove that $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$ is equal to $\sqrt{2}$.

(*Comp.* p. 21.)

(88) Simplify $\dfrac{1}{\sqrt[3]{x^2}+\sqrt[3]{ax}+\sqrt[3]{a^2}}$. Ans. $\dfrac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a}$.

(89) Simplify $\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$. Ans. $\dfrac{1}{12}\sqrt{30}+\dfrac{1}{4}\sqrt{2}+\dfrac{1}{6}\sqrt{3}$.

(90) Find the value of $\dfrac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}$. Ans. 3·1003.

(91) Find the value of $\dfrac{3+2\sqrt{7}}{5\sqrt{12}-6\sqrt{6}}$. Ans. 3·159.

(92) Simplify $\dfrac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}$. (*Comp.* p. 21.) Ans. $\sqrt{5}+1-\sqrt{5+2\sqrt{5}}$.

(93) Simplify $\sqrt{\dfrac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}$. (*Comp.* p. 22.) Ans. $\sqrt{2(2+\sqrt{2})}-\sqrt{2}-1$.

(94) Simplify $\dfrac{2\sqrt{2+\sqrt{3}}}{4+\sqrt{6}-\sqrt{2}}$. Ans. $\sqrt{6}-\sqrt{2}+\sqrt{3}-2$.

(95) Which is greater $\sqrt{5}+\sqrt{14}$ or $\sqrt{3}+3\sqrt{2}$? Ans. The former.

(96) Simplify $\dfrac{a\sqrt{a}+3a\sqrt{b}+3b\sqrt{a}+b\sqrt{b}}{\sqrt{a}+\sqrt{b}}$. Ans. $(\sqrt{a}+\sqrt{b})^2$.

(97) Simplify $\dfrac{\sqrt{1+a^2}-\sqrt{1-a^2}}{a^2+2-2\sqrt{1-a^4}}$. (*Comp.* p. 22.) Ans. $\dfrac{3\sqrt{1+a^2}+\sqrt{1-a^2}}{5a^2+4}$.

(98) Simplify the expressions

$$(b+\sqrt{2a-b^2})^4+(b-\sqrt{2a-b^2})^4; \quad (b+\sqrt{2a-b^2})^4-(b-\sqrt{2a-b^2})^4.$$

(1) Ans. $8(a^2+2ab^2+b^4)$; (2) Ans. $16ab\sqrt{2a-b^2}$.

(99) Simplify $(a+b\sqrt{-1})^4+(a-b\sqrt{-1})^4$. Ans. $2a^4-12a^2b^2+2b^4$.

(100) Prove that $5^{\frac{3}{4}}.\left\{\dfrac{\sqrt{5}\pm 1}{2}\right\}^{\frac{1}{2}}$ is equivalent to $\dfrac{5}{2}\sqrt{2\pm\dfrac{2}{5}\sqrt{5}}$; and

$5^{\frac{3}{4}}.\left\{\dfrac{\sqrt{5}\pm 1}{2}\right\}^{\frac{5}{2}}$ to $\dfrac{5}{2}\sqrt{10\pm\dfrac{22}{5}\sqrt{5}}$. (*Comp.* p. 23.)

(101) Find the value of $\left(\dfrac{x}{x-1}\right)^2+\left(\dfrac{x}{x+1}\right)^2$, when $x=\sqrt{\dfrac{n-1}{n+1}}$.

(*Comp.* p. 23.) Ans. $n(n-1)$.

(102) Find the value of $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$, when

$x=\dfrac{12ab}{a+b+\sqrt{\{(a+b)^2+12ab\}}}$. (*Comp.* p. 23.) Ans. 1.

(103) Find the value of $\dfrac{x^{\frac{1}{2}}+x}{1+x^{\frac{3}{2}}}$, when $x=\dfrac{a^2+a\sqrt{a^2+4}}{2}$. (*Comp.* p. 24.)

Ans. $\dfrac{2a}{\sqrt{(a^2+1+a\sqrt{a^2+4})^2-1}-2a}$.

(104) Simplify $\frac{n^3-3n+(n^2-1)\sqrt{n^2-4}-2}{n^3-3n+(n^2-1)\sqrt{n^2-4}+2}$. (*Comp.* p. 25.)

Ans. $\frac{n+1}{n-1}\sqrt{\frac{n-2}{n+2}}$.

(105) Simplify $\left\{\frac{2x^2+1+x\sqrt{4x^2+3}}{2x^2+3+x\sqrt{4x^2+3}}\right\}^{\frac{1}{2}}$. (*Comp.* p. 25.)

Ans. $\frac{x+\sqrt{4x^2+3}}{3\sqrt{x^2+1}}$.

(106) Simplify $\frac{xy-\sqrt{\frac{1-x^2}{1+x^2}\cdot\frac{1-y^2}{1+y^2}}}{1+xy\sqrt{\frac{1-x^2}{1+x^2}\cdot\frac{1-y^2}{1+y^2}}}$. (*Comp.* p. 25.)

Ans. $\frac{2xy-\sqrt{(1-x^4)(1-y^4)}}{1+x^2+y^2-x^2y^2}$.

(107) Simplify $\frac{1-ab+\sqrt{1+a^2}-a\sqrt{1+b^2}}{1-ab+\sqrt{1+b^2}-b\sqrt{1+a^2}}$. (*Comp.* p. 26.)

Ans. $\frac{1-a+\sqrt{1+a^2}}{1-b+\sqrt{1+b^2}}$.

(108) Find the value of $\sqrt{\frac{n^2-1}{n^2+1}}$, when $n^2=\frac{1+\sqrt{3}+\sqrt{2\sqrt{3}}}{2}$.

(*Comp.* p. 26.) Ans. $\sqrt{\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}}}$.

(109) Find the value of $\frac{1-ax}{1+ax}\cdot\sqrt{\frac{1+bx}{1-bx}}$, when $x=\frac{1}{a}\sqrt{\frac{2a}{b}-1}$.

(*Comp.* p. 27.) Ans. 1.

(110) If $x=\sqrt[3]{-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}$, find the value of $x^6+rx^3+\frac{q^3}{27}$.

Ans. 0.

(111) Find the value of $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$, when $x=\frac{2ab}{b^2+1}$. (*Comp.* p. 27.)

Ans. b.

(112) Find the value of $2(uv-\sqrt{1-u^2}.\sqrt{1-v^2})$, when $2u=x+x^{-1}$, and $2v=y+y^{-1}$. (*Comp.* p. 28.) Ans. $xy+(xy)^{-1}$.

(113) Find the value of $\frac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$, when $x=\frac{1}{2}\left\{\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right\}$.

(*Comp.* p. 28.) Ans. $a+b$.

(114) If $x = \frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}-1)$, and $y = \frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}+1)$, find the value of x^5+y^5.

Ans. 2.

(115) If $x = \left(\frac{a+b}{a-b}\right)^{\frac{2pq}{q-p}}$, find the value of $\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}(\sqrt[p]{x}+\sqrt[q]{x})$.

(*Comp.* p. 29.) Ans. $\left(\frac{a+b}{a-b}\right)^{\frac{q+p}{q-p}}$.

SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

(1) $20-4x+8 = 5x+10$, find the value of x. Ans. $x = 2$.

(2) $2x-24+3x-30-6x+66 = 0$, $x = 12$.

(3) $7x+20-3x = 60+4x-50+8x$, $x = 1\frac{1}{4}$.

(4) $x-3\frac{1}{2} = 3\frac{1}{2}+15x$, $x = -\frac{1}{2}$.

(5) $13\frac{3}{4}-\frac{x}{2} = 2x-8\frac{3}{4}$, $x = 9$.

(6) $\frac{3x}{4}+\frac{7x}{15}+\frac{11x}{6} = 366$, $x = 120$.

(7) $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}+\frac{x}{5} = 7\frac{5}{6}$, $x = 10$.

(8) $\frac{9}{2x}-4 = \frac{2}{3}$, $x = \frac{27}{28}$.

(9) $\frac{1}{x}+\frac{1}{2x}-\frac{1}{3x} = \frac{7}{3}$, $x = \frac{1}{2}$.

(10) $\frac{3x+1}{2}-\frac{2x}{3} = 10+\frac{x-1}{6}$, $x = 14$.

(11) $x+\frac{x-2}{3} = \frac{x-1}{2}+4$, $x = 5$.

(12) $\frac{2x}{7}+\frac{x-1}{6} = x-4$, $x = 7$.

(13) $\frac{3x-2}{4}+\frac{7x-2}{3} = 2x+\frac{4-x}{2}$, $x = 2$.

(14) $\frac{x}{2}+\frac{1}{1\frac{1}{2}} = \frac{x-2}{3}+\frac{x+3}{4}$, $x = 7$.

(15) $\frac{x-7\frac{1}{2}}{2} = \frac{3x-9}{4}+\frac{27-5x}{3}$, $x = 7\frac{7}{17}$.

(16) $x+\frac{10+x}{5}-5\frac{3}{4}=\frac{x-2}{3}+\frac{x}{4}$, $x=5$.

(17) $x-\frac{3x-2}{5}=3-\frac{2x-5}{3}$, $x=4$.

(18) $\frac{12x-2}{6}-\frac{18-4x}{3}=x+2$, $x=3\frac{4}{7}$.

(19) $5x-\frac{2x-1}{3}+1=3x+\frac{x+2}{2}+7$, $x=8$.

(20) $\frac{x+6}{4}-\frac{16-3x}{12}=\frac{25}{6}$, $x=8$.

(21) $\frac{7x+5}{3}-\frac{16+4x}{5}+6=\frac{3x+9}{2}$, $x=1$.

(22) $\frac{x+7}{3}-5\frac{3}{4}=\frac{2x+5}{7}+\frac{10-5x}{8}$, $x=8$.

(23) $\frac{2x+7}{7}-\frac{9x-8}{11}=\frac{x-11}{2}$, $x=7$.

(24) $\frac{3x-1}{7}+\frac{6-x}{4}-\frac{2x-4}{12}=2-\frac{x+2}{28}$, $x=5$.

(25) $\frac{7x+9}{8}-\frac{3x+1}{7}=\frac{9x-13}{4}-\frac{249-9x}{14}$, $x=9$.

(26) $\frac{3x+3}{16}+\frac{7x-4}{15}-\frac{7x+1}{20}=2$, $x=7$.

(27) $\frac{x}{4}-4\frac{1}{2}-\frac{12-x}{5\frac{1}{2}}+\frac{x}{2}=\frac{8\frac{1}{2}}{11}$, $x=8$.

(28) $\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}=23\frac{1}{3}$, $x=19$.

(29) $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$, $x=72$.

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(30) $52-5(2x-1)=27$, $x=3$.

(31) $4(x-3)-7(x-4)=6-x$, $x=5$.

(32) $20x-50(3x-4)=200-20(4x-5)-(3x-4)50$, $x=3$.

(33) $4(5x-3)-64(3-x)-3(12x-4)=96$, $x=6$.

(34) $10\left(x+\frac{1}{2}\right)-6x\left(\frac{1}{x}-\frac{1}{3}\right)=23$, $x=2$.

(35) $\frac{4}{5}x-\frac{5}{4}x+18=\frac{1}{9}(4x+1)$, $x=20$.

(36) $\frac{1}{3}(2x-10)-\frac{1}{11}(3x-40)=15-\frac{1}{5}(57-x)$, $x=17$.

(37) $\frac{1}{12}x-\frac{1}{8}(8-x)-\frac{1}{4}(5+x)+\frac{11}{4}=0$, $x=12$.

(38) $\frac{1}{7}\left(x-\frac{1}{2}\right)-\frac{1}{5}\left(\frac{2}{3}-x\right)=1\frac{13}{30}$, $x=4\frac{7}{9}$.

(39) $\frac{1}{4}\left(4+\frac{3}{2}x\right)-\frac{1}{7}\left(2x-\frac{1}{3}\right)=\frac{31}{28}$, $x=\frac{2}{3}$.

(40) $\frac{1}{14}\left(3x+\frac{2}{3}\right)-\frac{1}{7}\left(4x-6\frac{2}{3}\right)=\frac{1}{2}(5x-6)$, $x=1\frac{2}{5}$.

(41) $3x-4-\frac{4}{5}\cdot\frac{7x-9}{3}=\frac{4}{5}\left(6+\frac{x-1}{3}\right)$, $x=7\frac{1}{13}$.

(42) $\frac{1}{2}\left(\frac{2}{3}x+4\right)-\frac{7\frac{1}{2}-x}{3}=\frac{x}{2}\left(\frac{6}{x}-1\right)$, $x=3$.

(43) $3\frac{1}{3}\times\left\{28-\left(\frac{x}{8}+24\right)\right\}=3\frac{1}{2}\times\left\{2\frac{1}{3}+\frac{x}{4}\right\}$, $x=4$.

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(44) $\frac{10x+17}{18}-\frac{12x+2}{11x-8}=\frac{5x-4}{9}$, $x=4$.

(45) $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}$, $x=20$.

(46) $\frac{4x}{5-x}-\frac{20-4x}{x}=\frac{15}{x}$, $x=3\frac{2}{11}$.

(47) $\frac{5x+3}{x-1}+\frac{2x-3}{2x-2}=9$, $x=3\frac{1}{2}$.

(48) $\frac{30+6x}{x+1}+\frac{60+8x}{x+3}=14+\frac{48}{x+1}$, $x=3$.

(49) $\frac{1}{x-1}-\frac{2}{x+7}=\frac{1}{7(x-1)}$, $x=7$.

(50) $\frac{x}{x+1}-\frac{3x}{x+2}=-2$, $x=-\frac{4}{5}$.

(51) $\frac{6x+8}{2x+1}-\frac{2x+38}{x+12}=1$, $x=2$.

(52) $\frac{5x^2+x-3}{5x-4}=\frac{7x^2-3x-9}{7x-10}$, $x=3.$

(53) $\frac{4x-17}{9}-\frac{3\frac{2}{3}-22x}{33}=x-\frac{6}{x}\left(1-\frac{x^2}{54}\right)$, (*App.* p. 314.) $x=3.$

(54) $\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}=\frac{8x-30}{2x-7}+\frac{5x-4}{x-1}$, (*App.* p. 314.) $x=2\frac{1}{2}.$

(55) $\frac{x}{x-2}+\frac{x-9}{x-7}=\frac{x+1}{x-1}+\frac{x-8}{x-6}$, $x=4.$

(56) $\frac{1}{3}-\frac{7x-1}{6\frac{1}{2}-3x}=\frac{8}{3}\cdot\frac{x-\frac{1}{2}}{x-2}$, $x=1\frac{1}{13}.$

(57) $\frac{x}{2}-\frac{\frac{1}{3}(2x-3)-\frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)}=\frac{3}{2}\cdot\frac{x^2+2}{3x-2}$, $x=4\frac{1}{3}.$

(58) $\frac{3x}{2}-\frac{81x^2-9}{(3x-1)(x+3)}=3x-\frac{3}{2}\cdot\frac{2x^2-1}{x+3}-\frac{57-3x}{2}$, (*App.* p. 315.)
$x=10.$

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(59) $\frac{x}{a}+\frac{x}{b}=c$, $x=\frac{abc}{a+b}.$

(60) $\frac{ax}{b}+\frac{cx}{f}+g=qx+\frac{cx}{f}+h$, $x=b\cdot\frac{h-g}{a-bq}.$

(61) $\frac{a}{bx}+\frac{b}{ax}=a^2+b^2$, $x=\frac{1}{ab}.$

(62) $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax}$, $x=\frac{b}{a}(a-b+c).$

(63) $a+x+\sqrt{2ax+x^2}=b$, $x=\frac{(a-b)^2}{2b}.$

(64) $a+x+\sqrt{a^2+x^2}=b$, $x=\frac{b}{2}\left\{1-\frac{a}{b-a}\right\}.$

(65) $\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}$, $x=\frac{9a}{16}.$

(66) $a+x+\sqrt{a^2+bx+x^2}=b$, $x=b\cdot\frac{b-2a}{3b-2a}.$

(67) $\sqrt{a-x}+2\sqrt{a+x}=\sqrt{a-x+\sqrt{ax+x^2}}$, $x=\frac{64a}{1025}.$

(68) $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x},$ $\qquad x=\dfrac{(a-b)^2}{2a-b}.$

(69) $\sqrt{1+x+x^2}=a-\sqrt{1-x+x^2},$ $\qquad x=\dfrac{a}{2}\sqrt{1-\dfrac{3}{a^2-1}}.$

(70) $\sqrt{a+x}+\sqrt{a-x}=b\sqrt{a^2-x^2},$ $\qquad x=\dfrac{ab}{b^2-2}\sqrt{b^2-4}.$

(71) $\dfrac{a}{b+x}+\dfrac{a}{b-x}=c,$ $\qquad x=b\sqrt{1-\dfrac{2a}{bc}}.$

(72) $\dfrac{1}{a}\sqrt{a+x}+\dfrac{1}{x}\sqrt{a+x}=\dfrac{1}{b}\sqrt{x},$ (*Comp.* p. 29.) $\qquad x=\dfrac{ab^{\frac{2}{3}}}{a^{\frac{4}{3}}-b^{\frac{2}{3}}}.$

(73) $\sqrt{a^2+x^2}+\sqrt{a^2-x^2}=b,$ $\qquad x=b\sqrt[4]{\left(\dfrac{a}{b}\right)^2-\left(\dfrac{1}{2}\right)^2}.$

(74) $\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1,$ $\qquad x=\dfrac{16}{25}.$

(75) $1+\sqrt{1+x}-\sqrt{1+x+\sqrt{1-x}}=0,$ $\qquad x=-\dfrac{24}{25}.$

(76) $x+\sqrt{a^2+x^2}=\dfrac{na^2}{\sqrt{a^2+x^2}},$ $\qquad x=a.\dfrac{n-1}{\sqrt{2n-1}}.$

(77) $\sqrt{4a+x}+\sqrt{a+x}=2\sqrt{x-2a},$ $\qquad x=\dfrac{17a}{8}.$

(78) $\sqrt{1+x}+\sqrt{1+x+\sqrt{1-x}}=\sqrt{1-x},$ $\qquad x=-\dfrac{24}{25}.$

(79) $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=a\sqrt{\dfrac{x}{x+\sqrt{x}}},$ $\qquad x=\dfrac{1}{4}\left\{1-a+\dfrac{1}{1-a}\right\}^2.$

(80) $ax+\sqrt{a^2x^2+b^2}=\sqrt{b^2+ax\sqrt{4b^2+x^2}},$ $\qquad x=\dfrac{8ab}{\sqrt{1-16a^2}}.$

(81) $\dfrac{a^2-x^2}{a+\sqrt{a^2+x^2}}=b-\sqrt{a^2+x^2},$ $\qquad x=\dfrac{a^2}{a-b}\sqrt{3-\dfrac{2b}{a}}.$

(82) $\dfrac{ax-b^2}{\sqrt{ax}+b}=\dfrac{\sqrt{ax}-b}{n}-c,$ (*Comp.* p. 30.) $\qquad x=\dfrac{1}{a}\left\{b-\dfrac{nc}{n-1}\right\}^2.$

(83) $\sqrt{1+x+x^2}+\sqrt{1-x+x^2}=mx,$ $\qquad x=\dfrac{2}{m}\sqrt{\dfrac{m^2-1}{m^2-4}}.$

(84) $\sqrt{a^2-x^2}+x\sqrt{a^2-1}=a^2\sqrt{1-x^2},$ $\qquad x=\sqrt{\dfrac{a^2-1}{a^2+3}}.$

(85) $\sqrt[m]{a+x} = \sqrt[2m]{x^2+8ax+b^2}$, (*Comp.* p. 30.) $x = \frac{a^2-b^2}{6a}$.

(86) $\sqrt{x^2+2ax}+\sqrt{x^2-2ax} = \frac{nax}{\sqrt{x^2+2ax}}$, $x = a\left\{\frac{n-2}{2}+\frac{2}{n-2}\right\}$.

(87) $\frac{ax-1}{\sqrt{ax}+1} = 4+\frac{\sqrt{ax}-1}{2}$, (*Comp.* p. 30.) $x = \frac{81}{a}$.

(88) $\sqrt{(2a+x)^2+b^2}+\sqrt{(2a-x)^2+b^2} = 2a$, $x = \sqrt{a^2+\frac{b^2}{3}}$.

(89) $\sqrt{a+\sqrt{a^2-x^2}}+\sqrt{a-\sqrt{a^2-x^2}} = n\sqrt{\frac{a+x}{a+\sqrt{a^2-x^2}}}$, $x = n\sqrt{a-\frac{n^2}{4}}$.

(90) $\frac{1-ax}{1+ax}\cdot\sqrt{\frac{1+bx}{1-bx}} = 1$, (*Comp.* p. 30.) $x = \frac{1}{a}\sqrt{\frac{2a}{b}-1}$.

(91) $\frac{1+x+x^2}{1-x+x^2} = \frac{62}{63}\cdot\frac{1+x}{1-x}$, (*Comp.* p. 31.) $x = \frac{1}{5}$.

(92) $\left(\frac{a+x}{a-x}\right)^2 = 1+\frac{cx}{ab}$, (*Comp.* p. 31.) $x = a\left\{1\pm 2\sqrt{\frac{b}{c}}\right\}$.

(93) $\frac{1+x^3}{(1+x)^2}+\frac{1-x^3}{(1-x)^2} = a$, $x = \sqrt{\frac{a-2}{a+4}}$.

(94) $\sqrt{(1+x)^2-ax}+\sqrt{(1-x)^2+ax} = x$, $x = 2\sqrt{\frac{(a-1)(a-3)}{3}}$.

(95) $\frac{1}{\sqrt{1-x}+1}+\frac{1}{\sqrt{1+x}-1} = \frac{1}{x}$, $x = \frac{1}{2}\sqrt{3}$.

(96) $a+b = \frac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$, (*Comp.* p. 31.) $x = \frac{1}{2}\left\{\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right\}$.

(97) $\sqrt{a+x}+\sqrt{a-x} = \frac{2x}{\sqrt{a+\sqrt{a^2+x^2}}}$, $x = \frac{2\sqrt{6}}{5}.a$.

(98) $\frac{1+\sqrt{x^2-1}}{1+2a\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}-1}{x^2-2}$, (*Comp.* p. 32.) $x = \sqrt{1+4(a-1)^2}$.

(99) $\sqrt{x^2+2ax+b^2}+\sqrt{x^2-2ax+b^2} = \frac{2ax}{\sqrt{x^2-a^2}}$, $x = \frac{a}{b}\sqrt{a^2+b^2}$.

(100) $\sqrt{\frac{3a}{4}-x}+\sqrt{3ax-x} = \frac{3a}{2}\sqrt{1-4x}$, $x = \frac{1}{12}\cdot\frac{3a-1}{a+1}$.

(101) $\sqrt[3]{1+x}+\sqrt[3]{1-x} = \sqrt[3]{2}$, (See *App.* p. 317.) $x = 1$.

(102) $\dfrac{1+x}{1+x+\sqrt{1+x^2}}+\dfrac{1-x}{1-x+\sqrt{1+x^2}}=a,$ $\qquad x=\sqrt{(2-a)^2-1}.$

(103) $\sqrt{x^2+9}+\sqrt{x^2-9}=4+\sqrt{34},$ $\qquad x=5.$

(104) $\sqrt{\{a^2x^2+b\sqrt{(abx+4a^2x^2+b\sqrt{2abx+5a^2x^2})}\}}=ax+b,$ $\qquad x=-\dfrac{b}{2a}.$

(105) $\dfrac{1+x+\sqrt{2x+x^2}}{1-x+\sqrt{2x+x^2}}=1-ax,$ $\qquad x=\dfrac{(a+2)^2}{4a(a+1)}.$

(106) $\sqrt{1+a}.\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{1-a}.\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{4}}=2\sqrt[4]{1-a^2},$ (*Comp.* p. 32.) $\qquad x=-a.$

(107) $\sqrt{(1+a)^2+(1-a)x}+\sqrt{(1-a)^2+(1+a)x}=2a,$ $\qquad x=8.$

(108) $\dfrac{1+x-\sqrt{2x+x^2}}{1+x+\sqrt{2x+x^2}}=a.\dfrac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}},$ $\qquad x=\dfrac{1}{2}\left\{\sqrt[3]{a}+\dfrac{1}{\sqrt[3]{a}}\right\}-1.$

(109) $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b^2,$ (*App.* p. 315.) $\qquad x=\dfrac{a}{2b}(b-1)^2.$

(110) $\dfrac{a-x+\sqrt{2ax-x^2}}{a-x}=b,$ (*App.* p. 316.) $\qquad x=a-\dfrac{a}{\sqrt{1+(b-1)^2}}.$

(111) $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=b,$ $\qquad x=a.\dfrac{(1-\sqrt{b})^2}{1+b}.$

(112) $\dfrac{\sqrt{a}-\sqrt{a-\sqrt{a^2-ax}}}{\sqrt{a}+\sqrt{a-\sqrt{a^2-ax}}}=b,$ $\qquad x=a\left\{1-\left(\dfrac{2\sqrt{b}}{b+1}\right)^4\right\}.$

(113) $\dfrac{\sqrt{a^2+x^2}-a}{\sqrt{a^2-x^2}+a}=b,$ $\qquad x=\dfrac{2a\sqrt{b}}{1+b^2}\sqrt{1-b^2}.$

(114) $\sqrt{a+x}+\sqrt{a-x}=\sqrt[4]{a^2+x^2}+\sqrt[4]{a^2-x^2},$ $\qquad x=\dfrac{2a}{3}.\sqrt[4]{5}.$

(115) $\dfrac{a+x}{\sqrt{a}+\sqrt{a+x}}+\dfrac{a-x}{\sqrt{a}+\sqrt{a-x}}=\sqrt{a},$ $\qquad x=\dfrac{a}{2}.\sqrt{3}.$

(116) $\dfrac{\sqrt{1+x}-1}{\sqrt{1-x}+1}+\dfrac{\sqrt{1-x}+1}{\sqrt{1+x}-1}=a,$ $\qquad x=\dfrac{4}{a}\sqrt{1-\left(\dfrac{2}{a}\right)^2}.$

(117) $\dfrac{x+\sqrt{(x^2-x\sqrt{1-x})}}{x-\sqrt{(x^2-x\sqrt{1-x})}}=\dfrac{1}{\sqrt{1-x}},$ $\qquad x=\dfrac{16}{25}.$

(118) $(1+x)\sqrt{1+a}+(1-x)\sqrt{1-a}=2\sqrt{1+x^2},$ $\qquad x=\dfrac{1}{a}-\sqrt{\left(\dfrac{1}{a}\right)^2-1}.$

(119) $\sqrt[4]{x^4-1}+x\sqrt{x^4-1}=x^3$, $\qquad x=\sqrt[4]{\frac{1}{2}+\frac{1}{2}\sqrt{2}}.$

(120) $\sqrt{x}+2\sqrt{2a-x}=\sqrt{x+\sqrt{2a^2-3ax+x^2}}$,

$x=2a$, or a, or $\dfrac{961a}{1025}$.

(121) $\frac{2}{19}\sqrt{x^2+39x+374}-\sqrt{x^2+20x+51}=\sqrt{\frac{x+22}{x+17}}$, (*App.* p. 315.)

$x=78$.

(122) $\dfrac{243+324\sqrt{3x}}{16x-3}=(4\sqrt{x}-\sqrt{3})^2$, (*App.* p. 316.) $\qquad x=3$.

(123) $(x+a+\sqrt{x^2+2ax+b^2})^3+(x+a-\sqrt{x^2+2ax+b^2})^3=14(x+a)^3$,
(*App.* p. 318.) $\qquad x=-a$, or $\pm\sqrt{b^2-a^2}-a$.

(124) $b\left(\dfrac{a-b}{x}+1\right)\left(\dfrac{a-2b}{x}+1\right)=\dfrac{a^2}{x}-a$, (*App.* p. 319.)

$x=b$, or $\dfrac{a-b}{a+b}(a-2b)$.

(125) $\left(\dfrac{x+a}{x+b}\right)^{\frac{1}{2}}-\dfrac{a-b}{2(x+c)}=1$, (*App.* p. 319.) $\qquad x=\dfrac{4c(c-b)-b(a-b)}{a+3b-4c}$.

(126) $\dfrac{x+a}{x+b}=\left(\dfrac{2x+a+c}{2x+b+c}\right)^2$, (*App.* p. 320.) $\qquad x=\dfrac{c^2-ab}{a+b-2c}$.

(127) $\dfrac{a+c(a+x)}{a+c(a-x)}+\dfrac{a+x}{x}=\dfrac{a}{a-2cx}$, (*Comp.* p. 32.)

$x=-a$, or $\dfrac{a}{c}\cdot\dfrac{1+c}{3+2c}$.

(128) $\left(\dfrac{x-a}{x-b}\right)^3=\dfrac{x-2a+b}{x-2b+a}$, (*Comp.* p. 33.) $\qquad x=\dfrac{a+b}{2}$.

(129) $x+a+3\sqrt[3]{abx}=b$, (*App.* p. 321.) $\qquad x=(\sqrt[3]{b}-\sqrt[3]{a})^3$.

(130) $(x-a)\sqrt{x}-(x+a)\sqrt{b}=b(\sqrt{x}-\sqrt{b})$, (*App.* p. 322.)

$x=b$, or $(\sqrt{a}+\sqrt{b})^2$.

(131) $a+(b+\sqrt{x})\sqrt{x}=(b-\sqrt{x})\sqrt{2a+x}$, (*App.* p. 323.)

$x=\dfrac{a}{2}\sqrt[3]{\dfrac{a}{2b^2}}+\dfrac{ab}{2}\sqrt[3]{\dfrac{2}{ab}}-a$.

(132) $\dfrac{2a\sqrt{1+x^2}}{1-x+\sqrt{1+x^2}}=a+b$, (*App.* p. 323.)

$x=\dfrac{(a-b)^2}{8ab}\left\{\sqrt{1+\dfrac{8ab}{(a-b)^2}}-1\right\}^2$.

(133) $\dfrac{2x^2+1+x\sqrt{4x^2+3}}{2x^2+3+x\sqrt{4x^2+3}} = a$, (*App.* p. 325.) $\qquad x = \dfrac{3a-1}{\sqrt{(1-a)(9a-1)}}$.

(134) $\dfrac{(a^2-1)a+a^2x-x\sqrt{2a^2-1}}{(a^2-1)a+a^2x+x\sqrt{2a^2-1}} = (1-a^2)(a+x)^2-2ax$, (*App.* p. 326.)

$$x = \sqrt{1-a^2}.\frac{1\pm a\sqrt{1-a^2}}{a^2+\sqrt{2a^2-1}}.$$

(135) $\dfrac{1-ax+\sqrt{1+a^2}-a\sqrt{1+x^2}}{1-ax+\sqrt{1+x^2}-x\sqrt{1+a^2}} = a$, (*App.* p. 326.)

$$x = \frac{1}{a}, \text{ or } \frac{(a-1-\sqrt{1+a^2})^2-3a}{a(3a-4)}.$$

(136) $\left(\dfrac{2x+3}{2x-3}\right)^{\frac{1}{3}}+\left(\dfrac{2x-3}{2x+3}\right)^{\frac{1}{3}} = \dfrac{8}{13}.\dfrac{4x^2+9}{4x^2-9}$, (*App.* p. 329.) $\qquad x = \pm 1\tfrac{13}{14}$.

(137) $\left\{\dfrac{ab}{2}(b+c)^2+(b^2+c^2)cx\right\}.\sqrt{\dfrac{a^2}{4}(b-c)^4+(b^2+c^2)^2x^2}$

$$= \left\{\frac{ab}{2}(b-c)^2-(b^2+c^2)cx\right\}.\sqrt{\frac{a^2}{4}(b+c)^4+(b^2+c^2)^2x^2},$$

(*App.* p. 345.) $\qquad x = \dfrac{a}{2}.\dfrac{c^2-b^2}{c^2+b^2}$.

SIMPLE EQUATIONS OF TWO UNKNOWN QUANTITIES.

(1) $\left.\begin{aligned} 3x+5y &= 8, \\ 4x+3y &= 7, \end{aligned}\right\} \quad \begin{cases} x = 1, \\ y = 1. \end{cases}$

(2) $\left.\begin{aligned} 4x-3y &= 2, \\ 5x-2y &= 9\tfrac{1}{2}, \end{aligned}\right\} \quad \begin{cases} x = 3\tfrac{1}{2}, \\ y = 4. \end{cases}$

(3) $\begin{aligned} 9x-4y &= 8, \\ 13x+7y &= 101, \end{aligned} \quad \begin{cases} x = 4, \\ y = 7. \end{cases}$

(4) $\left.\begin{aligned} 45x+8y &= 350, \\ 21y-13x &= 132, \end{aligned}\right\} \quad \begin{cases} x = 6, \\ y = 10. \end{cases}$

(5) $\left.\begin{aligned} \frac{1}{8}x+\frac{1}{9}y &= 42, \\ \frac{1}{9}x+\frac{1}{8}y &= 43, \end{aligned}\right\} \quad \begin{cases} x = 144, \\ y = 216. \end{cases}$

(6) $\left.\begin{aligned} 2\tfrac{1}{4}.x+3\tfrac{1}{3}.y &= 48, \\ 4\tfrac{1}{2}.x+10y &= 126, \end{aligned}\right\} \quad \begin{cases} x = 8, \\ y = 9. \end{cases}$

(7) $\left.\begin{aligned} 2x-\frac{y-3}{5} &= \frac{5x-2}{2}, \\ 2y-\frac{x-5}{3} &= \frac{7y-7}{2}, \end{aligned}\right\} \quad \begin{cases} x = 2, \\ y = 3. \end{cases}$

$$(8)\quad \left.\begin{aligned} &\frac{x+11}{10}+\frac{y-4}{6}=x-7,\\ &\frac{x+5}{7}-\frac{y-7}{3}=3y-x,\end{aligned}\right\}\qquad \begin{cases} x=9,\\ y=4.\end{cases}$$

$$(9)\quad \left.\begin{aligned} &\frac{x+2}{7}+\frac{y-x}{4}=2x-8,\\ &\frac{2y-3x}{3}+2y=3x+4,\end{aligned}\right\}\qquad \begin{cases} x=5,\\ y=9.\end{cases}$$

$$(10)\quad \left.\begin{aligned} (x+5)(y+7)&=(x+1)(y-9)+112,\\ 2x+10&=3y+1,\end{aligned}\right\}\qquad \begin{cases} x=3,\\ y=5.\end{cases}$$

$$(11)\quad \left.\begin{aligned} &\frac{2x+y}{9}+\frac{7y+6x+11}{18}=\frac{19}{2}-\frac{5x-17}{6},\\ &\frac{3}{7}\{5x+3y+2\}=\frac{1}{2}\{9y+6\},\end{aligned}\right\}\qquad \begin{cases} x=7,\\ y=4.\end{cases}$$

$$(12)\quad \left.\begin{aligned} &\frac{3x-5y}{3}-\frac{2x-8y-9}{12}=\frac{y}{2}+\frac{1}{3}+\frac{1}{4},\\ &3\tfrac{1}{2}.\left\{\frac{x}{7}+\frac{y}{4}+1\tfrac{1}{3}\right\}=3\tfrac{1}{3}.\left\{4x-\frac{y}{8}-24\right\},\end{aligned}\right\}\qquad \begin{cases} x=7,\\ y=4.\end{cases}$$

$$(13)\quad \left.\begin{aligned} &x+y=38\tfrac{1}{5}-\frac{4x+12y}{10},\\ &\frac{1}{21x}+\frac{1}{22y}=\frac{452}{462xy},\end{aligned}\right\}\qquad \begin{cases} x=10,\\ y=11.\end{cases}$$

$$(14)\quad \left.\begin{aligned} &16x+6y-1=\frac{128x^2-18y^2+217}{8x-3y+2},\\ &\frac{10x+10y-35}{2x+2y+3}=5-\frac{54}{3x+2y-1},\end{aligned}\right\}\qquad \begin{cases} x=6,\\ y=5.\end{cases}$$

$$(15)\quad \left.\begin{aligned} &\frac{2}{3}\left(x-\frac{3}{5}y\right)+\frac{x+\frac{1}{5}y}{6}=\frac{1}{3}-\frac{1}{2}\left\{\frac{\frac{4}{5}y-2}{6}-(x-y)\right\},\\ &x-2y-\frac{3y-5x}{2}=\frac{11}{2}(x+y)+3(x-y),\end{aligned}\right\}\qquad \begin{cases} x=3,\\ y=-2\tfrac{1}{2}.\end{cases}$$

(16) $$\left.\begin{aligned} 2{\cdot}4x+0{\cdot}32y-\frac{0{\cdot}36x-0{\cdot}05}{0{\cdot}5} &= 0{\cdot}8x+\frac{2{\cdot}6+0{\cdot}005y}{0{\cdot}25}, \\ \frac{0{\cdot}04y+0{\cdot}1}{0{\cdot}3} &= \frac{0{\cdot}07x-0{\cdot}1}{0{\cdot}6}, \text{ (\textit{Comp.} p. 33.)} \end{aligned}\right\} \qquad \begin{cases} x = 10 \\ y = 5. \end{cases}$$

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(17) $$\left.\begin{aligned} ax &= by, \\ x+y &= c, \end{aligned}\right\} \qquad x = \frac{bc}{a+b}, \quad y = \frac{ac}{a+b}.$$

(18) $$\left.\begin{aligned} ax+by &= m, \\ a'x+b'y &= m', \end{aligned}\right\} \qquad x = \frac{b'm-bm'}{ab'-a'b}, \quad y = \frac{am'-a'm}{ab'-a'b}.$$

(19) $$\left.\begin{aligned} ax-by &= a^2, \\ bx-ay &= b^2, \end{aligned}\right\} \qquad x = a+\frac{b^2}{a+b}, \quad y = \frac{ab}{a+b}.$$

(20) $$\left.\begin{aligned} \frac{m}{x}+\frac{n}{y} &= a, \\ \frac{n}{x}+\frac{m}{y} &= b, \end{aligned}\right\} \qquad \begin{cases} x = \dfrac{m^2-n^2}{ma-nb} \\ y = \dfrac{m^2-n^2}{mb-na}. \end{cases}$$

(21) $$a(x+y)+b(x-y) = c(x+y)+d(x-y) = 1,$$
$$x = \frac{a-b-(c-d)}{2(ad-bc)}, \quad y = \frac{a+b-(c+d)}{2(bc-ad)}.$$

(22) $$\left.\begin{aligned} \frac{5\sqrt{x+y}}{x}+\frac{5\sqrt{x+y}}{y} &= 10\tfrac{2}{3}, \\ \frac{3\sqrt{x-y}}{y}-\frac{3\sqrt{x-y}}{x} &= \frac{4}{5}, \end{aligned}\right\} \text{ (\textit{Comp.} p. 34.)} \qquad \begin{cases} x = 2\tfrac{1}{2}, \\ y = 1\tfrac{1}{2}. \end{cases}$$

(23) $$\left.\begin{aligned} axy &= c(bx+ay), \\ bxy &= c(ax-by), \end{aligned}\right\} \text{ (\textit{Comp.} p. 35.)} \qquad \begin{cases} x = \dfrac{a^2+b^2}{a^2-b^2}\cdot c, \\ y = \dfrac{a^2+b^2}{2ab}\cdot c. \end{cases}$$

(24) $$\left.\begin{aligned} a(x^2+y^2)-b(x^2-y^2) &= 2a, \\ (a^2-b^2)(x^2-y^2) &= 4ab, \end{aligned}\right\} \qquad \begin{cases} x = \sqrt{\dfrac{a+b}{a-b}}, \\ y = \sqrt{\dfrac{a-b}{a+b}}. \end{cases}$$

(25) $$\left.\begin{aligned} \sqrt{y}-\sqrt{a-x} &= \sqrt{y-x}, \\ 2\sqrt{y-x}+2\sqrt{a-x} &= 5\sqrt{a-x}, \end{aligned}\right\} \qquad \begin{cases} x = \dfrac{4}{5}a, \\ y = \dfrac{5}{4}a. \end{cases}$$

(26) $$\left.\begin{aligned} \frac{x}{a}+\frac{y}{b} &= 1-\frac{x}{c}, \\ \frac{y}{a}+\frac{x}{b} &= 1+\frac{y}{c}, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \frac{(ab+ac-bc)abc}{a^2b^2+a^2c^2-b^2c^2}, \\ y &= \frac{(ac-ab-bc)abc}{a^2b^2+a^2c^2-b^2c^2}. \end{aligned}\right.$$

(27) $$\left.\begin{aligned} &(a^2-b^2)(3x+5y) = (4a-b)2ab, \quad (\textit{Comp. } \text{p. } 35.) \\ &a^2x-\frac{ab^2c}{a+b}+(a+b+c)by = b^2x+(a+2b)ab, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \frac{ab}{a-b}, \\ y &= \frac{ab}{a+b}. \end{aligned}\right.$$

(28) $$\left.\begin{aligned} x(bc-xy) &= y(xy-ac), \\ xy(ay+bx-xy) &= abc(x+y-c), \end{aligned}\right\} (\textit{App. } \text{p. } 331.) \qquad \left\{\begin{aligned} x &= \pm\sqrt{ac}, \\ y &= \pm\sqrt{bc}. \end{aligned}\right.$$

(29) $$\left.\begin{aligned} x^3+y(xy-1) &= 0, \\ y^3-x(xy+1) &= 0, \end{aligned}\right\} (\textit{App. } \text{p. } 332.) \qquad \left\{\begin{aligned} x &= \sqrt[4]{\frac{1}{2}(\sqrt{2}-1)}, \\ y &= \frac{1}{\sqrt[4]{2(\sqrt{2}-1)}}. \end{aligned}\right.$$

(30) $$\left.\begin{aligned} &\sqrt{x+y}+\sqrt{x-y} = \frac{1}{a}(xy-y\sqrt{x^2-y^2}), \\ &\sqrt[4]{x+y}+\sqrt[4]{x-y} = b, \end{aligned}\right\} (\textit{Comp. } \text{p. } 35.)$$

$$\left\{\begin{aligned} x &= \frac{1}{4}\left\{\frac{b^2}{2}+\frac{\sqrt[3]{2a^2}}{b^2}\right\}^2+\frac{1}{2}\sqrt[3]{2a^2}, \\ y &= \frac{b^2}{2}\cdot\sqrt[3]{\frac{a}{2}}+\frac{a}{b^2}. \end{aligned}\right.$$

(31) $$\left.\begin{aligned} (x^2-xy+y^2)(x^2+y^2) &= 91, \\ (x^2-xy+y^2)(x^2+xy+y^2) &= 133, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \pm 3, \\ y &= \pm 2. \end{aligned}\right.$$

(32) $$\left.\begin{aligned} &y+3\sqrt{y}\{\sqrt[3]{a+bx}-\sqrt[3]{y}\}\sqrt[3]{a+bx} = 2a, \\ &\frac{y-3\sqrt[3]{y}\,.\,\sqrt[3]{a^2-b^2x^2}}{\sqrt{y}-\sqrt[3]{a+bx}} = 2a\,.\,\sqrt[3]{a-bx}, \end{aligned}\right\} (\textit{App. } \text{p. } 343.)$$

$$x = \frac{a-1}{b},\ y = (\sqrt[3]{2a-1}+1)^3.$$

SIMPLE EQUATIONS OF THREE OR MORE UNKNOWN QUANTITIES.

(1) $$\left.\begin{aligned} 3x-y+z &= 17, \\ 5x+3y-2z &= 10, \\ 7x+4y-5z &= 3, \end{aligned}\right\} \quad \left\{\begin{aligned} x &= 4, \\ y &= 0, \\ z &= 5. \end{aligned}\right.$$

(2) $$\left.\begin{aligned} 3x+4z &= 57, \\ 5x+3y &= 65, \\ 2y-z &= 11, \end{aligned}\right\} \quad \left\{\begin{aligned} x &= 7, \\ y &= 10, \\ z &= 9. \end{aligned}\right.$$

(3) $$\left.\begin{array}{l} y+\frac{1}{2}x = 41, \\ x+\frac{1}{4}z = 20\tfrac{1}{2}, \\ y+\frac{1}{5}z = 34, \end{array}\right\} \quad \left\{\begin{array}{l} x = 18, \\ y = 32, \\ z = 10. \end{array}\right.$$

(4) $$\left.\begin{array}{l} x+\frac{1}{2}(y+z) = 102, \\ y+\frac{1}{3}(z+x) = 78, \\ z+\frac{1}{4}(x+y) = 61, \end{array}\right\} \quad \left\{\begin{array}{l} x = 10, \\ y = 5. \\ z = . \end{array}\right.$$

(5) $$\frac{x+y}{z} = \frac{44}{17}, \quad \frac{x-y}{z} = \frac{20}{17}, \text{ and } \frac{x^2-y^2}{z} = \frac{3520}{17},$$

$x = 128,\ y = 48,\ z = 68.$

(6) $4x-5y+mz = 7x-11y+nz = x+y+pz = 3,$ (*Comp.* p. 36.)

$x = 2,\ y = 1,\ z = 0.$

(7) $$\frac{6y-4x}{3z-7} = \frac{5z-x}{2y-3z} = \frac{y-2z}{3y-2x} = 1, \qquad x = 10,\ y = 7,\ z = 3.$$

(8) $$x+\frac{1}{2}(y+z) = y+\frac{2}{3}(x+z) = z+\frac{3}{4}(x+y) = x+y+z-4,$$

$x = 10,\ y = 6,\ z = 2.$

(9) $$\left.\begin{array}{l} \frac{2}{x}+\frac{1}{y} = \frac{3}{2}, \\ \frac{3}{z}-\frac{2}{y} = 2, \\ \frac{1}{x}+\frac{1}{z} = \frac{4}{3}, \end{array}\right\} \quad \left\{\begin{array}{l} x = 1, \\ y = -2, \\ z = 3. \end{array}\right.$$

(10) $$\left.\begin{array}{l} \frac{2}{x}-\frac{5}{3y}+\frac{1}{z} = 3\tfrac{4}{27}, \\ \frac{1}{4x}+\frac{1}{y}+\frac{2}{z} = 6\tfrac{11}{72}, \\ \frac{5}{6x}-\frac{1}{y}+\frac{4}{z} = 12\tfrac{1}{36}, \end{array}\right\} \quad \left\{\begin{array}{l} x = 6, \\ y = 9, \\ z = \frac{1}{3}. \end{array}\right.$$

(11) $$\left.\begin{array}{l} \frac{xy+86}{3x+2y} = \frac{4x-y}{5}, \\ \frac{xz+4\frac{1}{2}}{x+3z} = \frac{x-z}{2}, \\ \frac{yz+4\frac{1}{7}}{2y+z} = \frac{y+3z}{7}, \end{array}\right\} \quad \left\{\begin{array}{l} x = 6, \\ y = 1, \\ z = 3. \end{array}\right.$$

(12) $$\left.\begin{array}{l} \frac{3}{x}-\frac{4}{5y}+\frac{1}{z} = 7\tfrac{3}{5}, \\ \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z} = 10\tfrac{1}{6}, \\ \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z} = 16\tfrac{1}{10}, \end{array}\right\} \quad \left\{\begin{array}{l} x = \frac{1}{2}, \\ y = \frac{1}{3}, \\ z = \frac{1}{4}. \end{array}\right.$$

(13) $$\left.\begin{array}{l} \frac{x+z-3}{14}+\frac{4y-4z+5}{9} = \frac{x-4}{3}, \\ \frac{5x-6y+z}{7}+\frac{3x-y+z+1}{11} = \frac{2y+z-5}{3}, \\ \frac{5y-8z+4}{12}-\frac{x-2z+5}{6} = 2y+3z-2x, \end{array}\right\} \quad \left\{\begin{array}{l} x = 25, \\ y = 16, \\ z = 6. \end{array}\right.$$

(26) $\frac{a}{c}$

$$\left.\begin{aligned} x+y+z &= 0,\\ (a+b)x+(a+c)y+(b+c)z &= 0,\\ abx+acy+bcz &= 1, \end{aligned}\right\}\ \textit{(Comp. p. 37.)}\qquad \left\{\begin{aligned} x &= \frac{1}{(a-c)(b-c)},\\ y &= \frac{-1}{(a-b)(b-c)},\\ z &= \frac{1}{(a-b)(a-c)}. \end{aligned}\right.$$

(15)
$$\left.\begin{aligned} x-ay+a^2z-a^3 &= 0,\\ x-by+b^2z-b^3 &= 0,\\ x-cy+c^2z-c^3 &= 0, \end{aligned}\right\}\ \textit{(Comp. p. 37.)}\qquad \left\{\begin{aligned} x &= abc,\\ y &= ab+ac+bc,\\ z &= a+b+c. \end{aligned}\right.$$

(16)
$$\left.\begin{aligned} 5x-11\sqrt{y}+13\sqrt[3]{z} &= 22,\\ 4x+6\sqrt{y}+5\sqrt[3]{z} &= 31,\\ x-\sqrt{y}+\sqrt[3]{z} &= 2, \end{aligned}\right\}\qquad \left\{\begin{aligned} x &= 1,\\ y &= 4,\\ z &= 27. \end{aligned}\right.$$

(17)
$$\left.\begin{aligned} xy &= a(x+y),\\ xz &= b(x+z),\\ yz &= c(y+z), \end{aligned}\right\}\ \textit{(App. p. 332.)}\qquad \left\{\begin{aligned} x &= \frac{2abc}{ac+bc-ab},\\ y &= \frac{2abc}{ab+bc-ac},\\ z &= \frac{2abc}{ab+ac-bc}. \end{aligned}\right.$$

(18)
$$\left.\begin{aligned} x(x+y+z) &= a^2,\\ y(x+y+z) &= b^2,\\ z(x+y+z) &= c^2, \end{aligned}\right\}\ \textit{(App. p. 333.)}\qquad \left\{\begin{aligned} x &= \frac{a^2}{\sqrt{a^2+b^2+c^2}},\\ y &= \frac{b^2}{\sqrt{a^2+b^2+c^2}},\\ z &= \frac{c^2}{\sqrt{a^2+b^2+c^2}}. \end{aligned}\right.$$

(19)
$$\left.\begin{aligned} x(y+z) &= a,\\ y(x+z) &= b,\\ z(x+y) &= c, \end{aligned}\right\}\ \textit{(App. p. 334.)}\qquad \left\{\begin{aligned} x &= \sqrt{\frac{(a+b-c)(a+c-b)}{2(b+c-a)}},\\ y &= \sqrt{\frac{(a+b-c)(b+c-a)}{2(a+c-b)}},\\ z &= \sqrt{\frac{(b+c-a)(a+c-b)}{2(a+b-c)}}. \end{aligned}\right.$$

(20)
$$\left.\begin{aligned} 2\sqrt{x+\tfrac{1}{2}y+3z}+3\sqrt{x+5y-z}-4\sqrt{x+y+z} &= 10,\\ \tfrac{1}{2}\sqrt{4x+2y+12z}-\sqrt{2x+10y-2z}+\sqrt{2x+2y+2z} &= 3,\\ \sqrt{2x+y+6z}+2\sqrt{x+5y-z}-3\sqrt{x+y+z} &= 4\sqrt{2}, \end{aligned}\right\}\qquad \left\{\begin{aligned} x &= 7,\\ y &= 6,\\ z &= 5. \end{aligned}\right.$$

(21)
$$\left.\begin{aligned} x^{-2}y^{-1}z &= 1\tfrac{1}{2},\\ x^{-1}yz^2 &= 18,\\ xy^2z^3 &= 108, \end{aligned}\right\}\ \left\{\begin{aligned} x &= 1,\\ y &= 2,\\ z &= 3. \end{aligned}\right.$$

(22)
$$\left.\begin{aligned} x^{-1}y^{-1}z^{-1} &= (105)^{-1},\\ xy^{-1}z^{-1} &= 3\times(35)^{-1},\\ xyz^{-1} &= 2\tfrac{1}{7}, \end{aligned}\right\}\ \left\{\begin{aligned} x &= 3,\\ y &= 5,\\ z &= 7. \end{aligned}\right.$$

(23) $$\left.\begin{aligned} x^2y^3z^4 &= a, \\ x^3y^4z^2 &= b, \\ x^4y^2z^3 &= c, \end{aligned}\right\} \quad \begin{cases} x = \sqrt[27]{\dfrac{bc^{10}}{a^8}}, \\ y = \sqrt[27]{\dfrac{ab^{10}}{c^8}}, \\ z = \sqrt{\dfrac{ca^{10}}{b^8}}. \end{cases}$$

(24) $$\left.\begin{aligned} xyz &= 231, \\ xyw &= 420, \\ yzw &= 1540, \\ xzw &= 660, \end{aligned}\right\} \quad \begin{cases} x = 3, \\ y = 7, \\ z = 11, \\ w = 20. \end{cases}$$

(25) $x(y+z)^2 = 1+a^3$; $x+y = \frac{3}{2}+z$; and $yz = \frac{3}{16}$; (*App.* p. 333.)

$$x = 1+a, \quad y = \frac{1}{4}-\frac{a}{2}\pm\frac{1}{2}\sqrt{(a-1)^2+a}, \quad z = \frac{a}{2}-\frac{1}{4}\pm\frac{1}{2}\sqrt{(a-1)^2+a}.$$

(26) $$\left.\begin{aligned} x+2y+3z+4u &= 27, \\ 3x+5y+7z+u &= 48, \\ 5x+8y+10z-2u &= 65, \\ 7x+6y+5z+4u &= 53, \end{aligned}\right\} \quad \begin{cases} x = 1, \\ u = 2, \\ y = 3, \\ z = 4. \end{cases}$$

(27) $$\left.\begin{aligned} 7x-2z+3u &= 17, \\ 4y-2z+t &= 11, \\ 5y-3x-2u &= 8, \\ 4y-3u+2t &= 9, \\ 3z+8u &= 33, \end{aligned}\right\} \quad \begin{cases} x = 2, \\ y = 4, \\ z = 3, \\ u = 3, \\ t = 1. \end{cases}$$

(28) Shew that the following equations are not sufficient to determine x, y, z. (Art. 198.)

(1) $$\left.\begin{aligned} 3x-2y+5z &= 14, \\ 2x+y-8z &= 10, \\ 8x-3y+2z &= 38. \end{aligned}\right\}$$

(2) $$\left.\begin{aligned} 3x-2y+5z &= 14, \\ 6x-4y-3z &= 15, \\ 9x-6y-7z &= 20. \end{aligned}\right\}$$

(3) $$\left.\begin{aligned} 2x-y+2z &= 8, \\ 6y+13-\frac{x}{2} &= 4x-1+3z, \\ \frac{x}{4}+z-y+\frac{1}{2} &= \frac{1}{4}-\frac{2y}{8}+\frac{5z}{6}. \end{aligned}\right\} \quad (\textit{Comp. p. 38.})$$

QUADRATIC EQUATIONS OF ONE UNKNOWN QUANTITY.

(1) $5x^2-12x+2 = 11$, $\quad x = 3$, or $-\frac{3}{5}$.

(2) $x^2 = 5x+6000$, $\quad x = 80$, or -75.

(3) $3x^2-53x+34=0,$ $\qquad x=17, \text{ or } \frac{2}{3}.$

(4) $45x^2-25x=1000,$ $\qquad x=5, \text{ or } -4\tfrac{4}{9}.$

(5) $110x^2-21x+1=0,$ $\qquad x=\frac{1}{10}, \text{ or } \frac{1}{11}.$

(6) $780x^2-73x+1=0,$ $\qquad x=\frac{1}{13}, \text{ or } \frac{1}{60}.$

(7) $125x^2-7x=17\tfrac{1}{5},$ $\qquad x=\frac{2}{5}, \text{ or } -\frac{43}{125}.$

(8) $\frac{x+1}{x-1}-\frac{x-1}{x+1}=1,$ $\qquad x=2\pm\sqrt{5}.$

(9) $\frac{x+2}{x-1}-\frac{4-x}{2x}=2\tfrac{1}{3},$ $\qquad x=3, \text{ or } -\frac{4}{5}.$

(10) $\frac{3}{3-x}+\frac{2}{2-x}=5,$ $\qquad x=2\pm\frac{1}{5}\sqrt{10}.$

(11) $4(x-1)-\frac{x-1}{2x}=3\tfrac{3}{4},$ $\qquad x=2, \text{ or } \frac{1}{16}.$

(12) $\frac{x}{x+1}+\frac{x+1}{x}=2\tfrac{1}{6},$ $\qquad x=2, \text{ or } -3.$

(13) $\frac{1}{x-1}-\frac{1}{x+3}=\frac{1}{35},$ $\qquad x=11, \text{ or } -13.$

(14) $\frac{2x-1}{2x+1}+\frac{13}{11}=\frac{3x+5}{3x-5},$ $\qquad x=5, \text{ or } -\frac{1}{6}.$

(15) $\frac{12}{5-x}+\frac{8}{4-x}=\frac{32}{x+2},$ $\qquad x=2, \text{ or } 4\tfrac{6}{13}.$

(16) $\frac{1}{3}+\frac{1}{3+x}-\frac{1}{3+2x}=0,$ $\qquad x=3(-1\pm\frac{1}{2}\sqrt{2}).$

(17) $\frac{x^4+3x^3+6}{x^2+x-4}=x^2+2x+15,$ $\qquad x=2, \text{ or } -2\tfrac{7}{13}.$

(18) $2\sqrt{3x+7}=9+\sqrt{2x-3},$ $\qquad x=14, \text{ or } 2\tfrac{12}{25}.$

(19) $\sqrt{(x^2+\tfrac{1}{2}\sqrt{x^2+1664})}=x+1,$ $\qquad x=10, \text{ or } -11\tfrac{1}{15}.$

(20) $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2},$ $\qquad x=2, \text{ or } 3.$

(21) $x^2+\sqrt{x^2-5}=11,$ $\qquad x=\pm 3, \text{ or } \pm\sqrt{14}.$

(22) $x+2-\frac{6}{x+2}=1$, $\quad x=1$, or -4.

(23) $x^2+x+1=\frac{42}{x^2+x}$, $\quad x=2$, or -3.

(24) $9x+2x\sqrt{9x+4}=15x^2-4$, $\quad x=1\frac{1}{3}$, or $-\frac{1}{3}$, or &c.

(25) $x^2-x+3\sqrt{2x^2-3x+2}=\frac{x}{2}+7$, $\quad x=2$, or $-\frac{1}{2}$, or &c.

(26) $x(\sqrt{x}+1)^2=102(x+\sqrt{x})-2576$, $\quad x=49$, or 64, or &c.

(27) $(9+5\sqrt{3})x^2-(15+7\sqrt{3})x+6=0$, $\quad x=3-\sqrt{3}$, or $2-\sqrt{3}$.

(28) $5\sqrt{62+3x}-\frac{1}{2}\sqrt{95\frac{2}{3}-5x}=41$, $\quad x=6\frac{1}{3}$, or &c.

(29) $\frac{3\sqrt{x}-x^{\frac{3}{2}}}{x+2}=\frac{1\frac{2}{3}+3\sqrt{x}-2x}{2\sqrt{x}-3}$, $\quad x=4$, or $\frac{1}{25}$.

(30) $\frac{1}{x^2+11x-8}+\frac{1}{x^2+2x-8}+\frac{1}{x^2-13x-8}=0$, (*Comp.* p. 38.)

$x=\pm 8$, or ± 1.

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(31) $\frac{x^2}{a^2}+\frac{x}{b}=2.\frac{a^2}{b^2}$, $\quad x=\frac{a^2}{b}$, or $-\frac{2a^2}{b}$.

(32) $\frac{2x(a-x)}{3a-2x}=\frac{a}{4}$, $\quad x=\frac{3a}{4}$, or $\frac{a}{2}$.

(33) $\frac{nx+b}{\sqrt{x}}=\frac{na+b}{\sqrt{a}}$, $\quad x=a$, or $\frac{b^2}{n^2a}$.

(34) $\sqrt{a+x}+\sqrt{a-x}=\frac{12a}{5\sqrt{a+x}}$, $\quad x=\frac{4a}{5}$, or $\frac{3a}{5}$.

(35) $\frac{a+x}{\sqrt{a-x}}+\frac{a-x}{\sqrt{a+x}}=2\sqrt{a}$, $\quad x=\pm a\sqrt{8\sqrt{2}-11}$.

(36) $ax+2\sqrt{n^2x+nax^2}=(3x-1).n$, $\quad x=\frac{n}{n-a}$, or $\frac{n}{9n-a}$.

(37) $x+\sqrt{x^2-ax+b^2}=\frac{x^2}{a}+b$, $\quad x=0$, or a, or $\frac{a}{2}\left\{1\pm\sqrt{5-\frac{8b}{a}}\right\}$.

(38) $\left(\frac{x}{x-1}\right)^2+\left(\frac{x}{x+1}\right)^2=n(n-1)$, (*Comp.* p. 39.)

$$x=\pm\sqrt{\frac{n}{n-2}},\ \text{or}\ \pm\sqrt{\frac{n-1}{n+1}}.$$

(39) $\frac{x^2}{8a}+\frac{2x}{3}=\sqrt{\frac{x^3}{3a}+\frac{x^2}{4}-\frac{a}{2}}$, (*Comp.* p. 39.) $x=6a$, or $-\frac{2a}{3}$.

(40) $(x+m)^{\frac{3}{2}}-m^{\frac{3}{2}}=3m^{\frac{1}{2}}x$, $x=0$, or $\{3\pm2\sqrt{3}\}.m$.

(41) $a+x+\sqrt{2ax+x^2}=\frac{3x^2-6ax}{4\sqrt{2ax+x^2}}$, $x=\frac{8a}{5}$, or $\left(\frac{2}{3}\sqrt{3}-1\right)2a$.

(42) $\left(x-\frac{ab}{x}\right)^2=\frac{a}{2}(a+b)\left(1+\frac{a^2}{x^2}\right)$, (*Comp.* p. 40.)

$$x=\sqrt{2ab+a^2},\ \text{or}\ \sqrt{\frac{a}{2}(b-a)}.$$

(43) $mqx^2-mnx+pqx-np=0$, $x=\frac{n}{q}$, or $-\frac{p}{m}$.

(44) $\sqrt{x+\sqrt{2x-1}}-\sqrt{x-\sqrt{2x-1}}=\frac{3}{5}\sqrt{\frac{10x}{x+\sqrt{2x-1}}}$, $x=2\frac{1}{2}$, or $\frac{5}{8}$.

(45) $(x-a)^2+2\sqrt{x}.(x-a)=a^2+\sqrt{x}$, $x=0$, or 1, or $2a+1\frac{1}{2}\pm\sqrt{2a+1\frac{1}{4}}$.

(46) $x^2(a+x)+bx(a+x)=b-(x+2b)$, $x=-b$, or $\pm\sqrt{\frac{a^2}{4}-1}-\frac{a}{2}$.

(47) $\sqrt[4]{a+x}+\sqrt[4]{a-x}=b$, $x=\pm\sqrt{a^2-\left(b^2\pm\sqrt{a+\frac{b^4}{2}}\right)^4}$.

(48) $\sqrt{2x^2-1}+\sqrt{1-a^2}.\sqrt{1-x^2}=ax$, $x=\left\{\frac{5a^2-6\pm2\sqrt{2}.a(a^2-1)}{8a^2-9}\right\}^{\frac{1}{2}}$.

(49) $2x^3-x-2x\sqrt{1-x^2}=1\frac{1}{2}$, $x=\pm\frac{1}{2}\sqrt{3}$, or $\frac{1}{4}(1\pm\sqrt{7})$.

(50) $\frac{(1+x)^2}{1+x^3}+\frac{(1-x)^2}{1-x^3}=a$, $x=\left\{\frac{2}{a}-\frac{1}{2}\pm\sqrt{\left(\frac{2}{a}\right)^2-\frac{3}{4}}\right\}^{\frac{1}{2}}$.

(51) $\frac{1+x^3}{(1+x)^3}+\frac{1-x^3}{(1-x)^3}=a$, $x=\frac{\pm\sqrt{a+1}+\sqrt{3}}{\sqrt{a-2}}$.

(52) $16(x^2+2)^{\frac{3}{2}}+\frac{3}{\sqrt{x^2+2}}=32x^2+48$, $x=\pm\frac{1}{2}$, or &c.

(53) $(x+3)^2-2(x^2+3)=2x(x+1)^2$, $\qquad x=1$, or -3, or $-\frac{1}{2}$.

(54) $\sqrt{1+a}.\sqrt{1-x}-\sqrt{1-a}.\sqrt{1+x}=2a$, $\qquad x=a(3-4a^2)$, or $-a$.

(55) $(1-x)\sqrt{x^2+x^3}-(1+x)\sqrt{x^2-x^3}=\frac{\sqrt{\{2+2\sqrt{1-x^2}\}}}{a}$,

$$x=\left\{\frac{1}{2}-\frac{1}{a}\pm\sqrt{\frac{1}{4}+\frac{1}{a}}\right\}^{\frac{1}{2}}.$$

(56) $\frac{\sqrt{1+a^2}-a\sqrt{1+x^2}}{\sqrt{1+x^2}-x\sqrt{1+a^2}}=a$, $\qquad x=\frac{1}{a}$, or $\frac{1}{a}.\frac{1-3a^2}{a^2-3}$.

(57) $(a+x)\sqrt{a^2+x^2}=6(a-x)^2$, $\qquad x=(9\pm4\sqrt{2}).\frac{a}{7}$.

(58) $\frac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=\frac{x}{a-x}$, $\qquad x=a$, or $\frac{a}{5}$.

(59) $\sqrt{x^2+1}-\sqrt{x^2-1}=\frac{\sqrt{2x}}{\sqrt[8]{x^4+1}}$, (*Comp.* p. 40.) $\qquad x=\sqrt[8]{\frac{2\pm\sqrt{5}}{4}}$.

(60) $n\sqrt{a^2+x^2}+(n-1)\sqrt{x^2-2(n-1)a^2}=2n-1$,

$$x=\sqrt{(n\pm n-1.\sqrt{1-a^2})^2-a^2}.$$

(61) $\frac{na+n.\frac{n-1}{2}.ra}{1+nr}=p$,

$$n=\left(\frac{1}{2}-\frac{1}{r}\right)+\left(\frac{1}{2}+\frac{p}{a}\right)-\frac{1}{2}\pm\sqrt{\left(\frac{1}{2}-\frac{1}{r}\right)^2+\left(\frac{1}{2}+\frac{p}{a}\right)^2-\frac{1}{2^2}}.$$

(62) $(1+x+x^2)^2=\frac{a+1}{a-1}(1+x^2+x^4)$, (*App.* p. 319.) $\qquad x=\frac{}{2}\pm\sqrt{\left(\frac{a}{2}\right)^2-1}$.

(63) $a+x+\sqrt{2ax+x^2}=\sqrt{ax-x^2}+\sqrt{2a^2-ax-x^2}$, (*App.* p. 323.)

$$x=\{1\pm\frac{1}{2}\sqrt{2}\}.\frac{a}{2}.$$

(64) $\frac{x-\sqrt{x^2-a^2}}{\sqrt{x+\sqrt{x^2-a^2}}}=\sqrt[4]{x^2-a^2}.\{\sqrt{x^2+ax}-\sqrt{x^2-ax}\}$, (*App.* p. 324.)

$$x=\pm\frac{\sqrt{a}}{2}.\left\{\frac{a}{\sqrt[4]{a^2+2}}\pm\sqrt[4]{a^2+2}\right\}.$$

(65) $\dfrac{2x+\sqrt{2(1+x)}}{1-x} = a+\dfrac{1}{a}$, (*Comp.* p. 41.) $x = 1$, or $\dfrac{a^2+\dfrac{1}{a^2}}{\left(\sqrt{a}+\dfrac{1}{\sqrt{a}}\right)^4}$.

(66) $(a+b)\sqrt{a^2+b^2+x^2}-(a-b)\sqrt{a^2+b^2-x^2} = a^2+b^2$, (*Comp.* p. 42.)

$$x = \sqrt{\pm\frac{\sqrt{3}}{2}(a^2-b^2)-ab}.$$

(67) $\dfrac{1}{ax}\left(\dfrac{a^3+x^3}{5}\right)^2+12\tfrac{2}{3}.\left(\dfrac{ax}{5}\right)^2 = \dfrac{(a^2+x^2)^2}{6}$, (*Comp.* p. 42.)

$$x = a, \text{ or } (2\pm\sqrt{3}).a.$$

(68) $\sqrt{(b^2+c^2)(a^2+x^2)} = n(ac+bx)$, (*Comp.* p. 43.) $x = \dfrac{b\mp\sqrt{n^2-1}.c}{c\pm\sqrt{n^2-1}.b}.a$.

(69) $x^{2m}-1 = a+\dfrac{a}{x^m}$, $x = \sqrt[m]{\dfrac{1}{2}\pm\sqrt{a+\dfrac{1}{4}}}$.

(70) $\sqrt[5]{(a+x)^2}+\sqrt[5]{(a-x)^2} = 3\sqrt[5]{a^2-x^2}$, (*Comp.* p. 43.) $x = \pm\dfrac{11}{25}\sqrt{5}.a$.

(71) $\dfrac{(27a+8x)^{\frac{2}{3}}}{15x^{\frac{13}{15}}}+\dfrac{8x^{\frac{2}{15}}}{3\sqrt[3]{27a+8x}} = \dfrac{8}{5\sqrt[5]{x}}$, (*Comp.* p. 44.)

$$x = \frac{3}{32}.\{3\pm\sqrt{21}\}.a.$$

(72) $a^2b^2x^{\frac{1}{q}}-4(ab)\ .x^{\frac{p+q}{2pq}} = (a-b)^2.x^{\frac{1}{p}}$, (*Comp.* p. 44.) $x = \left(\dfrac{1}{\sqrt{b}}\pm\dfrac{1}{\sqrt{a}}\right)^{\frac{4pq}{p-q}}$.

(73) $\sqrt[2pq]{x^{p+q}}-\dfrac{1}{2}.\dfrac{a^2-b^2}{a^2+b^2}.(\sqrt[p]{x}+\sqrt[q]{x}) = 0$, (*Comp.* p. 45.) $x = \left(\dfrac{a\pm b}{a\mp b}\right)^{\frac{2pq}{q-p}}$.

(74) $(a^{4m}+1)(x^{\frac{1}{2}}-1)^2 = 2(x+1)$, $x = \left\{\dfrac{a^{2m}\pm 1}{a^{2m}\mp 1}\right\}^2$.

(75) $\sqrt[m]{(1+x)^2}-\sqrt[m]{(1-x)^2} = \sqrt[m]{1-x^2}$, (*App.* p. 320.) $x = \dfrac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}$.

(76) $x-1 = 2+\dfrac{2}{\sqrt{x}}$, (*App.* p. 317.) $x = 1$, or 4.

(77) $x^3-3x = 2$, (*App.* p. 318.) $x = 2$, or -1.

(78) $x^2-\dfrac{2}{3x} = 1\tfrac{4}{9}$, (*App.* p. 318.) $x = -\dfrac{2}{3}$, or $\dfrac{1\pm\sqrt{10}}{3}$.

(79) $x-3=\frac{3+4\sqrt{x}}{x}$, $x=\frac{7\pm\sqrt{13}}{2}$.

(80) $x^2+\frac{1}{x^2}+x+\frac{1}{x}=4$, $x=1$, or $\frac{-3\pm\sqrt{5}}{2}$.

(81) $x^2-8(x+1)\sqrt{x}+18x+1=0$, $x=7\pm4\sqrt{3}$.

(82) $\frac{1}{3}x^2(x^2+3)=20+5\frac{1}{3}.x$, $x=3$, or -2.

(83) $x+7\sqrt[3]{x}=22$, $x=8$, or $\{-1\pm\sqrt{-10}\}^3$, or &c.

(84) $(x+3)^2-2(x^2+3)=2x(x+1)^2$, $x=1$, or -3, or $-\frac{1}{2}$.

(85) $\sqrt{x}-\frac{8}{x}=\frac{7}{\sqrt{x}-2}$. $x=1$, or 16.

(86) $x^2(x^2-23)-10x(x^2-24)=649$, $x=\frac{1}{2}\{5\pm3\sqrt{29}\}$.

(87) $2x^{\frac{3}{2}}(x^3+a^3)^{\frac{1}{2}}=2x^2(x+2a)+a^2(x-a)$, $x=\frac{a}{2}$, or $-a$.

(88) $x^2-7=\sqrt{\{x^2-42x+89\}}$, $x=2$, or -5, or &c.

(89) $(x^2-5)^2=(x-3)^2+(x+1)^2$, $x=3$, or -1, or $\pm\sqrt{6}-1$.

(90) $\frac{x}{2}+\frac{63}{\sqrt[4]{x}}=\frac{220\frac{1}{2}}{\sqrt{x}}+49\sqrt{x}-1196$,

$x=2401$, or $\frac{1}{16}\{-7\pm\sqrt{61}\}^4$, or $\frac{1}{16}\{-7\pm\sqrt{37}\}^4$.

(91) $\frac{1+x^3}{(1+x)^3}=a$, $x=\frac{1+2a\pm\sqrt{12a-3}}{2(1-a)}$.

(92) $\frac{1+x^4}{(1+x)^4}=a$, (*App.* p. 321), $x=p\pm\sqrt{p^2-1}$,

where $p=\frac{2a\pm\sqrt{2(1+a)}}{2(1-a)}$.

(93) $\frac{1+x^5}{(1+x)^5}=a$,

$$x=\frac{\sqrt{4a+1}(\sqrt{5}+\sqrt{4a+1})+[10(6a-1)+2\sqrt{5}(4a+1)^{\frac{3}{2}}]^{\frac{1}{2}}}{4(1-a)}.$$

(94) $\sqrt{x^2+\sqrt{x}}=x-1+3\sqrt{x}$, $x=\frac{1}{4}$, or $\frac{31\pm5\sqrt{37}}{18}$.

(95) $x-2\sqrt{x+2} = 1+\sqrt[4]{x^3-3x+2}$, (*App.* p. 325.)

$$x = \frac{1}{2}(3\pm\sqrt{13}), \text{ or } 9\pm4\sqrt{7}.$$

(96) $2x\sqrt{1-x^4} = a(1+x^4)$, (*App.* p. 327.)

$$x = \pm\frac{1}{a}\{(\sqrt{1+a^2}-1)(\sqrt{1-a^2}+1)\}^{\frac{1}{2}}.$$

(97) $\left(x-\frac{1}{3}\right)^2-\frac{25}{9} = \dfrac{3x^2+\frac{4}{9}}{2\left(x-\frac{1}{3}\right)+\sqrt{x\left(x-\frac{8}{3}\right)}}$, (*App.* p. 328.)

$$x = 3, \text{ or } -\frac{1}{3}, \text{ or } \frac{2}{3}(2\pm\sqrt{13}).$$

(98) $\dfrac{(n-1)(a^4+a^2x^2+x^4)}{(n+1)(a^4-a^2x^2+x^4)} = \left(2-\frac{1}{n}\right)\left(\frac{ax}{a^2-x^2}\right)^2$, (*App.* p. 328.)

$$x = \frac{a}{2}\left\{\sqrt{\frac{5n-3}{n-1}}+\sqrt{\frac{n+1}{n-1}}\right\}, \text{ or } \frac{a}{2}\left\{\sqrt{\frac{1}{n}+2}+\sqrt{\frac{1}{n}-2}\right\}.$$

(99) $a^2x^2-b^2-2x(a-cx)\sqrt{\dfrac{a^2-b^2}{1-x^2}+\left(\dfrac{a-cx}{1-x^2}\right)^2}+\dfrac{1+x^2}{1-x^2}(a-cx)^2 = 0$,

(*App.* p. 329.) $x = \dfrac{a\mp b}{c\pm a}$.

(100) $(1-x)\sqrt{a\left(1+\frac{1}{x}\right)-2} = \sqrt{x+1}+\sqrt{3x-1}$, (*App.* p. 330.)

$$x = \frac{\pm\sqrt{a+1}-1}{\pm\sqrt{a-1}+1}.$$

QUADRATIC EQUATIONS OF TWO OR MORE UNKNOWN QUANTITIES.

(1) $\left.\begin{aligned} x^2-xy &= 10, \\ (x-y)^2 &= 4, \end{aligned}\right\}$ $\left\{\begin{aligned} x &= \pm5, \\ y &= \pm3. \end{aligned}\right.$

(2) $\left.\begin{aligned} x^2+y^2 &= 13, \\ x+y &= 5, \end{aligned}\right\}$ $\left\{\begin{aligned} x &= 3, \text{ or } 2, \\ y &= 2, \text{ or } 3. \end{aligned}\right.$

(3) $\left.\begin{aligned} x^2-y^2 &= 9, \\ xy-y^2 &= 4, \end{aligned}\right\}$ $\left\{\begin{aligned} x &= \pm5, \\ y &= \pm4. \end{aligned}\right.$

(4) $\left.\begin{aligned} 2x^2+3xy &= 26, \\ 3y^2+2xy &= 39, \end{aligned}\right\}$ $\left\{\begin{aligned} x &= \pm2, \\ y &= \pm3. \end{aligned}\right.$

(5) $\left.\begin{aligned} xy+xy^2 &= 12, \\ x+xy^3 &= 18, \end{aligned}\right\}$ $\left\{\begin{aligned} x &= 2, \text{ or } 16, \\ y &= 2, \text{ or } \frac{1}{2}. \end{aligned}\right.$

(6) $\left.\begin{aligned} x^2-xy &= 27y, \\ xy-y^2 &= 3x, \end{aligned}\right\}$ $\left\{\begin{aligned} x &= 13\tfrac{1}{2}, \text{ or } 6\tfrac{3}{4}, \\ x &= 4\tfrac{1}{2}, \text{ or } 2\tfrac{1}{4}. \end{aligned}\right.$

(7) $2y-3x = 3x^2+2(y-11)^2 = 14,$ $\quad x = 2,$ or $1\frac{1}{5}$; $y = 10,$ or $8\frac{4}{5}.$

(8) $\left.\begin{array}{r} 2y+3x = 8, \\ 3y^2+2x^2 = 11, \end{array}\right\}$ $\quad \begin{cases} x = 2, \text{ or } 2\frac{4}{35}. \\ y = 1, \text{ or } \dfrac{29}{35}. \end{cases}$

(9) $\left.\begin{array}{r} x^2+xy+4y^2 = 6, \\ 3x^2+8y^2 = 14, \end{array}\right\}$ $\quad \begin{cases} x = \pm 2, \text{ or } \pm\frac{1}{5}\sqrt{10}, \\ y = \pm\dfrac{1}{2}, \text{ or } \mp\frac{2}{5}\sqrt{10}. \end{cases}$

(10) $\left.\begin{array}{r} 2x^2+3xy+y^2 = 20, \\ 5x^2+4y^2 = 41, \end{array}\right\}$ $\quad \begin{cases} x = \pm 1, \text{ or } \pm\dfrac{13}{21}\sqrt{21}, \\ y = \pm 3, \text{ or } \pm\dfrac{2}{21}\sqrt{21}. \end{cases}$

(11) $\left.\begin{array}{r} x-y = a, \\ y^2+ay+bx = 0, \end{array}\right\}$ $\quad \begin{cases} x = a-b, \text{ or } 0, \\ y = -b, \text{ or } -a. \end{cases}$

(12) $\left.\begin{array}{r} \sqrt[3]{x}+\sqrt[3]{y} = 3, \\ x+y = 9, \end{array}\right\}$ $\quad \begin{cases} x = 8, \text{ or } 1, \\ y = 1, \text{ or } 8. \end{cases}$

(13) $\left.\begin{array}{r} x+y = 5, \\ (x^2+y^2).(x^3+y^3) = 455, \end{array}\right\}$ (*Comp.* p. 45.) $\quad \begin{cases} x = 3, \text{ or } 2, \\ y = 2, \text{ or } 3. \end{cases}$

(14) $\left.\begin{array}{r} \sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}} = \dfrac{61}{\sqrt{xy}}+1, \\ \sqrt[4]{x^3y}+\sqrt[4]{xy^3} = 78, \end{array}\right\}$ $\quad \begin{cases} x = 81, \text{ or } 16, \\ y = 16, \text{ or } 81. \end{cases}$

(15) $\left.\begin{array}{r} x^4+y^4 = 641, \\ x^3y+y^3x = 290, \end{array}\right\}$ $\quad \begin{cases} x = 5, \\ y = 2. \end{cases}$

(16) $\left.\begin{array}{r} x-y = 8, \\ x^4-y^4 = 14560, \end{array}\right\}$ (*App.* p. 331.) $\quad \begin{cases} x = 11, \\ y = 3. \end{cases}$

(17) $\left.\begin{array}{r} x+y = a, \\ x^4+y^4 = 14x^2y^2, \end{array}\right\}$ $\quad \begin{cases} x = \dfrac{a}{2}(\pm\sqrt{3}+1), \text{ or } \dfrac{a}{6}(3\pm\sqrt{3}), \\ y = \dfrac{a}{2}(\mp\sqrt{3}+1), \text{ or } \dfrac{a}{6}(3\mp\sqrt{3}). \end{cases}$

(18) $\sqrt{ax}+\sqrt{by} = \frac{1}{2}(x+y) = a+b,$ (*Comp.* p. 46.) $\quad \begin{cases} x = (\sqrt{a}\pm\sqrt{b})^2. \\ y = (\sqrt{a}\mp\sqrt{b})^2. \end{cases}$

(19) $\left.\begin{array}{r} xz = y^2 \\ (x+y)(z-x-y) = 3, \\ (x+y+z)(z-x-y) = 7, \end{array}\right\}$ (*Comp.* p. 46.) $\quad \begin{cases} x = 1, \text{ or } 9, \\ y = 2, \text{ or } -6, \\ z = 4. \end{cases}$

(20) $$\left.\begin{aligned} x+y+z &= 216, \\ \sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z} &= 12, \\ \sqrt[3]{y^2}+\sqrt[3]{z^2} &= \sqrt[3]{x^2}+\sqrt[3]{yz}-12, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 125, \\ y &= 64, \text{ or } 27, \\ z &= 27, \text{ or } 64. \end{aligned}\right.$$

(21) $$\left.\begin{aligned} (x-2)^2+(y-3)^2+(z-1)^2 &= 24, \\ xy+xz+yz &= 63, \\ 2x+3y+z &= 30, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 6, \text{ or } 3\tfrac{1}{3}, \\ y &= 5, \text{ or } 6\tfrac{1}{3}, \\ z &= 3, \text{ or } 4\tfrac{1}{3}. \end{aligned}\right.$$

(22) $$\left.\begin{aligned} x^2+xy+y^2 &= 37, \\ x^2+xz+z^2 &= 28, \\ y^2+yz+z^2 &= 19, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \pm 4, \text{ or } \pm\frac{10}{3}\sqrt{3}, \\ y &= \pm 3, \text{ or } \pm\frac{1}{3}\sqrt{3}, \\ z &= \pm 2, \text{ or } \mp\frac{8}{3}\sqrt{3}. \end{aligned}\right.$$

(23) $$\left.\begin{aligned} x^3+y^3+xy(x+y) &= 13, \\ (x^2+y^2)x^2y^2 &= 468, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 3, \text{ or } -2, \\ y &= -2, \text{ or } 3. \end{aligned}\right.$$

(24) $$\left.\begin{aligned} \frac{y}{x}-\frac{x}{x+y} &= \frac{x^2-y^2}{y}, \\ \frac{x}{y}-\frac{x+y}{x} &= \frac{y}{x}, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= -\frac{1}{9}, \\ y &= -\frac{1}{18}. \end{aligned}\right.$$

(25) $$\left.\begin{aligned} \frac{x}{y}-\frac{y}{x} &= \frac{x+y}{x^2+y^2}, \\ \frac{x^2}{y^2}-\frac{y^2}{x^2} &= \frac{x-y}{y^2}, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \frac{1}{4}\{1\pm\sqrt{2}\}, \\ y &= \frac{1}{4}. \end{aligned}\right.$$

(26) $$\left.\begin{aligned} \frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}} &= x, \\ \frac{x}{y} &= \sqrt{\frac{1+x}{1-y}}, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \frac{1}{4}, \\ y &= \frac{1}{5}. \end{aligned}\right.$$

(27) $$\left.\begin{aligned} x^2-6\sqrt{x^2y} &= 27, \\ x-2\sqrt{xy} &= 3, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 9, \\ y &= 1. \end{aligned}\right.$$

(28) $$\left.\begin{aligned} (x+y)^2 &= x^4+x^2y^2+y^4, \\ x^4+4y^4 &= 4xy(2y^2-x^2), \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \sqrt{\frac{8\pm 2\sqrt{3}}{13}}, \\ y &= \sqrt{\frac{11\pm 6\sqrt{3}}{13}}. \end{aligned}\right.$$

(29) $$\left.\begin{aligned} x+\sqrt{x^2-y^2} &= \frac{8}{y}(\sqrt{x+y}+\sqrt{x-y}), \\ (x+y)^{\frac{3}{2}}-(x-y)^{\frac{3}{2}} &= 26, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 5, \\ y &= \pm 4. \end{aligned}\right.$$

(30) $$\left.\begin{aligned} x+\sqrt{x^2-y^2} &= \frac{a}{y}\left(\sqrt{x+y}+\sqrt{x-y}\right),\\ \sqrt[4]{x+y}+\sqrt[4]{x-y} &= y, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \left\{\frac{a^2+1}{4}+\frac{a^2}{a^2+1}\right\}\sqrt[8]{\frac{4}{a^2+1}},\\ y &= a\sqrt[8]{\frac{4}{a^2+1}}. \end{aligned}\right.$$

(31) $$\left.\begin{aligned} x^2-y^2 &= 17ay+3y^2+4x\sqrt{2ay-y^2},\\ x^2+y^2 &= 2ay+8\sqrt{y}.(x\sqrt{a}-y\sqrt{y}), \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 7a,\\ y &= a. \end{aligned}\right.$$

(32) $$\left.\begin{aligned} x-\sqrt{x^2-y^2} &= \frac{a}{y}\left(\sqrt{x+y}+\sqrt{x-y}\right),\\ (x+y)^{\frac{3}{2}}-(x-y)^{\frac{3}{2}} &= b, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \frac{b+2a}{3\sqrt[3]{4a}},\\ y &= \sqrt[6]{4a}.\sqrt{\frac{b-a}{3}}. \end{aligned}\right.$$

(33) $$\left.\begin{aligned} \frac{x+y-\sqrt{x^2+y^2}}{x+y+\sqrt{x^2+y^2}} &= \frac{2x}{a},\\ \frac{x}{y} &= \sqrt{\frac{a+x}{a-y}}, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \frac{-11\pm3\sqrt{17}}{16}.a,\\ y &= \frac{13\mp3\sqrt{17}}{8}.a. \end{aligned}\right.$$

(34) $$\left.\begin{aligned} x+\sqrt{xy}+y+\sqrt{x}+\sqrt[4]{xy}+\sqrt{y} &= 210,\\ x-\sqrt{xy}+y+\sqrt{x}-\sqrt[4]{xy}+\sqrt{y} &= 126, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 144, \text{ or } 9, \text{ or } 92\mp32\sqrt{7},\\ y &= 9, \text{ or } 144, \text{ or } 92\pm32\sqrt{7}. \end{aligned}\right.$$

(35) $$\left.\begin{aligned} \sqrt{x}-\sqrt{y} &= \sqrt{y+2},\\ x+8 &= 8\sqrt{y+2}, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 4,\\ y &= \frac{1}{4}. \end{aligned}\right.$$

(36) $$\left.\begin{aligned} x^4+y^4 &= 1+2xy+3x^2y^2,\\ x^3+y^3 &= 2y^2x+2y^2+x+1, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 2, \text{ or } -1,\\ y &= 1. \end{aligned}\right.$$

(37) $$\left.\begin{aligned} x^2+y^2 &= 3xy,\\ x^5+y^5 &= 2, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}+1),\\ y &= \frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}-1). \end{aligned}\right.$$

or equal values of y.

'(38) $$\left.\begin{aligned} x^2-xy &= 6,\\ x^2+y^2 &= 61, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \pm6, \text{ or } \pm\frac{1}{2}\sqrt{2},\\ y &= \pm5, \text{ or } \mp\frac{11}{2}\sqrt{2}. \end{aligned}\right.$$

(39) $$\left.\begin{aligned} x^{x+y} &= y^{4a},\\ y^{x+y} &= x^{a}, \end{aligned}\right\} \text{ (App. p. 336.)} \qquad \left\{\begin{aligned} x &= \tfrac{1}{2}(4a+1\mp\sqrt{8a+1}),\\ y &= \tfrac{1}{2}(-1\pm\sqrt{8a+1}). \end{aligned}\right.$$

'(40) $$\left.\begin{aligned} (\sqrt{x})^{\sqrt[4]{x}+\sqrt[4]{y}} &= (\sqrt{y})^{\frac{8}{3}},\\ (\sqrt{y})^{\sqrt[4]{x}+\sqrt[4]{y}} &= (\sqrt{x})^{\frac{2}{3}}, \end{aligned}\right\} \text{ (Comp. p. 47)} \qquad \left\{\begin{aligned} x &= \left\{-\frac{1}{2}\pm\frac{1}{6}\sqrt{57}\right\}^8,\\ y &= \left\{-\frac{1}{2}\pm\frac{1}{6}\sqrt{57}\right\}^4. \end{aligned}\right.$$

(41) $$\left.\begin{aligned} 5ax+12y(a-x) &= 0, \\ x^2-y^2+a^2 &= 0, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= \frac{4a}{3}, \text{ or } \frac{3a}{4}, \\ y &= \frac{5a}{3}, \text{ or } -\frac{5a}{4}. \end{aligned}\right.$$

(42) $$\left.\begin{aligned} x^4y^2-4 &= 4xy^2-\frac{y^6}{4}, \\ x^3-3 &= xy(x-y), \end{aligned}\right\}$$ (See *App.* p. 338.) $$\left\{\begin{aligned} x &= 1, \\ y &= 2. \end{aligned}\right.$$

(43) $$\left.\begin{aligned} \sqrt{x^2+\sqrt[3]{x^4y^2}}+\sqrt{y^2+\sqrt[3]{x^2y^4}} &= a, \\ x+y+3\sqrt[3]{bxy} &= b, \end{aligned}\right\}$$ (*Comp.* p. 48.) $$\left\{\begin{aligned} x &= \frac{1}{8}\{b^{\frac{1}{3}}+\sqrt{2a^{\frac{2}{3}}-b^{\frac{2}{3}}}\}^3, \\ y &= \frac{1}{8}\{b^{\frac{1}{3}}-\sqrt{2a^{\frac{2}{3}}-b^{\frac{2}{3}}}\}^3. \end{aligned}\right.$$

(44) $$\left.\begin{aligned} (x^2+y^2).\frac{y}{x} &= 8\tfrac{2}{3}, \\ (x^2-y^2).\frac{x}{y} &= 7\tfrac{1}{2}, \end{aligned}\right\}$$ (*Comp.* p. 49.) $$\left\{\begin{aligned} x &= \pm 3, \\ y &= \pm 2. \end{aligned}\right.$$

(45) $$\left.\begin{aligned} (x^2+y^2)(x+y) &= 15xy, \\ (x^4+y^4)(x^2+y^2) &= 85x^2y^2, \end{aligned}\right\}$$ (*Comp.* p. 49.) $$\left\{\begin{aligned} x &= 2, \text{ or } 4, \\ y &= 4, \text{ or } 2. \end{aligned}\right.$$

(46) $$\left.\begin{aligned} (x-2)y-\sqrt{xy}.(y^2-1) &= 2y^2-x, \\ \frac{xy}{4} &= \frac{\sqrt{xy}-12}{xy-18}, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 8, \text{ or } -12, \\ y &= 2, \text{ or } -\frac{4}{3}. \end{aligned}\right.$$

(47) $$\left.\begin{aligned} 5-2\sqrt{y+2} &= \frac{9x^2}{64}-(\sqrt{x}-3\sqrt{y})^2, \\ \frac{7}{y}-10\sqrt{\frac{x}{y}} &= x-16, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 4, \\ y &= \frac{1}{4}. \end{aligned}\right.$$

(48) $$\left.\begin{aligned} \sqrt{x+y}+\sqrt[3]{x-y} &= \sqrt[3]{a}, \\ \sqrt[3]{x^2+y^2}+\sqrt[3]{x^2-y^2} &= \sqrt[3]{a^2}, \end{aligned}\right\}$$ (*Comp.* p. 50.) $$\left\{\begin{aligned} x &= \frac{1}{2}(1\pm\sqrt{3})a, \\ y &= \sqrt{1\pm\frac{11}{18}\sqrt{3}}.a. \end{aligned}\right.$$

(49) $$\left.\begin{aligned} (b+x)x+ay &= (x+y)^2-2\sqrt{xy}.\sqrt{a-x}.\sqrt{b-y}, \\ \sqrt{y}-\sqrt{a-x} &= \sqrt{\frac{nx}{y}}, \end{aligned}\right\}$$ (*Comp.* p. 51.)

$$x = \left(\frac{b-n}{b}\right)^2\left\{\sqrt{n}\pm\sqrt{\frac{ab}{b-n}-(b-n)}\right\}^2, \quad y = b-n.$$

(50) $$\left.\begin{aligned} x^2+y^2 &= x+\frac{a}{2}, \\ x^3(x-2)+6y^2x(x-1)+y^4 &= \frac{b}{2}-x, \end{aligned}\right\}$$ (*Comp.* p. 52.)

$$\left\{\begin{aligned} x &= \frac{1}{2}\pm\frac{1}{4}\sqrt{4a+2\mp2\sqrt{8a^2-4a-8b+1}}, \\ y &= \pm\frac{1}{4}\sqrt{4a+2\pm2\sqrt{8a^2-4a-8b+1}}. \end{aligned}\right.$$

(51) $$\left.\begin{aligned} &y\sqrt{(a-x)(x-b)}+x\sqrt{(a-y)(b-y)} \\ &\quad = 2\{b\sqrt{(a-x)(a-y)}+a\sqrt{(x-b)(b-y)}\}, \\ &xy = 4ab, \end{aligned}\right\}$$ (*Comp.* p. 53.)

$$\left\{\begin{aligned} x &= 2(a\pm b+\sqrt{a^2\pm ab+b^2}), \\ y &= 2(b\pm a\mp\sqrt{a^2\pm ab+b^2}). \end{aligned}\right.$$

(52) $$\left.\begin{aligned} x^ma^n+y^nb^m &= 2(ax)^{\frac{m}{2}}.(by)^{\frac{n}{2}}, \\ xy &= ab, \end{aligned}\right\}$$ (*Comp.* p. 53.)

$$\left\{\begin{aligned} x &= b^{\frac{2n}{m+n}}\{a^{\frac{m-n}{2}}\pm(a^{m-n}-b^{m-n})^{\frac{1}{2}}\}^{\frac{2}{m+n}}, \\ y &= a^{\frac{2m}{m+n}}\{b^{\frac{n-m}{2}}\mp(b^{n-m}-a^{n-m})^{\frac{1}{2}}\}^{\frac{2}{m+n}}. \end{aligned}\right.$$

(53) $$\left.\begin{aligned} \frac{\sqrt{y^2+1}+1}{y} &= \frac{\sqrt{x+9}+3}{\sqrt{x}}, \\ x(y+1)^2 &= 36\left(y^2+\frac{16}{9}\right), \end{aligned}\right\}$$ (*App.* p. 336.) $$\left\{\begin{aligned} x &= \frac{3}{2}(19\pm\sqrt{105}), \\ y &= \frac{1}{6}(3\pm\sqrt{105}). \end{aligned}\right.$$

(54) $$\left.\begin{aligned} \frac{y}{2x}+\frac{2}{3}.\frac{y-\sqrt{x-1}}{y^2-2\sqrt{x^2-1}} &= \frac{\sqrt{x+1}}{x}, \\ \frac{1}{4}y^4 &= y^2x-1, \end{aligned}\right\}$$ (*App.* p. 338.) $$\left\{\begin{aligned} x &= 1\tfrac{1}{4}, \\ y &= 2. \end{aligned}\right.$$

(55) $$\left.\begin{aligned} a(1-xy) &= x\sqrt{1-y^2}, \\ \sqrt{x}(1-xy) &= y-x, \end{aligned}\right\}$$ (*App.* p. 339.)

$$x = \frac{1}{2}(a^2\pm a\sqrt{a^2+4}),\ y = \pm1,\ \text{or}\ \pm\frac{a^2-1}{a^2+1},\ \text{or}\ \frac{2a}{\sqrt{(a^2+1+a\sqrt{a^2+4})^2-1-2a}}.$$

(56) $$\left.\begin{aligned} (a+b).\frac{y\sqrt{1-x^2}-x\sqrt{1-y^2}}{xy+\sqrt{1-x^2}.\sqrt{1-y^2}} &= (a-b).\frac{y\sqrt{1-x^2}+x\sqrt{1-y^2}}{xy-\sqrt{1-x^2}.\sqrt{1-y^2}}, \\ a(2y^2-1)+b(2x^2-1) &= c, \end{aligned}\right\}$$ (*Comp.* p. 54.)

$$x = \frac{1}{2}\sqrt{\frac{(b+c)^2-a^2}{bc}},\ y = \frac{1}{2}\sqrt{\frac{(a+c)^2-b^2}{ac}}.$$

(57) $$\left.\begin{array}{l}\dfrac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}(x^{\frac{2}{3}}-1)+\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(2x^{\frac{2}{3}}-1)=\dfrac{4y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(y^{\frac{1}{3}}+x^{\frac{1}{3}})+\dfrac{3y}{x^{\frac{1}{3}}}+2,\\ \dfrac{x^{\frac{4}{3}}}{y^{\frac{4}{3}}}-\dfrac{2x^{\frac{2}{3}}}{y}-\dfrac{2x^{\frac{1}{3}}}{y^{\frac{1}{3}}}=\dfrac{133}{36}\cdot\dfrac{1}{y^{\frac{2}{3}}}-\dfrac{2}{x^{\frac{1}{3}}}-\dfrac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}},\end{array}\right\}$$

(*Comp.* p. 55.) $\begin{cases}x=27,\ \text{or } -8,\\ y=8,\ \text{or } -27.\end{cases}$

(58) $$\left.\begin{array}{l}y^4=x^2(ay-bx),\\ x^2=ax-by,\end{array}\right\}$$ (*App.* p. 340.) $x=y=a-b,$

$$\text{or } x=a-\frac{b}{2}(m\pm\sqrt{m^2-4}),\ y=\frac{1}{2}\{a-\frac{b}{2}(m\pm\sqrt{m^2-4})\}(m\pm\sqrt{m^2-4}),$$

$$\text{where } m=\frac{1}{2b}(a-b\pm\sqrt{a^2+2ab+5b^2}).$$

(59) $$\left.\begin{array}{l}x^4=mx+ny,\\ y^4=my+nx,\end{array}\right\}$$ (*Comp.* p. 56.) $x=y=\sqrt[3]{m+n},$

$$\text{or } x=\sqrt[3]{m+\frac{n}{2}(a\mp\sqrt{a^2-4})},\quad y=\sqrt[3]{m+\frac{n}{2}(a\pm\sqrt{a^2-4})},$$

$$\text{where } a=\frac{-(m+n)\pm\sqrt{m^2-2mn+5n^2}}{2n}.$$

(60) $$\left.\begin{array}{l}x^5+y^5=xy(x+y)^3,\\ y^2\sqrt{x}=(x+y)^{\frac{3}{2}},\end{array}\right\}$$ (*App.* p. 341.) $$\begin{cases}x=\dfrac{5}{2}\sqrt{10\pm\dfrac{22}{5}\sqrt{5}},\\ y=\dfrac{5}{2}\sqrt{2+\dfrac{2}{5}\sqrt{5}}.\end{cases}$$

(61) $$\left.\begin{array}{l}(x^6+1)y=(y^2+1)x^3,\\ (y^6+1)x=9(x^2+1)y^3,\end{array}\right\}$$ (*App.* p. 342.)

$$x=\frac{1}{2}\{\sqrt{\sqrt[3]{3}+3}+\sqrt{\sqrt[3]{3}-1}\};\ y=\frac{1}{2}\{\sqrt[3]{3}.\sqrt{\sqrt[3]{3}+3}\pm\sqrt{\sqrt[3]{9}-1}\}.$$

(62) $$\left.\begin{array}{l}(2+4xy-3x^2)^2=2-4x^2y^2+3x^4,\\ 5y^2+\dfrac{27x^2}{32}=\dfrac{9xy}{2}+\dfrac{2xy+1}{x^2},\end{array}\right\}$$ (*App.* p. 343.)

$$x=2,\ y=1-\frac{1}{4}\sqrt{6}.$$

(63) $$\left.\begin{aligned}\frac{3+2x^2-4x^4}{x^2-1} &= y^2(1-2y^2),\\ (2x^2-1)(2y^2-1) &= 3,\end{aligned}\right\}\quad (\textit{App.}\ \text{p. } 346.)$$

$$x = \pm\frac{1}{2}\sqrt{5},\ \text{or} \pm\frac{1}{2}\sqrt{\frac{1}{2}(1\pm\sqrt{33})};\ y = \pm\frac{1}{2}\sqrt{6},\ \text{or} \pm\frac{1}{2}\sqrt{5\pm\sqrt{33}}.$$

(64) $$\left.\begin{aligned}&30\sqrt{\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{1}{3}}y^{\frac{1}{3}}}}+40\sqrt{\frac{x^{\frac{2}{3}}y^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}} = 241,\\ &\left\{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}\right\}.\left\{3x^{\frac{4}{3}}y^{\frac{2}{3}}+\frac{91}{216}\sqrt{x^2+x^{\frac{1}{3}}y^{\frac{4}{3}}}\right\} = \left(\frac{5}{6}\right)^3-x^2-y^2,\quad (\textit{Comp.}\ \text{p. } 57.)\end{aligned}\right\}$$

$$x = \pm\frac{8}{27},\ \text{or} \pm\frac{1}{8};\ y = \pm\frac{1}{8},\ \text{or} \pm\frac{8}{27}.$$

(65) $$\left.\begin{aligned}(x^4+2bx^2y+a^2y^2)(y^4+2bxy^2+a^2x^2) &= 4(a^2-b^2)(b+c)^2(xy)^2,\\ x^3+y^3 &= 2cxy,\end{aligned}\right\}$$

(*Comp.* p. 58.) $$\left.\begin{aligned}x &= \sqrt[3]{p^2}.\sqrt[3]{c\pm\sqrt{c^4-p^2}},\\ y &= \sqrt[3]{p^2}.\sqrt[3]{c\mp\sqrt{c^2-p^2}},\end{aligned}\right\}\ \text{where } p^2 = a^2-2bc-2b^2.$$

(66) $$\left.\begin{aligned}&a^2-x^2 = 3xy,\\ &(\sqrt{y}-\sqrt{x})(a-x) = 3\sqrt{x}(x+y),\end{aligned}\right\}\quad (\textit{Comp.}\ \text{p. } 59.)$$

$$x = -\frac{a}{2},\ \text{or} \pm a\sqrt{-2},\ y = -\frac{a}{2},\ \text{or} \pm\frac{a}{\sqrt{-2}}.$$

(67) $$\left.\begin{aligned}(1-x^2)^2.(1+y^2)-(1+x^2)^2.(1-y^2) &= 4x^2\sqrt{1+y^4},\\ 4xy &= \sqrt{2}(1-x^2)(1-y^2),\end{aligned}\right\}\quad (\textit{Comp.}\ \text{p. } 60.)$$

$$x = \frac{\mp\sqrt{2\sqrt{3}-2}.\sqrt[4]{2\sqrt{3}}\pm 2}{\sqrt{2\sqrt{3}}\mp(\sqrt{3}-1)},\ y = \pm\sqrt{\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}}}.$$

MISCELLANEOUS QUESTIONS ON EQUATIONS.

(1) WHAT is the quadratic equation whose roots are 17 and $\frac{2}{3}$?

Ans. $3x^2-53x+34 = 0$.

(2) What is the quadratic equation whose roots are 3 and $-\frac{3}{5}$?

Ans. $5x^2-12x-9 = 0$.

(3) The trinomial ax^2+bx+c becomes 42, when $x=4$; 22, when $x=3$; and 8, when $x=2$; what are the values of a, b, c?

Ans. $a=3$; $b=-1$; and $c=-2$.

(4) Solve the equation $x^2-5x-24=0$, without completing the square.

Ans. $x=8$, or -3.

(5) Prove that in *simple* equations of two unknown quantities there is only one pair of values of the unknown quantities which will satisfy the two equations.

(6) If x_1, x_2, represent the two values of x which satisfy the equation $ax^2+bx+c=0$, prove that $\frac{x_1}{x_2}+\frac{x_2}{x_1}=\frac{b^2-2ac}{ac}$; and verify this formula by the equation $2x^2-7x+3=0$.

(7) If α, β, represent the two roots of the equation

$$x^2-(1+a)x+\frac{1}{2}(1+a+a^2)=0,$$

shew that $\alpha^2+\beta^2=a$.

(8) Shew that $x+\frac{1}{x}$ cannot be less than 2, whatever positive quantity be substituted for x.

(9) If a proposed equation be reduced to the form $ax=ax+c$, what conclusion is to be drawn as to the value of x?

Ans. $x=\infty$.

(10) Shew that there can be no more than *three* distinct values of x which satisfy the equation $ax^3+bx^2+cx+d=0$, and that their sum $=\frac{-b}{a}$.

(11) Find the sum of the four roots of the equation

$$a(px^2+qx+r)^2+b(px^2+qx+r)+c=0.$$

Ans. $\frac{-2q}{p}$.

(12) Is $2\{(x-a)(x-b)+(a-x)(a-b)+(b-x)(b-a)\}=(a-b)^2+(x-a)^2+(x-b)^2$ an '*Identity*' or an '*Equation*'?

Ans. The former.

(13) Given $x+y+z=\frac{14}{3}x=\frac{7}{2}y$, find $\frac{x+y+z}{z}$. (*Comp.* p. 61.) Ans. 2.

(14) Given $x-y=7z$, and $x-z=4y$, find $\frac{y-z}{x}$. (*Comp.* p. 61.) Ans. $\frac{1}{9}$.

(15) Given $x^2+y^2=123z$, and $x^2-y^2=27z$, find $\frac{xy}{z}$. (*Comp.* p. 62.)

Ans. 60.

(16) Given that

$$\left.\begin{aligned}ab-\tfrac{1}{2}(a+b)(p+q)+pq&=0,\\ \text{and } cd-\tfrac{1}{2}(c+d)(p+q)+pq&=0;\end{aligned}\right\}\ (\textit{Comp.}\ \text{p. }62.)$$

shew that $\left(\frac{p-q}{2}\right)^2=\frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}$.

(17) If the same value of x satisfies both the equations $ax^2+bx+c=0$, and $a'x^2+b'x+c'=0$, what is the relation subsisting between a, b, c, a', b', c'? (*Comp.* p. 63.) Ans. $(ac'-a'c)^2=(ab'-a'b)(bc'-b'c)$.

(18) Find the relation subsisting between the coefficients in the equations $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$, $a_3x+b_3y=c_3$, that they may be satisfied by the same values of x and y. (*Comp.* p. 63.)

Ans. $a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)=0$.

(19) Find the relations subsisting between the coefficients, when the equations

$$ax+by+cz=a'x+b'y+c'z=a''x+b''y+c''z=1,$$

are equivalent to no more than two distinct equations. And shew, that in this case the values of x, y, z, assume the form $0 \div 0$. (*Comp.* p. 64.)

Ans. $\frac{a-a'}{a'-a''}=\frac{b-b'}{b'-b''}=\frac{c-c'}{c'-c''}$.

(20) Eliminate b and c from the equations, $a+b+c=-p$, $ab+ac+bc=q$, $abc=-r$. Ans. $a^3+pa^2+qa+r=0$.

(21) Find the value of $ax^2+by^2+cz^2$, when $ax^3=by^3=cz^3$, and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{d}$. (*Comp.* p. 65.) Ans. $(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}})^3.d^2$.

(22) Eliminate a, b, c, from the equations

$$\left(\frac{x}{a}\right)^m+\Bigl(\frac{y^m}{b}+\left(\frac{z}{c}\right)^m=1=\left(\frac{a}{d}\right)^n+\left(\frac{b}{d}\right)^n+\left(\frac{c}{d}\right)^n, \text{ and } \frac{x^m}{a^{m+n}}=\frac{y^m}{b^{m+n}}=\frac{z^m}{c^{m+n}}.$$

(*Comp.* p. 65.) Ans. $x^{\frac{mn}{m+n}}+y^{\frac{mn}{m+n}}+z^{\frac{mn}{m+n}}=d^{\frac{mn}{m+n}}$.

EQUATIONAL PROBLEMS.

(1) I HAVE six times as many shillings as half-crowns, and together they amount to £8. 10*s*. How many have I of each?

Ans. 20 half-crowns, 120 shillings.

(2) Divide £25 among 3 persons, *A*, *B*, *C*, so that *B* shall have twice as much, and *C* three times as much, as *A*.

Ans. To *A*, £4. 3*s*. 4*d*. To *B*, £8. 6*s*. 8*d*. To *C*, £12. 10*s*.

(3) Divide 20 into two such parts that one of them shall be exactly 20 times as great as the other. Ans. $19\frac{1}{21}$, $\frac{20}{21}$.

(4) A boy is exactly one-third the age of his father, and has a brother one-sixth his own age—the ages of all three amount to 50 years. What is the age of each? Ans. 36, 12, 2, years.

(5) Of two brothers whose ages differ by 20 years, one is as much above 25 as the other is below it. What is the age of each?

Ans. 35, and 15, years.

(6) A is twice as old as B. Twenty-two years ago he was four times as old as B. What is A's age? Ans. 66 years.

(7) Nine years ago A was three times as old as B, but now he is only twice as old. Required the respective ages of A and B.
Ans. A is 36, B is 18.

(8) Seven years ago a father was four times as old as his son, but in 7 years more he will be only twice as old. What is the age of each?
Ans. 35, and 14, years.

(9) The ages of a father and his son together are 80 years; and if the age of the son be doubled, it will exceed the father's age by 10 years. Find the age of each. Ans. 50, and 30.

(10) A certain party is composed of three times as many men as women; and when four men have left together with their wives, there remain *four times* as many men as women. How many were there of each sex at first? Ans. 36 men, 12 women.

(11) Divide the number 90 into two such parts, that if half of the greater part be added to the double of the smaller, the result is the original number 90. Ans. 60, and 30.

(12) Find two consecutive numbers, such that the half and fifth parts of the first taken together shall be equal to the third and fourth parts of the second taken together. Ans. 5, and 6.

(13) The product of two numbers is 180, but if the lesser of the two be increased by 1, the product is increased by 20. What are the numbers?
Ans. 9, and 20.

(14) Find two numbers in the ratio of 4 to 5, such that if 6 be added to the greater number, and 1 to the smaller, the square roots of the resulting numbers shall differ by 1. Ans. 24, and 30; or 8, and 10.

(15) There is a certain number of which the cube root is one-fifth of the square root: find it. Ans. 15625.

(16) A certain fraction becomes $\frac{1}{3}$ if 1 be added to its numerator; but if 1 be added to its denominator, it becomes $\frac{1}{4}$. What is the fraction?

Ans. $\frac{4}{15}$.

(17) A certain fraction becomes $\frac{4}{5}$ if 1 be taken from its denominator, and added to the numerator; but, if 1 be taken from its numerator and added to the denominator, it becomes $\frac{1}{2}$. What is the fraction?

Ans. $\frac{7}{11}$.

(18) The numerator of a certain fraction being multiplied by 3, and the numerator and denominator added together for a new denominator, the resulting fraction $= \frac{1}{2}$. Find the original fraction. Ans. $\frac{1}{5}$.

(19) A person distributed £5 among 36 persons, old men and widows, giving 3*s.* each to the men, and 2*s.* 6*d.* each to the women. How many were there of each?

Ans. 20 men, 16 women.

(20) A person distributed p shillings among n persons, giving 9*d.* to some, and 15*d.* to the rest. How many were there of each?

Ans. $\frac{1}{2}(5n-4p)$ at 9*d.*, and $\frac{1}{2}(4p-3n)$ at 15*d.*

(21) Divide the number n into two such parts, that the quotient of the greater divided by the less shall be q with a remainder r.

Ans. $\frac{nq+r}{1+q}$, $\frac{n-r}{1+q}$.

(22) Divide the fraction $\frac{8}{5}$ into two parts, so that the numerators of the two parts taken together shall be equal to their denominators taken together.

Ans. $\frac{1}{2}$, $\frac{11}{10}$.

(23) Divide 30 in two such parts that one is the square of the other.

Ans. 5, 25.

(24) A certain number of sovereigns, shillings, and sixpences together amount to £8. 6*s.* 6*d.*, and the amount of the shillings is a guinea less than that of the sovereigns, and a guinea and a half more than that of the sixpences. Find the numbers of each coin.

Ans. 4 sovereigns, 59 shillings, 55 sixpences.

(25) What is the number from the n^{th} part of which if a be taken, a times the remainder is equal to b?

Ans. $n\left(a+\frac{b}{a}\right)$.

(26) A person sells a acres more than the m^{th} part of his estate, and there remain b acres less than the n^{th} part. Of how many acres does the whole estate consist?

Ans. $\frac{mn(a-b)}{mn-(m+n)}$.

(27) A labourer is engaged for n days, on condition that he receives p pence for every day he works, and pays q pence for every day he is idle. At the end of the time he receives a pence. How many days did he work, and how many was he idle?

Ans. He worked $\frac{nq+a}{p+q}$, and was idle $\frac{np-a}{p+q}$, days.

(28) A person, being asked what o'clock it was, answered that it was between 5 and 6, and that the hour and minute hands were together. Required the time of day. (*Comp.* p. 66.)

Ans. 27m. $16\frac{4}{11}$s. past 5.

(29) Find the time after h o'clock at which the hour and minute hands of a watch are distant d of the minute divisions from each other. (*Comp.* p. 67.)

Ans. $\frac{12}{11}(5h\pm d)$.

(30) There are two places 154 miles distant from each other, from which two persons A and B set out at the same instant with a design to meet on the road, A travelling at the rate of 3 miles in 2 hours, and B at the rate of 5 miles in 4 hours. How long and how far did each travel before they met? Ans. A travelled 84 miles, and 56 hours,
B .... 70 ....

(31) Find a number, such, that whether it is divided into *two* or *three* equal parts, the continued product of the parts shall be the same.
Ans. $6\frac{3}{4}$.

(32) A person bought a certain number of sheep for £94: having lost 7 of them, he sold one-fourth of the remainder at prime cost for £20. How many sheep had he at first? Ans. 47.

(33) A farmer buys m sheep for £p, and sells n of them at a gain of £5 per cent.: how must he sell the remainder that he may clear 10 per cent. on the whole?
Ans. $\frac{22m-21n}{20m(m-n)}.p$£ each.

(34) A boy at a fair spends his money in oranges. If he had received 5 more for his money, they would have cost a half-penny each less; but if 3 less, a half-penny each more. How much did he spend? Ans. 2*s.* 6*d.*

(35) A hare is 80 of its own leaps before a greyhound, and takes 3 leaps for every 2 taken by the greyhound, but the latter passes over as much ground in one leap as the former does in two. How many leaps will the hare have taken before it is caught? (*Comp.* p. 67.)
Ans. 240.

(36) A courier passing through a certain place (P) travels at the rate of 5 miles in 2 hours. Four hours afterwards another passes through the same place travelling the same way at the rate of 7 miles in two hours. How far from the place (P) is the first overtaken by the second?
Ans. 35 miles.

(37) A person has just a hours at his disposal; how far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour? Ex. $a = 2$, $b = 12$, $c = 4$.

(1) Ans. $\frac{abc}{b+c}$ miles. (2) Ans. 6 miles.

(38) A banker has two kinds of money, silver and gold; and a pieces of silver, or b pieces of gold, make up the same sum S. A person comes, and wishes to be paid the sum S with c pieces of money: how many of each must the banker give him? (*Comp.* p. 67.)
Ans. Of silver, $a.\frac{c-b}{a-b}$; of gold, $b.\frac{a-c}{a-b}$.

(39) A certain number being divided into both n and $\overline{n+1}$ equal parts, the product of the n parts is n times the product of the $\overline{n+1}$ parts. Find the number.
Ans. $\left(\frac{n+1}{n}\right)^{n+1}$.

(40) Two travellers, A and B, set out from two places P and Q at the same time; A from P intending to pass through Q, and B from Q intending to travel the same way. After A had overtaken B, and they had computed their travels, it was found that the distance A had travelled together with the distance B had travelled made up 30 miles; that A had passed through Q 4 hours before; and that B at his rate of travelling was 9 hours' journey distant from P. Required the distance between the two places P and Q. Ans. 6 miles.

(41) The rent of a farm is paid in certain fixed numbers of quarters of wheat and barley: when wheat is at 55*s.* and barley at 33*s.* per quarter, the portions of rent by wheat and barley are equal to one another: but when wheat is at 65*s.* and barley at 41*s.* per quarter, the rent is increased by £7. What is the corn-rent?

Ans. 6 qrs. of wheat; 10 qrs. of barley.

(42) A constable in pursuit of a thief at a uniform pace finds by inquiry that the thief is travelling $1\frac{1}{2}$ miles per hour quicker than himself; he therefore doubles his speed after the first 4 hours, and takes the thief at the end of 6 hours and 20 minutes from the time of his starting. Given that the thief had a start of 1 hour, and never varied his speed, find the rates of travelling of the two parties, and the distance at which the capture took place.

Ans. Constable's speed at first, $8\frac{1}{4}$ miles per hour,
Thief's speed throughout $9\frac{3}{4}$ ....
Required distance $71\frac{1}{2}$ miles.

(43) At an election where each elector may give two votes to different candidates, but only one to the same, it is found on casting up the poll, that of the candidates A, B, C, A had 158 votes, B had 132, C had 58. Now 26 voted for A only, 30 for B only, and 28 for C only. How many voted for A and B jointly; how many for A and C; and how many for B and C? Ans. For A and B, 102; A and C, 30; B and C, 0.

(44) During a panic there was a run on two bankers, A and B; B stopped payment at the end of three days, in consequence of which the alarm increased, and the daily demand for cash on A being tripled, A failed at the end of two more days. Now if A and B had joined their capitals together, they might both have stood the run as it was at first for 7 days, at the end of which time B would have been indebted to A £4000. What was the daily demand for cash on A at the beginning of the run?

Ans. £2000.

(45) There is a waggon with a mechanical contrivance by which the difference of the number of revolutions of the wheels on a journey is noted. The circumference of the fore-wheel is a feet, and of the hind-wheel b feet; what is the distance gone over, when the fore-wheel has made n revolutions more than the hind-wheel? Ans. $\frac{abn}{b-a}$ feet.

(46) A person has two casks containing a certain quantity of wine in each. He wishes to have an equal quantity in each; and in order to have this, he pours out of the first cask into the second as much as the second contained at first; then he pours from the second into the first as much as was left in the first; and then again from the first into the second as much as was left in the second. At last there are exactly a gallons in each cask. How many gallons were in each cask at first?

Ans. $\frac{11a}{8}$, and $\frac{5a}{8}$.

(47) A person rows from Cambridge to Ely (a distance of 20 miles) and back again in ten hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the velocity of the stream, and the times of going and returning.

(1) Ans. $\frac{5}{6}$ of a mile per hour. (2) Ans. 4 hours. (3) Ans. 6 hours.

(48) The Gas Company engage to light a shop, for 6 days in a week, with 5 large and 3 small burners, but having by them only one large burner, they supply the deficiency with 5 small ones. The shopkeeper, not finding this light sufficient, procures two small burners more, and at the same time agrees for the lights to burn double the usual time on Saturday nights, for which additional gas he was to pay £1. 11*s*. How much did he pay a year altogether? Ans. £5. 5*s*.

(49) A shopkeeper, on account of bad book-keeping, knows neither the weight nor the prime cost of a certain article which he had purchased. He only recollects, that if he had sold the whole at 30*s*. per lb., he would have gained £5 by it, and if he had sold it at 22*s*. per lb., he would have lost £15 by it. What was the weight and prime cost of the article?

Ans. Weight 50lbs. Cost 28*s*. per lb.

(50) A book is so printed, that each page contains a certain number of lines, and each line a certain number of letters. If we wished each page to contain 3 lines more, and each line 4 letters more, then there would be 224 letters more than before in a page; but if we wished to have 2 lines less in each page, and 3 letters less in each line, then the page would contain 145 letters less than at first. How many lines are there in each page, and how many letters in each line? Ans. 29 lines, and 32 letters.

(51) When wax candles are half-a-crown a pound, a composition is invented of such a nature, that a candle made of it will burn two-thirds of the time in which a wax candle of the same thickness and one-fourth as heavy again will continue burning. Supposing the two candles give an equally bright light, what must be charged per lb. for the composition that it may be as *cheap* as wax? (*Comp.* p. 68.) Ans. 2*s*. 1*d*.

(52) *A*, *B*, *C*, *D*, *E* play together on the condition that he who loses shall give to all the rest as much as they have already. First *A* loses, then *B*, then *C*, then *D*, then *E*. All lose in turn, and yet at the end of the-

fifth game they all have the same sum, viz. £32. How much had each before they began to play?

Ans. *A* £81, *B* £41, *C* £21, *D* £11, *E* £6.

(53) A man at his death leaves property to the amount of £5850 to be divided among three sons, four daughters, and his widow, in manner following:—viz. the share of two sons is to be equal to that of three daughters, and the mother's share half that of a son and a daughter taken together. Find each person's share.

Ans. Each son £900, daughter £600, mother £750.

(54) *A* and *B* engaged to reap equal quantities of wheat, and *A* began half an hour before *B*. They stopped at 12 o'clock, and rested an hour, observing that just half the whole work was done. *B*'s part was finished at 7 o'clock, and *A*'s at a quarter before 10. Supposing them to have laboured uniformly, determine the times at which they commenced. (*Comp.* p. 69.)

Ans. *A* at $\frac{1}{2}$ past 4, *B* at 5 o'clock.

(55) A cistern can be filled by three different pipes; by the 1st in $1\frac{1}{3}$ hours, by the 2nd in $3\frac{1}{3}$ hours, and by the third in 5 hours. In what time will this cistern be filled when all three pipes are opened at once?

Ans. 48 minutes.

(56) If *A* and *B* together can perform a piece of work in *a* days, *A* and *C* together the same in *b* days, and *B* and *C* together in *c* days; find the time in which each can perform it separately. (*App.* p. 350.)

Ans. A in $\frac{2abc}{ac+bc-ab}$, B in $\frac{2abc}{ab+bc-ac}$, C in $\frac{2abc}{ab+ac-bc}$, days.

(57) In a tithe-commutation the rent-charge was apportioned so as to be 3*s*. an acre, and in the 1st year the rates payable on the rent-charge wanted £6 of 10 per cent. on the whole receipts. The next year the rates were doubled, and amounted to 15 per cent. on the receipts; what was the number of acres?

Ans. 1600.

(58) *A* and *B* drink from a cask of beer for 2 hours, after which *A* falls asleep, and *B* drinks the remainder in 2 hours and 48 minutes: but if *B* had fallen asleep, and *A* had continued to drink, it would have taken him 4 hours and 40 minutes to finish the cask. In what time would each singly be able to drink the whole?

Ans. *A* in 10 hours, *B* in 6 hours.

(59) There is a number composed of two figures, of which the figure in the units' place is triple of that in the tens', and if 36 be added to the number the sum is expressed by the same digits reversed. What is the number?

Ans. 26.

(60) The fore-wheel of a coach makes 6 revolutions more than the hind-wheel in going 120 yards; but, if the circumference of each wheel be increased 1 yard, the fore-wheel will make only 4 revolutions more than the hind-wheel in the same distance. Find the circumference of each wheel. (*App.* p. 351.)

Ans. 4, and 5, yards.

(61) The mail-train upon a railway starts a certain time after a luggage-train from the same terminus, and the time is so adjusted that,

before arriving at the other terminus, the trains will exactly escape collision and no more. It happens, however, that from an accident to the engine the speed of the luggage train is suddenly reduced one-half after performing two-thirds of its journey, and a collision takes place a miles from the end of it. The proper speeds of the trains being m and n miles per hour, $(m>n)$ find the length of the railway, and the difference of times of starting. (*Comp.* p. 69.)

(1) Ans. $3\left(2-\frac{n}{m}\right)a$. (2) Ans. $3.\frac{m-n}{mn}\left(2-\frac{n}{m}\right)a$.

(62) Three persons divide a certain sum of money amongst them in the following manner:—A takes the n^{th} part of the whole together with $\frac{a}{n}$£; B takes the n^{th} part of the remainder together with $\frac{a}{n}$£; C takes the n^{th} part of what now remains together with $\frac{a}{n}$£; and then nothing remains. Find the sum.

Ans. $\frac{3n^2-3n+1}{(n-1)^3}.a$£.

(63) Find two numbers whose product is equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes. (*App.* p. 353.)

Ans. $\frac{1}{2}\sqrt{5}$, and $\frac{1}{4}(5+\sqrt{5})$.

(64) A pack of p cards is distributed into n heaps, so that the number of pips on the lowest cards, together with the number of cards laid upon them, is the same number m for each heap, and the number of cards remaining is found to be r; required the number of pips on all the lowest cards. (*Comp.* p. 70.) Ans. $(m+1)n+r-p$.

(65) If a men or b boys can dig m acres in n days, find the number of boys whose assistance will be required to enable $\overline{a-p}$ men to dig $\overline{m+p}$ acres in $\overline{n-p}$ days. (*Comp.* p. 70.)

Ans. $\frac{pb}{a}\left(1+\frac{m+n}{n-p}.\frac{a}{m}\right)$.

(66) Supposing the sum of 51 cards in a common pack to be $10n+a$, (where $a<10$), prove the value of the last card to be $10-a$, the court-cards reckoning for 10, and the others for as much as is the number of pips upon each. Find also the value of n. (*Comp.* p. 71.) Ans. $n = 33$.

(67) If a oxen in m weeks eat b acres of grass, and c oxen eat d acres in n weeks, how many oxen will eat e acres in p weeks, supposing the grass to grow uniformly? (*Comp.* p. 72.)

Ans. $\left\{\frac{m-p}{m-n}.\frac{nc}{d}-\frac{n-p}{m-n}.\frac{ma}{b}\right\}\frac{e}{p}$.

(68) The distance between two places is a, and on the first day $\frac{1}{m}$th of the journey from one to the other is performed; on the 2nd day $\frac{1}{n}$th

of the remainder; then $\frac{1}{m}$th and $\frac{1}{n}$th of the remainders alternately on succeeding days. Find the distance gone over in $2p$ days. (*Comp.* p. 73.)

Ans. $a\left\{1-\left(1-\frac{1}{m}\right)^p\cdot\left(1-\frac{1}{n}\right)^p\right\}$.

(69) Two labourers A and B, whose rates of working are as 3 to 5, were employed to dig a ditch: A worked 12 hours and B 10 hours a day: B being called away, A worked one day alone in order to complete the work: when they were paid, B received as many pence more than A as the number of days they worked together. Now had B been called away a day sooner, A would have received 3*s.* 11*d.* more than B at the conclusion of the work. What are their respective daily wages on supposition that each is paid in proportion to the work performed? (*App.* p. 354.)

Ans. A's daily wages 1*s.* 6*d.* B's 2*s.* 1*d.*

(70) To complete a certain work A requires m times as long a time as B and C together; B requires n times as long as A and C together; and C requires p times as long as A and B together. Compare the times in which each would do it, and prove that

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1. \quad (\textit{Comp. } \text{p. } 73.)$$

(71) $S_1, S_2, S_3, \ldots S_{n+1}$ are $\overline{n+1}$ stones placed in a straight line a yard from each other, and X is another assumed station in the same line produced; two persons set out from S_1, the one to carry the stones separately to S_1, and the other to X; find the distance from S_{n+1} to X, that the latter may travel exactly twice as far as the former. (*Comp.* p. 74.)

Ans. $\frac{n(n+2)}{2n+1}$.

(72) A steam-boat sets out from London 3 miles behind a wherry, and having got to the same distance a-head it overtakes a barge *floating* down the stream, and reaches Gravesend $1\frac{1}{2}$ hours afterwards. Having waited to land the passengers $\frac{1}{5}$th of the time of coming down, it starts to return, and meets the wherry in $\frac{3}{4}$ of an hour, the barge being then $5\frac{1}{4}$ miles a-head of the steam-boat, and arrives at London in the same time that the wherry was in coming down. Find the distance between London and Gravesend, and the rate of each vessel. (*App.* p. 356.)

(1) Ans. 30 miles. (2) Ans. 9, and 3, miles per hour.

(73) Two clocks are striking the hour together, and are heard to strike 19 times. There is a difference of two seconds in their time, and one strikes every three, the other every four, seconds. What is the hour they strike? it being observed that, when the clocks strike in the same second, the sounds cannot be distinguished, so as to determine whether one or both strike in that second, and that this is the case with the last stroke of the faster clock. (*Comp.* p. 75.) Ans. 11 o'clock.

(74) A and B travelled on the same road and at the same rate to London. At the 50th mile-stone from London A overtook a flock of geese,

which travelled at the rate of 3 miles in 2 hours; and 2 hours afterwards he met a stage-waggon which travelled at the rate of 9 miles in 4 hours. *B* overtook the flock of geese at the 45th mile-stone from London, and met the stage-waggon 40 minutes before he came to the 31st mile-stone. Where was *B*, when *A* reached London? (*App.* p. 357.)

Ans. 25 miles from London.

(75) The hold of a vessel partly full of water (which is uniformly increased by a leak) is furnished with two pumps worked by *A* and *B*, of whom *A* takes three strokes to two of *B*'s, but four of *B*'s throw out as much water as five of *A*'s. Now *B* works for the time in which *A* alone would have emptied the hold. *A* then pumps out the remainder, and the hold is cleared in 13 hrs. 20 min. Had they worked together, the hold would have been emptied in 3 hrs. 45 min., and *A* would have pumped out 100 gallons more than he did. Required the quantity of water in the hold at first, and the horary influx at the leak. (*Comp.* p. 76.)

Ans. Quantity in the hold 1200 gallons.
Horary influx 120 gallons.

INEQUALITIES.

(1) If $4x-7<2x+3$, and $3x+1>13-x$, find the integral value of x.

Ans. $x=4$.

(2) What is the integral value of x, when $\frac{1}{4}(x+2)+\frac{1}{3}x<\frac{1}{2}(x-4)+3$, and $>\frac{1}{2}(x+1)+\frac{1}{3}$?

Ans. $x=5$.

(3) Which is greater $x-y$, or $(\sqrt{x}-\sqrt{y})^2$? Ans. The former, if $x>y$.

(4) Shew that $\frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{a}+\frac{1}{b}$, if $a+b$ be positive. (*Comp.* p. 78.) ✓

(5) Which is greater $\frac{1}{a+\frac{1}{b}}$, or $\frac{1}{a+\frac{1}{b+\frac{1}{c}}}$?

Ans. The latter.

(6) Shew that $\sqrt{a^2-b^2}+\sqrt{a^2-(a-b)^2}>a$, if $a>b$. (*Comp.* p. 78.) ✓

(7) If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, which is greater, xy, or $ac+bd$? (*Comp.* p. 79.)

Ans. xy.

✓(8) Which is greater, n^3+1, or n^2+n? (*Comp.* p. 79.) ✓

Ans. n^3+1, unless $n=1$.

(9) Which is greater, $3(1+a^2+a^4)$, or $(1+a+a^2)^2$? (*Comp.* p. 79.)

Ans. The former, unless $a=1$.

(10) Which is greater, $2(1+a^2+a^4)$, or $3(a+a^3)$?

Ans. The former, unless $a = 1$.

(11) Shew that $b\left(\frac{a^3}{b^3}+\frac{b}{a}+2\right) >$ or $< a\left(\frac{b^3}{a^3}+\frac{a}{b}+2\right)$, according as $a >$ or $< b$. (*Comp.* p. 79.)

(12) Shew that $\frac{n^2-n+1}{n^2+n+1}$ lies between 3 and $\frac{1}{3}$ for all real values of n. (*Comp.* p. 80.)

(13) Shew that $abc > (a+b-c)(a+c-b)(b+c-a)$, unless $a = b = c$. (*Comp.* p. 80.)

(14) Shew that $abc > (2a-b)(2b-c)(2c-a)$, unless $a = b = c$, each of the factors being *positive.* (*Comp.* p. 81.)

(15) Shew that $ab(a+b)+ac(a+c)+bc(b+c)$ is between $6abc$, and $2(a^3+b^3+c^3)$; a, b, c, being positive quantities. (*Comp.* p. 82.)

(16) Shew that $(a+b+c)^3 > 27abc$, and $< 9(a^3+b^3+c^3)$, unless $a = b = c$. (*Comp.* p. 82.)

(17) If $a<x$, shew that $(x+a)^3-x^3<7ax^2$. (*Comp.* p. 83.)

(18) Shew that $\sqrt[3]{n} > \sqrt[4]{n+1}$, for all values of n not less than 3. (*Comp.* p. 83.)

(19) Shew that $(a^2+b^2+1)(c^2+d^2+1) > (ac+bd+1)^2$, unless $a = c$, and $b = d$. (*Comp.* p. 83.)

(20) Shew that $\frac{1}{4}(a+b+c+d) > \sqrt[4]{abcd}$, unless $a = b = c = d$. (*Comp.* p. 84.)

(21) If a_1, a_2, a_3, ...a_n, be positive quantities, shew that

$$\frac{n-1}{2}(a_1+a_2+a_3+\ldots+a_n) > \sqrt{a_1a_2}+\sqrt{a_1a_3}+\sqrt{a_2a_3}+\&c.$$ (*Comp.* p. 84.)

(22) Shew that $\frac{1}{a^n+a^{-n}-1} < 1$, for all *real* values of a. (*Comp.* p. 85.)

(23) The double of a certain number increased by 7 is not greater than 19, and its triple diminished by 5 is not less than 13. What is the number?

Ans. 6.

RATIOS, PROPORTION, AND VARIATION.

(1) WHICH is greater, $3 : 5$, or $5 : 8$? $\frac{24, 25}{40}$ Ans. The latter.

(2) Prove that $a : b$ is a greater ratio than $ax : bx+y$, and a less ratio than $ax : bx-y$, if y be positive.

(3) Prove that $a^3+b^3 : a^2+b^2 > a^2+b^2 : a+b$, unless $a = b$.

(4) Find the ratio *compounded* of $a : x$, $x : y$, and $y : b$. Ans. $a : b$.

(5) Find the ratio compounded of $x+a : x+b$, and $a(x+b) : b(x+a)$. Ans. $a : b$.

(6) What quantity must be added to each of the terms of the ratio $a : b$, that it may become the ratio $c : d$? Ans. $\frac{ad-bc}{c-d}$.

(7) Prove that, if $a : b$ is a greater ratio than $c : d$, $a+c : b+d$ is a less ratio than $a : b$, but a greater than $c : d$. (*Comp.* p. 85.)

(8) What is the proportion deducible from the equation $ab = a^2-x^2$? Ans. $a : a+x :: a-x : b$.

(9) What is the proportion deducible from the equation $x^2+y^2 = 2ax$? Ans. $x : y :: y : 2a-x$.

(10) Four given numbers are represented by a, b, c, d; required the quantity which added to each will make them proportionals. Ans. $\frac{bc-ad}{a-b-c+d}$.

(11) If four numbers be proportionals, shew that there is no number which, being added to each, will leave the resulting four numbers proportionals. ∵ by prev. ansʳ $bc = ad$ ∴ $x = 0$.

(12) If $a : b = c : d$, shew that $\frac{2a+3b}{4a+5b} = \frac{2c+3d}{4c+5d}$. NB ✓

(13) If four quantities of the same kind be proportionals, prove that the greatest and least together are greater than the other two together.

(14) If $(a+b)^2 : (a-b)^2 :: b+c : b-c$, shew that $a : b :: \sqrt{2a-c} : \sqrt{c}$.

(15) If $x : y :: a^3 : b^3$, and $a : b :: \sqrt[3]{c+x} : \sqrt[3]{d+y}$, shew that $cy = dx$.

(16) If $a : b :: c : d$, shew that $a(a+b+c+d) = (a+b)(a+c)$.

(17) If $a : b :: c : d$, shew that $\sqrt{a-b} : \sqrt{c-d} :: \sqrt{a}-\sqrt{b} : \sqrt{c}-\sqrt{d} :: \sqrt{a}+\sqrt{b} : \sqrt{c}+\sqrt{d}$.

(18) If $(a+b+c+d)(a-b-c+d) = (a-b+c-d)(a+b-c-d)$, shew that $a : b :: c : d$.

(19) If $a : b :: c : d$, shew that

$$\frac{1}{ma}+\frac{1}{nb}+\frac{1}{pc}+\frac{1}{qd} = \frac{1}{bc}.\left\{\frac{a}{q}+\frac{b}{p}+\frac{c}{n}+\frac{d}{m}\right\}.$$ (*Comp.* p. 86.)

(20) If $a_1 : b_1 :: a_2 : b_2 :: a_3 : b_3 ::$ &c., shew that (*Comp.* p. 86.)

$$(a_1^2+a_2^2+a_3^2+\&c.)(b_1^2+b_2^2+b_3^2+\&c.) = (a_1b_1+a_2b_2+a_3b_3+\&c.)^2,$$

and $$\sqrt{a_1+a_2+a_3+\&c.)(b_1+b_2+b_3+\&c.)} = \sqrt{a_1b_1}+\sqrt{a_2b_2}+\sqrt{a_3b_3}+\&c.$$

(21) If the difference between a and b be small when compared witl either of them, shew that the ratio $\sqrt[m]{a}-\sqrt[m]{b} : \sqrt[n]{a}-\sqrt[n]{b}$ is nearly equal tc $n\sqrt[m]{a} : m\sqrt[n]{a}$. (*Comp.* p. 86.)

(22) If $\dfrac{\sqrt[n]{x}-m\sqrt[n]{y}}{\sqrt[n]{x}+m\sqrt[n]{y}}=\dfrac{\sqrt[2n]{x}-m\sqrt[2n]{x-y}}{\sqrt[2n]{x}+m\sqrt[2n]{x-y}}$, shew that $x : y :: 1\pm\sqrt{5} : 2$. (*Comp.* p. 87.)

(23) Find the number to which if 1 and 3 be successively added, the resulting numbers are in the proportion of 2 : 7. Ans. $-\frac{1}{5}$.

$x+1 : x+3 :: 2 : 7$

(24) Find two numbers in the ratio 3 : 4, and of which the sum : the sum of their squares :: 7 : 50. Ans. 6, and 8.

(25) Distribute s soldiers among t towns in proportion to their respective populations $p_1, p_2, p_3, \dots p_t$.

Ans. $\frac{p_1}{P}s, \quad \frac{p_2}{P}s, \quad \frac{p_3}{P}s, \dots \frac{p_t}{P}s$, where $P = p_1+p_2+p_3+\dots+p_t$.

(26) If m shillings in a row reach as far as n sovereigns, and a pile of p shillings be as high as a pile of q sovereigns, compare the values of equal bulks of gold and silver. (*Comp.* p. 87.)

Ans. Val. of gold : val. of silver :: $20n^2q : m^2p$.

(27) A person in a railway carriage observes that another train running on a parallel line in the opposite direction occupies 2 seconds in passing him—but, if the two trains had been proceeding in the same direction, it would have taken 30 seconds to pass him. Compare the speeds of the two trains. (*Comp.* p. 88.) Ans. 7 : 8.

(28) A person, having travelled 56 miles on a railway and the rest of his journey by a coach, observed that in the train he had performed one-fourth of his whole journey in the time the coach took to go 5 miles, and that at the instant he arrived at home the train would be 35 miles farther than he was from the station at which it left him. Compare the rates of the coach and train. (*Comp.* p. 88.) Ans. 1 : $3\frac{1}{2}$.

(29) Given that $y \propto x$, and when $x=2$, $y=10$, required the equation between x and y. Ans. $y = 5x$.

(30) Given that $y^2 \propto a^2-x^2$; and when $x=\sqrt{a^2-b^2}$, $y=\frac{b^2}{a}$; find the equation between x and y. Ans. $y=\frac{b}{a}\sqrt{a^2-x^2}$.

(31) Given that $s \propto t^2$, when f is constant; and $s \propto f$, when t is constant: also $2s=f$, when $t=1$. Find the equation between f, s, and t.

Ans. $s=\frac{1}{2}ft^2$.

(32) Given that y is equal to the sum of two quantities, one of which varies as x, and the other varies inversely as x^2; and when $x = 1, 2$, $y = 6, 5$, respectively. Find the equation between x and y. Ans. $y = 2x + \frac{4}{x^2}$.

(33) Given that y is equal to the sum of three quantities, the 1st of which is invariable, the 2nd varies as x, and the third varies as x^2. Also when $x = 1, 2, 3$, $y = 6, 11, 18$, respectively. Express y in terms of x. Ans. $y = 3 + 2x + x^2$.

(34) Given that $x \propto \frac{1}{y^m}$, and $y \propto \frac{1}{z^n}$; also when $x = a$, $z = c$; find the equation between x and z. (*Comp.* p. 89.) Ans. $az^{mn} = c^{mn}x$.

(35) If $y = r + s$, whilst $r \propto x$, and $s \propto \sqrt{x}$; and if, when $x = 4$, $y = 5$, and when $x = 9$, $y = 10$; shew that $6y = 5(x + \sqrt{x})$. (*Comp.* p. 89.)

(36) If $a + b \propto a - b$, prove that $a^2 + b^2 \propto ab$; and if $a \propto b$, prove that $a^2 - b^2 \propto ab$.

(37) If $\frac{x}{y} \propto x + y$, and $\frac{y}{x} \propto x - y$, shew that $x^2 - y^2$ is invariable.

(38) If $ax + by = cx + dy$, prove that $x \propto y$.

(39) If $a \propto b$, and $b \propto c$, shew that $(a^2 + b^2)^{\frac{3}{2}} \propto c^3$.

(40) If $A \propto B$, and $B \propto C$, shew that
$mA + nB + pC \propto m'\sqrt{AB} + n'\sqrt{BC} + p'\sqrt{AC}$. (*Comp.* p. 90.)

(41) If $A \propto B$, and $B \propto \sqrt{AC}$, shew that $C \propto \sqrt[3]{A^2B} + \sqrt[3]{AB^2}$, and that
$m\sqrt[4]{AB} - n\sqrt[4]{BC} \propto p\sqrt{A} + q\sqrt{B}$. (*Comp.* p. 90.)

(42) If $mA + nB \propto pA - qB$, and $m'A - n'C \propto p'B + q'C$; shew that
$(\alpha_1\sqrt{A} - \beta_1\sqrt{B} + \gamma_1\sqrt{C})^2 \propto \alpha_2 A + \beta_2 B + \gamma_2 C$. (*Comp.* p. 91.)

(43) A sphere of metal is known to have a hollow space about its centre in the form of a concentric sphere, and its weight is $\frac{7}{8}$ of the weight of a solid sphere of the same substance and radius; compare the inner and outer radii, having given that the weights of spheres of the same substance $\propto (\text{radii})^3$. (*Comp.* p. 92.) Ans. 1 : 2.

(44) Two globes of gold whose radii are r, r', are melted, and formed into a single globe: find its radius, having given that the volume of a globe $\propto (\text{radius})^3$. Ans. $\sqrt[3]{r^3 + r'^3}$.

(45) A locomotive engine without a train can go 24 miles an hour, and its speed is diminished by a quantity which varies as the square root of the number of waggons attached. With four waggons its speed is 20 miles an hour. Find the greatest number of waggons which the engine can move. (*Comp.* p. 92.) Ans. 143.

(46) The value of diamonds $\propto$ the square of their weight, and the square of the value of rubies $\propto$ the cube of their weight. A diamond of a carats is worth m times the value of a ruby of b carats, and both together are worth c£. Required the values of a diamond and of a ruby, each weighing n carats. (*Comp.* p. 92.)

Ans. Value of diamond $= \dfrac{mcn^2}{(m+1)a^2}$; value of ruby $= \dfrac{cn^{\frac{3}{2}}}{(m+1)b^{\frac{3}{2}}}$.

(47) If the prices of two trees containing p and q cubic feet of timber be a£ and b£, required the price of a tree containing r cubic feet, supposing the values of the timber and bark to be respectively proportional to the m^{th} and n^{th} powers of the quantity of timber in the tree. (*Comp.* p. 93.)

Ans. $\dfrac{(bp^n-aq^n)r^m+(aq^m-bp^m)r^n}{p^nq^m-p^mq^n}$.

ARITHMETICAL PROGRESSION.

(1) Find the 64^{th} term of the series 4, $6\frac{1}{2}$, 9, &c. Ans. $161\frac{1}{2}$.

(2) Find the 30^{th} term of the series -27, -20, -13, &c. Ans. 176.

(3) Find the sum of 50 terms in A.P. of which the first is 3, and the ast 199. Ans. 5050.

(4) Find the sum of 100 consecutive whole numbers beginning rom 1. Ans. 5050.

(5) Find the 7^{th} term, and the sum of 7 terms, of the series $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, &c.

(1) Ans. $-\frac{1}{2}$, (2) Ans. 0.

(6) Find the 6^{th} term, and the sum of 6 terms, of the series $\frac{2}{3}, \frac{7}{15}, \frac{4}{15}$, &c.

(1) Ans. $-\frac{1}{3}$, (2) Ans. 1.

(7) Find the n^{th} term of $1+3+5+7+$&c. Ans. $2n-1$.

and of $2+2\frac{1}{3}+2\frac{2}{3}+$&c. Ans. $\frac{1}{3}(n+5)$.

and of $13+12\frac{2}{3}+12\frac{1}{3}+$&c. Ans. $\frac{1}{3}(40-n)$.

(8) Sum the following series:—

(1) $1+2+3+4+\ldots$ to n terms. Ans. $n.\dfrac{n+1}{2}$.

(2) $1+3+5+7+\ldots$ to n terms. Ans. n^2.

(3) $2+2\frac{1}{3}+2\frac{2}{3}+\ldots$ to n terms. Ans. $\frac{n}{6}(n+11)$.

(4) $5+4\frac{3}{4}+4\frac{1}{2}+$&c. to 21 terms. 21st term is 0. Ans. $52\frac{1}{2}$.

(5) $13+12\frac{2}{3}+12\frac{1}{3}+$&c. to n terms. nth term $\frac{40-n}{3}$ Ans. $\frac{n}{6}(79-n)$.

(6) $2\frac{1}{2}+2\frac{5}{6}+3\frac{1}{6}+$&c. to 13 terms. Ans. $58\frac{1}{2}$.

(7) $\frac{1}{2}-\frac{2}{3}-\frac{11}{6}-$&c. to n terms. Ans. $\frac{n}{12}(13-7n)$.

(8) $\frac{1}{4}+\frac{3}{8}+\frac{1}{2}+$&c. to n terms. Ans. $\frac{n}{16}(n+3)$.

(9) $\frac{5}{7}+1+1\frac{2}{7}+$&c. to n terms. Ans. $\frac{n}{7}(n+4)$.

(10) $\frac{1}{3}+\frac{5}{6}+\frac{4}{3}+$&c. to n terms. Ans. $\frac{n}{12}(3n+1)$.

(11) $-9-7-5-$&c. to 20 terms. Ans. 200.

(12) $\frac{8}{9}+\frac{29}{63}+\frac{2}{63}+$&c. to 7 terms. Ans. $-2\frac{7}{9}$.

(13) $\frac{5}{7}+\frac{2}{9}+$&c. to 101 terms. Ans. $-2412\frac{7}{9}$.

(14) $2\frac{1}{3}+3\frac{5}{24}+4\frac{1}{12}+$&c. to 24 terms. Ans. $297\frac{1}{2}$.

(15) $\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+$&c. to n terms. Ans. $\frac{n}{a+b}.\left\{na-\frac{n+1}{2}b\right\}$.

(16) $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+$&c. to n terms. kth T. = 0. Ans. $\frac{n-1}{2}$.

(17) $\left(\frac{1}{a}-\frac{n}{x}\right)+\left(\frac{1}{a}-\frac{n-1}{x}\right)+\left(\frac{1}{a}-\frac{n-2}{x}\right)+$&c. to n terms.

Ans. $\frac{n}{a}-n.\frac{n+1}{2}.\frac{1}{x}$.

(18) $(s+a)+(s+a+d)+(s+a+2d)+$...................... to n terms, where $\{2a+(n-1)d\}\frac{n}{2}$. Ans. $(n+1)s$.

(9) Prove the following forms in Arithmetical Progression:

(1) $a=\frac{b}{2}\pm\sqrt{\left(l+\frac{b}{2}\right)^2-2bs}$;

(2) $b=2.\left\{\frac{l}{n-1}-\frac{s}{n(n-1)}\right\}$; (3) $b=\frac{(l+a)(l-a)}{2s-(l+a)}$;

(4) $s=\frac{l+a}{2b}(l-a+b)$; (5) $n=\frac{l-a}{b}+1$.

(10) Given the first term (a) of a series in Arithmetical Progression, the common difference (b), and (s) the sum of the series to n terms; find n.

$$\text{Ans. } n = \frac{1}{2} - \frac{a}{b} \pm \sqrt{\frac{2s}{b} + \left(\frac{1}{2} - \frac{a}{b}\right)^2}.$$

Explain the meaning of the two signs in the value of n.

(11) How many terms of the series 19, 18, 17, &c. amount to 124?

Ans. 8, or 31.

(12) The sum of a series in Arithmetical Progression is 72, the first term 17, and the common difference -2; find the number of terms, and explain the double answer. Ans. 6, or 12.

(13) Given $s = 40$, $a = 7$, and $b = 2$. Find n. Ans. $n = 4$, or -10.

(14) In any Arith^c. Prog^n. of which a is the first term, and $2a$ the common difference, prove that the number of terms, which must be taken to make a sum s, is $\sqrt{\frac{s}{a}}$, s being assumed such that $\frac{s}{a}$ is *any* square number, but no other.

(15) Find the series in A.P. in which 7 and 5 are the 5th and 7th terms respectively. (*Comp.* p. 94.) Ans. 11, 10, 9, 8, 7, 6, 5, &c.

(16) Insert 40 Arith^c. Means between 0 and 20.

Ans. $\frac{20}{41}$, $\frac{40}{41}$, $1\frac{19}{41}$, $1\frac{39}{41}$, &c. ......$19\frac{21}{41}$.

(17) Insert 15 Arith^c. Means between 3 and 47.

Ans. $5\frac{3}{4}$, $8\frac{1}{2}$, $11\frac{1}{4}$, &c.......$44\frac{1}{4}$.

(18) If the Arith^c. Mean between a and b be twice as great as the Geom^c. Mean, shew that $a : b :: 2+\sqrt{3} \,.\, 2-\sqrt{3}$. (*Comp.* p. 94.)

(19) If the Arith^c. Mean between a and b be m times as great as the Harmonic Mean, shew that $a : b :: \sqrt{m}+\sqrt{m-1} : \sqrt{m}-\sqrt{m-1}$. (*Comp.* p. 95.)

(20) Shew that if the same number of Arith^c. Means be inserted between every two contiguous terms of an A.P. the whole will be in A.P. (*Comp.* p. 95.)

(21) Find the series in A.P. having 29 terms, of which the first is 3 and the last 17. Ans. 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, &c....17.

244 (22) Find the series in A.P. of $\overline{n+7}$ terms of which the sum of the first n terms is 40, the sum of the next 4 is 86, and that of the last 3 is 96.

Ans. 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35.

(23) The first two terms of an A.P. are $189\frac{1}{3}$ and $103\frac{3}{4}$; and the sum of all the terms is $-147\frac{3}{4}$; what is the last term, and what the number of terms?

(1) Ans. $-238\frac{7}{12}$. (2) Ans. 6.

(24) Divide $\frac{n}{7}(n+4)$ into n parts, such that each part shall exceed the one immediately preceding by a fixed quantity. (*Comp.* p. 96.)

Ans. $\frac{5}{7}$, 1, $1\frac{2}{7}$, &c.

(25) The n^{th} term of an Arith^c^. Prog^n^. is $\frac{1}{6}(3n-1)$, prove that the sum of n terms is $\frac{n}{12}(3n+1)$, and find the series. (*Comp.* p. 96.)

Ans. $\frac{1}{3}, \frac{5}{6}, \frac{4}{3}, \frac{11}{6}$, &c.

(26) The sum of the first n terms of a series in Arith^c^. Prog^n^. is $\left(na-\frac{n+1}{2}.b\right)\frac{n}{a+b}$, find the series. Ans. $\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+\frac{5a-3b}{a+b}+$&c.

(27) There are two series in Arith^c^. Prog^n^., the sums of which to n terms are as $13-7n : 3n+1$; prove that their first terms are as $3 : 2$, and their second terms as $-4 : 5$. (*Comp.* p. 97.)

(28) The $\overline{n+1}^{\text{th}}$ term of a series in Arith^c^. Prog^n^. is $\frac{ma-nb}{a-b}$, required the sum of the series to $\overline{2n+1}$ terms. Ans. $\frac{ma-nb}{a-b}.(2n+1)$.

(29) Shew that the sum of the $\overline{m-n}^{\text{th}}$ and $\overline{m+n}^{\text{th}}$ terms of any Arith^c^. Prog^n^. is equal to twice the m^{th} term.

(30) The sum of a certain number of terms of the series 21, 19, 17, &c. is 120. Find the last term, and the number of terms.

$=\{42-(n-1)2\}\frac{n}{2}$; $n^2-22n+120=0$
$(n-10)(n-12)=0$ (1) Ans. 3, or -1. (2) Ans. 10, or 12.

(31) In the two series 127, 120, 113, ... 1, and 2, 5, 8, ... 56, shew that the number of terms is the same in both, and find the number.

$\left.\begin{array}{l}127+(n-1)\times-7=1\\ 2+(n-1)3=56\end{array}\right\}n=19$ in both cases. Ans. 19.

(32) Determine the relation between a, b, and c, that they may be the p^{th}, q^{th}, and r^{th} terms of an Arithmetical Progression. (*Comp.* p. 97.)

Ans. $(q-r)a+(r-p)b+(p-q)c=0$.

(33) In an Arithmetic Progression, if the $\overline{p+q}|^{\text{th}}$ term $=m$, and the $\overline{p-q}|^{\text{th}}$ term $=n$, shew that the p^{th} term $=\frac{1}{2}(m+n)$, and the q^{th} term $=m-(m-n)\frac{p}{2q}$. (*Comp.* p. 97.)

(34) If the m^{th} term of an Arithmetic Progression $= n$, and the n^{th} term $= m$, of how many terms will the sum be $\frac{1}{2}(m+n)(m+n-1)$, and wha will be the last of them? (*Comp.* p. 98.)

(1) Ans. $m+n$, or $m+n-1$. (2) Ans. 0, or 1.

(35) The sum of m terms of an Arith$^{\text{c}}$. Prog$^{\text{n}}$. is n, and the sum of n terms is m; shew that the sum of $\overline{m+n}$ terms is $-\overline{m+n}$, and the sum o $\overline{m-n}$ terms is $(m-n)\left(1+\frac{2n}{m}\right)$. (*Comp.* p. 99.)

(36) In an Arithmetic Progression a is the first term, b the common difference, and S_n the sum of n terms, prove that (*Comp.* p. 100.)

$$S_n+S_{n+1}+S_{n+2}+\&\text{c. to } n \text{ terms} = (3n-1)\frac{na}{2}+(7n-2)(n-1)\frac{nb}{6}.$$

(37) $S_1, S_2, S_3, \ldots\ldots S_p$ are the sums of p Arithmetic Progressions to n terms; the first terms are 1, 2, 3, &c. and the common differences 1, 3, 5, 7, &c. Shew that $S_1+S_2+S_3+\ldots+S_p = (np+1)\frac{np}{2}$. (*Comp.* p. 101.)

(38) How far does a person travel in gathering up 200 stones placed in a straight line at intervals of 2 feet from each other—supposing that he fetches each stone singly and deposits it in a basket which is in the same line produced 20 yards distant from the nearest stone—and that he starts from the basket? Ans. $19\frac{41}{66}$ miles.

(39) In the two series, 2, 5, 8, &c. and 3, 7, 11, &c. each continued to 100 terms, find how many terms are identical. (*Comp.* p. 101.) Ans. 25.

GEOMETRICAL AND HARMONICAL PROGRESSION.

(1) FIND the 12^{th} term of the series 30, 15, $7\frac{1}{2}$, &c. Ans. $\frac{15}{1024}$.

(2) The first two terms of a series in Geom$^{\text{c}}$. Prog$^{\text{n}}$. are $\frac{3}{7}$ and $\frac{8}{9}$; find the Common Ratio, and the third term. (1) Ans. $2\frac{2}{27}$. (2) Ans. $1\frac{205}{243}$.

(3) Find the Common Ratio and the fifth term of $3\frac{3}{8}$, $2\frac{1}{4}$, $1\frac{1}{2}$, &c.

(1) Ans. $\frac{2}{3}$. (2) Ans. $\frac{2}{3}$.

(4) Sum the following series:

(1) $1-2+4-8+\&\text{c.}$ to n terms. Ans. $\frac{1}{3}\{1-(-2)^n\}$.

(2) $\frac{1}{5}-\frac{2}{15}+\frac{4}{45}-\&\text{c.}$ to n terms. Ans. $\frac{3}{25}\left\{1-\left(-\frac{2}{3}\right)^n\right\}$.

(3) $1-\frac{1}{4}+\frac{1}{16}-$&c. to n terms. Ans. $\frac{4}{5}\left\{1-\left(-\frac{1}{4}\right)^n\right\}$.

(4) $\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+$&c. to n terms. Ans. $\frac{2}{3}\left\{\frac{3^n}{2^n}-1\right\}$.

(5) $5+\frac{15}{7}+\frac{45}{49}+\ldots$ *in inf.* Ans. $8\frac{3}{4}$.

(6) $\frac{5}{3}+1+\frac{3}{5}+\ldots$ *in inf.* Ans. $4\frac{1}{6}$.

(7) $\frac{2}{3}-\frac{1}{2}+\frac{3}{8}+\ldots$ *in inf.* Ans. $\frac{8}{21}$.

(8) $2+\sqrt[4]{8}+\sqrt{2}+$&c. to 12 terms. Ans. $\frac{7}{4-2\sqrt[4]{8}}$.

(9) $\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{1}{2-\sqrt{2}}+\frac{1}{2}+\ldots$ *in inf.* Ans. $4+3\sqrt{2}$.

(10) $a^p+a^{p+q}+a^{p+2q}+$&c. to n terms. Ans. $a^p.\frac{a^{nq}-1}{a^q-1}$.

(11) $x-y+\frac{y^2}{x}-\frac{y^3}{x^2}+$&c. to n terms. Ans. $\frac{x^2}{x+y}.\left\{1-\left(-\frac{y}{x}\right)^n\right\}$.

(12) $\frac{a}{x}\sqrt{\frac{3}{2}}+\sqrt{\frac{a}{x}}+\sqrt{\frac{2}{3}}+$&c....... Ans. $\frac{1}{6^{\frac{1}{2}}.(3a)^{\frac{n-3}{2}}x}.\frac{(2x)^{\frac{n}{2}}-(3a)^{\frac{n}{2}}}{(2x)^{\frac{1}{2}}-(3a)^{\frac{1}{2}}}$.

(5) Prove the following forms in Geometric Progression :—

(1) $l(s-l)^{n-1}-a(s-a)^{n-1}=0$, (2) $r=\frac{s-a}{s-l}$,

(3) $s=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}$, (4) $r^n-\frac{s}{s-l}r^{n-1}+\frac{l}{s-l}=0$.

(6) Of each of the following series find the n^{th} term, and the sum of n terms :—

(1) $1+5+13+29+61+$&c. (1) Ans. $2^{n+1}-3$. (2) Ans. $4(2^n-1)-3n$.

(2) $2+6+14+30+$&c. (1) Ans. $2^{n+1}-2$. (2) Ans. $4(2^n-1)-2n$.

(3) $1+3+7+15+31+$&c. (1) Ans. 2^n-1. (2) Ans. $2(2^n-1)-n$.

(4) $3+6+11+20+$&c. (1) Ans. 2^n+n. (2) Ans. $2^{n+1}+\frac{1}{2}\{n^2+n-4\}$.

(5) $\frac{1}{1}+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+$&c. (1) Ans. $\frac{2^n-1}{2^{n-1}}$. (2) Ans. $2(n-1)+\frac{1}{2^{n-1}}$.

(6) $2+4+\frac{14}{3}+\frac{44}{9}+\frac{134}{27}+$&c. (1) Ans. $5-\frac{1}{3^{n-2}}$. (2) Ans. $5n-\frac{9}{2}\left(1-\frac{1}{3^n}\right)$.

(7) Sum $(s-a)+(s-\overline{a+ar})+(s-\overline{a+ar+ar^2})+\&c.$ to n terms, and *in inf.*, where $s=a.\frac{r^n-1}{r-1}$. (*Comp.* p. 102.)

(1) Ans. $\frac{nar^n}{r-1}-\frac{ar}{(r-1)^2}(r^n-1)$. (2) Ans. $\frac{ar}{(1-r)^2}$.

(8) Find the sum of $(1^3-1)+(2^3-2)+(3^3-3)+\&c.$ to n terms.

Ans. $\frac{1}{4}(n-1)n(n+1)(n+2)$.

(9) Given $x-1+2(x-2)+3(x-3)+\&c.$ to 6 terms $=14$; find x.

Ans. $x=5$.

(10) Find the sum of $1+2^2+3+4^2+5+6^2+\&c.$ to x terms, when x is an odd number.

Ans. $\frac{1}{12}(x+1)(2x^2+x+3)$.

(11) The first term of a Geometric Series, continued *in inf.*, is 1, and any term is equal to the sum of all the succeeding terms; find the series.

Ans. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots$

(12) Insert three Geometric Means between $\frac{1}{9}$ and 9. Ans. $\frac{1}{3}$, 1, 3.

(13) Insert seven Geometric Means between 2 and 13122.

Ans. 6, 18, 54, 162, 486, 1458, 4374.

(14) Insert two Geometric Means between $\frac{16}{27}$ and 2. Ans. $\frac{8}{9}$, $\frac{4}{3}$.

(15) The sum of an infinite Geometric Series is 3, and the sum of its first two terms is $2\frac{2}{3}$; find the series.

Ans. $2+\frac{2}{3}+\frac{2}{9}+\ldots$, or $4-\frac{4}{3}+\frac{4}{9}-\ldots\ldots$

(16) There are four numbers in A.P., which being increased respectively by 1, 1, 3, and 9, are in G.P.; what are the numbers?

Ans. 1, 3, 5, 7.

(17) In a Geometric Progression, if the $\overline{p+q}^{\text{th}}$ term $=m$, and the $\overline{p-q}^{\text{th}}$ term $=n$, shew that the p^{th} term $=\sqrt{mn}$, and the q^{th} term $=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$. Also, if P be the p^{th} term, and Q the q^{th} term, shew that the n^{th} term $=\left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$. (*Comp.* p. 102.)

(18) Find the relation between a, b, and c, that they may be the p^{th}, q^{th}, and r^{th} terms of a Geometric series. (*Comp.* p. 103.)

Ans. $a^{q-r}b^{r-p}c^{p-q}=1$.

(19) Required the sum of the first p terms of the series whose n^{th} term is $na+a^n$.
Ans. $\frac{1}{2}p(p+1)a+a.\frac{a^p-1}{a-1}$.

(20) Given $\frac{2}{3}$, $\frac{5}{7}$, the first two terms of an A.P., find the sum of 15 terms; and if the same quantities be the first two terms of a G.P. find the sum of 15 terms.
(1) Ans. 15. (2) Ans. $\frac{28}{3}.\left\{\left|\frac{15}{14}\right|^{15}-1\right\}$.

(21) If a, b, c, d are quantities in G.P., prove that $a^2+b^2+c^2>(a-b+c)^2$; and that $(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2$. (*Comp.* p. 103.)

(22) If there be any number of quantities in G.P., r the common ratio, and S_m the sum of the first m terms, prove that the sum of the products of every two $=\frac{r}{r+1}.S_m.S_{m-1}$. (*Comp.* p. 104.)

(23) Prove that in any G.P., in which the common ratio is positive the sum of the first and last terms is greater than the sum of any other two terms taken equidistant from the beginning and end of the series.

(24) If S_1, S_2, S_3,...S_n be the sums of n terms of n Geom$^{\text{c}}$. Prog$^{\text{ns}}$., of which all the first terms are 1, and common ratios 1, 2, 3,...n, respectively, shew that

$$S_1+S_2+2S_3+3S_4+\ldots+(n-1)S_n=1^n+2^n+3^n+\ldots+n^n.$$

(25) The first two terms of a series in Harmonical Progression are a, b; continue the series to three more terms.
Ans. a, b, $\frac{ab}{2a-b}$, $\frac{ab}{3a-2b}$, $\frac{ab}{4a-3b}$.

(26) Given a and b the first two terms of an Harmonic Progression find the n^{th} term.
Ans. $\frac{ab}{(n-1)a-(n-2)b}$.

(27) Insert two Harmonic Means between 6 and 24. Ans. 8, 12.

(28) Insert six Harmonic means between 3 and $\frac{6}{23}$.
Ans. $1\frac{1}{5}$, $\frac{3}{4}$, $\frac{6}{11}$, $\frac{3}{7}$, $\frac{6}{17}$, $\frac{3}{10}$.

(29) If a, b, c, be three terms in Harmonical Progression, a and c being supposed to have the same sign, shew that $a^2+c^2>2b^2$. (*Comp.* p. 105.)

(30) The sum of three consecutive terms in Harmonical Progression is $1\frac{1}{12}$, and the first term is $\frac{1}{2}$; find the series.
Ans. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. or $\frac{1}{2}$, $\frac{7}{4}$, $\frac{-7}{6}$, &c.

(31) Shew that the Arith$^{\text{c}}$., Geom$^{\text{c}}$., and Harm$^{\text{c}}$. Means between any two quantities are themselves in Geom$^{\text{c}}$. Prog$^{\text{n}}$.

(32) From each of three quantities in H.P., what quantity must be taken that the three resulting quantities may be in G.P.?

Ans. Half the 2nd term.

(33) If a, b, c be in A.P., and a, mb, c in G.P., then a, m^2b, c are in H.P.

(34) If between any two quantities there be inserted $\overline{2n-1}$ Arith^{c}., Geom^{c}., and Harm^{c}. Means, the n^{th} Means are in G.P. (*Comp.* p. 105.)

(35) There are four quantities, of which the 1st three are in Arith^{c}., and the last three in Harm^{c}., Prog^{n}., prove that the 1st : 2nd :: 3rd : 4th.

(36) If there be five quantities a_1, a_2, a_3, a_4, a_5, such that a_1, a_2, a_3, are in A.P., a_2, a_3, a_4, in G.P., and a_3, a_4, a_5 in H.P., shew that a_1, a_3, a_5, are in G.P. (*Comp.* p. 106.)

(37) Find the relation between a, b, and c, that they may be the p^{th}, q^{th}, and r^{th} terms of an Harm^{c}. Prog^{n}. (*Comp.* p. 107.)

Ans. $(p-q)ab+(r-p)ac+(q-r)bc = 0$.

(38) If a, b, c be in G.P., shew that $\log_a n$, $\log_b n$, $\log_c n$, are in H.P. (*Comp.* p. 107.)

(39) Find the sum of $2p$ terms of the series whose n^{th} term is $n\{[(-1)^n+1]n+2\}$. (*Comp.* p. 107.)

Ans. $\frac{2p}{3}(2p+1)(2p+5)$.

PERMUTATIONS AND COMBINATIONS.

(1) THE number of Permutations of n things taken four together = six times the number taken three together; find n. Ans. $n = 9$.

(2) The number of Permutations of 15 things taken r together = ten times the number taken $\overline{r-1}$ together; find r. Ans. $r = 6$.

(3) How many days can 5 persons be placed in different positions about a table at dinner? Ans. 120.

(4) The number of Permutations of $\overline{2n+1}$ things taken $\overline{n-1}$ together : number of Permutations of $\overline{2n-1}$ things n together :: 3 : 5; find n.

Ans. $n = 4$.

(5) How many different sums may be formed with a guinea, a half-guinea, a crown, a half-crown, a shilling, and a sixpence? (*Comp.* p. 108.)

Ans. 63.

(6) In the Permutations formed out of a, b, c, d, e, f, g, taken all together, how many begin with ab? How many with abc? How many with $abcd$? (*Comp.* p. 109.)

(1) Ans. 120. (2) Ans. 24. (3) Ans. 6.

(7) Of the Combinations of 10 letters, *a*, *b*, *c*, &c. taken 5 together, in how many will *a* occur? (*Comp.* p. 109.) Ans. 126.

(8) How many different Permutations can be formed from the letters of the word '*Algebra*' taken altogether? Ans. 2520.

How many from the letters of '*Proposition*'? Ans. 1663200.

(9) At an election, where every voter may vote for any number of candidates not greater than the number to be elected, there are 4 candidates, and 3 members to be chosen; in how many ways may a man vote? (*Comp.* p. 109.) Ans. 14.

(10) From a company of 50 policemen 4 are taken every night to guard the police-station. On how many different nights can a different selection be made; and on how many of these will any particular man be engaged? (1) Ans. 230300. (2) Ans. 18424.

(11) How many combinations are there of 52 things taken 13 together; that is, how many different hands may a person hold at the game of whist? Ans. 635013559600.

(12) Find the number of different triangles into which a polygon of n sides may be divided by joining the angular points. (*Comp.* p. 109.)

Ans. $\frac{1}{6}n(n-1)(n-2)$.

(13) Prove that the total number of different combinations of n things taken 1, 2, 3, ... n at a time is 2^n-1. (*Comp.* p. 198.)

(14) The total number of combinations of $2n$ things = 65×the total number of combinations of n things; find n. Ans. $n = 6$.

(15) Shew that the total number of combinations that can be formed out of $\overline{n+1}$ things is more than twice the number that can be formed out of n things.

(16) If there be any unknown number of beans in a bag, prove that the chance of bringing out an odd number taken at random is greater than that of bringing out an even number, excluding the case of bringing out none at all. (*Comp.* p. 110.)

(17) In how many ways can 8 persons be seated at a round table, so that all shall not have the same neighbours in any two arrangements? (*Comp.* p. 110.) Ans. 2520.

(18) Out of 17 consonants and 5 vowels, how many words can be made having two consonants and one vowel in each? (*Comp.* p. 111.) Ans. 4080.

(19) Find the number of words of 6 letters which can be formed from an alphabet of 19 consonants, and 5 vowels, each word containing two vowels and no more. Ans. 27907200.

(20) Find the number of combinations that can be formed out of the letters of the word '*Notation*' taken 3 together. (*Comp.* p. 111.) Ans. 22.

(21) If there be two dice, one of which has n and the other $\overline{n+r}$ faces, each die being marked in the usual manner from 1 upwards, find the number of different throws which can be made with them. (*Comp.* p. 111.)

Ans. $\frac{1}{2}n(n+1)+nr$.

(22) Find the number of Permutations with repetitions (that is, allowing quantities which recur to be combined as if they were different) of n things taken r together. Ans. n^r.

(23) Find the total number of different combinations of n things taken 1, 2, 3, ... n together, of which there are p of one sort, q of another, r of another, &c. (*Comp.* p. 112.) Ans. $(p+1)(q+1)(r+1)$.&c.-1.

BINOMIAL THEOREM, &c.

(1) EXPAND $(a+b)^8$, $(a-b)^7$, $(2x-3y)^5$, $(5-4x)^4$, and $\left(1-\frac{1}{x}\right)^7$.

(1) Ans. $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$.

(2) Ans. $a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7$.

(3) Ans. $32x^5-240x^4y+720x^3y^2-1080x^2y^3+810xy^4-243y^5$.

(4) Ans. $625-2000x+2400x^2-1280x^3+256x$.

(5) Ans. $1-\frac{7}{x}+\frac{21}{x^2}-\frac{35}{x^3}+\frac{35}{x^4}-\frac{21}{x^5}+\frac{7}{x^6}-\frac{1}{x^7}$.

(2) Required the coefficient of x^5 in the expansion of $(x+a)^8$. Ans. $56a^3$.

(3) Expand $(\sqrt{a}\pm\sqrt{b})^4$. Ans. $a^2+6ab+b^2\pm4(a+b)\sqrt{ab}$.

(4) Expand $\left(1-\frac{x}{2}\right)^{-2}$. Ans. $1+x+\frac{3}{4}x^2+\frac{1}{2}x^3+\frac{5}{16}x^4+\ldots$

(5) Find the 5th term of the expansion of $(a^2-b^2)^{12}$. Ans. $495a^{16}b^8$.

(6) Find the 7th term of the expansion of $(a^3+3ab)^9$.

Ans. $61236a^{15}b^6$.

(7) Find the 5th term of the expansion of $(3a^2-7x^3)^8$.

Ans. $13613670a^8x^{12}$.

(8) Find the 6th term of the expansion of $(ax+by)^{10}$.

Ans. $252a^5b^5x^5y^5$.

(9) Find the middle term of the expansion of $(a^m+x^m)^{12}$.

Ans. $924a^{6m}x^{6m}$.

(10) Find the middle term of $(a^{\frac{1}{8}}+b^{\frac{1}{8}})^8$. Ans. $70a^{\frac{1}{2}}b^{\frac{1}{2}}$.

(11) Find the two middle terms of $(a+x)^{13}$. Ans. $1716a^7x^6$, $1716a^6x^7$.

(12) Expand $\frac{1}{\sqrt[3]{ax-x^2}}$. Ans. $\frac{1}{(ax)^{\frac{1}{3}}}+\frac{1}{3}\cdot\frac{x^{\frac{2}{3}}}{a^{\frac{4}{3}}}+\frac{2}{9}\cdot\frac{x^{\frac{5}{3}}}{a^{\frac{7}{3}}}+\frac{14}{81}\cdot\frac{x^{\frac{8}{3}}}{a^{\frac{10}{3}}}+\ldots\ldots$

(13) Expand $\frac{1}{(\sqrt[3]{a}-\sqrt[3]{x})^6}$. Ans. $\frac{1}{a^2}+\frac{6x^{\frac{1}{3}}}{a^{\frac{7}{3}}}+\frac{21x^{\frac{2}{3}}}{a^{\frac{8}{3}}}+\frac{56x}{a^3}+\ldots\ldots$

(14) Find the middle term of the expansion of $(1+x)^{2n}$.

Ans. $\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots\ n}.2^nx^n$.

(15) Find the r^{th} term of the expansion of $(3a-2x)^{\frac{1}{5}}$.

Ans. $-\frac{1.4.9\ldots(5r-11)}{1.2.3\ldots\ (r-1)}.(3a)^{\frac{6}{5}-r}.\left(\frac{2x}{5}\right)^{r-1}$.

(16) Expand $\frac{1}{\sqrt[5]{a^5-x^5}}$ to 5 terms; and find the $\overline{5+r}|^{\text{th}}$ term.

(1) Ans. $\frac{1}{a}+\frac{1}{5}\cdot\frac{x^5}{a^6}+\frac{3}{25}\cdot\frac{x^{10}}{a^{11}}+\frac{11}{125}\cdot\frac{x^{15}}{a^{16}}+\frac{44}{625}\cdot\frac{x^{20}}{a^{21}}+\ldots\ldots$

(2) Ans. $\frac{1.6.11.16\ldots\ldots(16+5r)}{1.2.\ 3\ .\ 4\ \ldots\ldots\ (4+r)}\cdot\frac{1}{5^{4+r}}\cdot\frac{x^{20+5r}}{a^{21+5r}}$.

(17) Find the term which involves $a^{13}b^7$ in the expansion of $(a^3+3ab)^9$.
Ans. $78732a^{13}b^7$.

(18) Expand $(a+2b-c)^3$ by means of the Binomial Theorem.
Ans. $a^3+6a^2b+12ab^2+8b^3-3a^2c-12abc-12b^2c+3ac^2+6bc^2-c^3$.

(19) Find the term which involves a^4b^2 in $(7a^2-3ab+4b^2)^3$.
Ans. $777a^4b^2$.

(20) Find which is the greatest term of the expansion of $\left(1+\frac{5}{6}\right)^{\frac{3}{2}}$.

Ans. The 2nd.

(21) Find which is the greatest coefficient in $(1+x)^{\frac{11}{\sqrt{2}}}$. Ans. The 5th.

(22) At what term does the series for $\left(1+\frac{9}{10}\right)^{\frac{3}{2}}$ begin to converge?

Ans. The 3rd.

(23) Find an approximation to the cube root of 31 by the Binomial Theorem.
Ans. 3·14138.

(24) Find an approximate value of $\sqrt[7]{108}$. (*Comp.* p. 112.)
Ans. 1·9520.

(25) Find the sum of $1+\frac{a}{2}+\frac{\beta}{3}+\frac{\gamma}{4}+$&c. 1, a, β, γ, &c. being the coefficients taken in order of the expansion of $(a+b)^n$. (*Comp.* p. 113.)

Ans. $\frac{2^{n+1}-1}{n+1}$.

(26) Prove that the coefficient of the $\overline{r+1}^{\text{th}}$ term of $(1+x)^{n+1}$ is equal to the sum of the coefficients of the r^{th} and $\overline{r+1}^{\text{th}}$ terms of $(1+x)^n$.

(27) If a, b, c, d, be the 6th, 7th, 8th, 9th, terms respectively of an expanded binomial, shew that $\frac{b^2-ac}{c^2-bd}=\frac{4a}{3c}$. (*Comp.* p. 113.)

(28) Find the coefficient of x^n in the expansion of $\frac{(1-2x)^2}{(1-x)^4}$. (*Comp.* p. 114.)

Ans. $\frac{1}{6}(n-6)(n^2-1)$.

(29) If r be the greatest whole number contained in $\frac{p}{q}$, $(a+x)^{\frac{p}{q}}$ has the first $\overline{r+2}$ terms of its expansion positive, and the $\overline{r+m}^{\text{th}}$ of the same sign as $(-1)^m$; but $(a-x)^{\frac{p}{q}}$ has the first $\overline{r+1}$ terms alternately positive and negative, and all the rest of the same sign as $(-1)^{r+1}$. (*Comp.* p. 114.)

(30) Given that the coefficient of the $\overline{p+1}^{\text{th}}$ term of the expansion of $(1+x)^n$ is equal to that of the $\overline{p+3}^{\text{th}}$ term, shew that $p=\frac{n}{2}-1$. (*Comp.* p. 115.)

(31) What is the meaning of $(1+x)^{\sqrt{2}}$? Has the Binomial Theorem been proved in any sense to extend to such a quantity?

(32) If $x>a$, prove that the sum of all the terms of the expansion of $(x+a)^n$, after the first two, is less than $(2^n-n-1)ax^{n-1}$. (*Comp.* p. 116.)

(33) If $\frac{p(p-1)(p-2)\ldots\ldots(p-r+1)}{1\,.\,2\,.\,3\;\ldots\ldots\;r}$ be represented by p_r, prove that $(p+q)_r=p_r+p_{r-1}q_1+p_{r-2}q_2+\ldots\ldots+p_1p_{r-1}+q_r$. (*Comp.* p. 116.)

(34) If $\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{1\,.\,2\,.\,3\;\ldots\ldots\;r}$ be represented by ${}_nP_r$, shew that

$${}_nP_r+{}_{n+1}P_{r-1}={}_{n+1}P_r. \quad (\textit{Comp. p. 117.})$$

(35) Prove that

$$\frac{1.3.5\ldots(2r-1)}{\underline{|r}}+\frac{1.3.5\ldots(2r-3)}{\underline{|r-1}}\cdot\frac{3}{\underline{|1}}+\frac{1.3.5\ldots(2r-5)}{\underline{|r-2}}\cdot\frac{3.5}{\underline{|2}}+\ldots\ldots$$

is equal to $2^r(1+r)$. (*Comp.* p. 117.)

(36) Shew that if t_r denote the middle term of $(1+x)^{2r}$, then will

$$t_0+t_1+t_2+\ldots\ldots = (1-4x)^{-\frac{1}{2}}.\quad (\textit{Comp.}\ \text{p. } 118.)$$

(37) If x be very small compared with 1, prove that

$$\frac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{1+x+\sqrt{1+x}} = 1-\frac{5}{6}x \text{ nearly.}\quad (\textit{Comp.}\ \text{p. } 119.)$$

(38) If $c = a-b$, and is very small compared with a and b, shew that $\dfrac{a^2b^2}{(a^2-a^2x^2+b^2x^2)^{\frac{3}{2}}} = a-2c+3cx^2$ nearly. (*Comp.* p. 119.)

(39) If x be nearly equal to 1, shew that $\dfrac{mx^m-nx^n}{m-n} = x^{m+n}$ nearly. (*Comp.* p. 120.)

(40) If $a>b$, prove that $a^n-b^n>nb^{n-1}(a-b)$ and $<na^{n-1}(a-b)$.

(41) If $u = \dfrac{a^p}{z^m}+\dfrac{b^q-a^p}{z^n}$, and z is very nearly equal to 1, shew that $u = b^q.z^{(n-m)\frac{a^p}{b^q}-n}$ very nearly. (*Comp.* p. 120.)

(42) If $\dfrac{x}{a} = 1+h$, h being very small, find the value of

$$\frac{\left(2ax-x^2+\dfrac{b^2x^2}{a^2}\right)^{\frac{1}{2}}}{(a^2+b^2)^{\frac{1}{2}}}.\quad (\textit{Comp.}\ \text{p. } 120.)\qquad \text{Ans. } 1+\frac{b^2h}{a^2+b^2} \text{ nearly.}$$

(43) If N represent the n^{th} term of the expansion of a^x, find n when the series begins to converge at that term; and shew that the sum of all the succeeding terms is less than $\dfrac{Nn}{n-x.\log_e a}$. (*Comp.* p. 121.)

(44) Required the term involving x^4 in the expansion of $(1+x+x^2)^5$.
Ans. $45x^4$.

(45) Required the term involving x^5 in the expansion of $(1+2x+3x^2+4x^3+\ldots)^6$.
Ans. $4368x^5$.

(46) Required the term involving $a^2b^2c^2$ in the expansion of $(a+b+c)^6$.
Ans. $90a^2b^2c^2$.

(47) Required the term involving $b^2c^3e^4f$ in the expansion of $(a+2b+3c+4d+5e+6f)^{10}$.
Ans. $5103000000b^2c^3e^4f$.

(48) Find the coefficient of x^4 in $(2+\sqrt{x}-x)^5$. (*Comp.* p. 122.) Ans. 0.

(49) Find the terms which involve $\dfrac{yz^7}{x^2}$, and y^5z, in $\left(\dfrac{x^2}{y}-\dfrac{y^2}{z}+\dfrac{z^2}{x}\right)^6$. (*Comp.* p. 122.)
(1) Ans. $-\dfrac{30yz^7}{x^2}$, (2) Ans. $-60y^5z$.

(50) Find the coeff^t. of $\frac{1}{x^3}$ in the expansion of $\left(a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}\right)^6$.
(*Comp.* p. 123.) Ans. $6b^5d+15b^4c^2+120ab^3cd+60ab^2c^3+90a^2b^2d^2+15a^2c^4-$
$60a^3cd^2+180a^2bc^2$

(51) Find the coeff^t. of x^7 in the expansion of $(a+bx+cx^2+\&\text{c. } in\ inf.)$
(*Comp.* p. 124.) Ans. $\left.\begin{matrix}4a^3h+12a^2bg+12a^2cf+12ab^2f+4b^3e+12a^2de\\+24abce+12ac^2d+12abd^2+4bc^3+12b^2cd\end{matrix}\right\}$

(52) Find the coefficient of x^3 in $(a+bx+cx^2)^{\frac{3}{2}}$. (*Comp.* p. 124.)
Ans. $\frac{3}{4}\cdot\frac{bc}{a^{\frac{1}{2}}}-\frac{b^3}{16a^{\frac{3}{2}}}$

(53) Find the coefficient of x^4 in $(1+bx+cx^2+dx^3+\&\text{c.})^{-4}$. (*Comp.* p. 125.
Ans. $35b^4-60b^2c+10c^2+20bd-4e$

(54) In the expansion of $\sqrt[3]{a+bx+cx^2+dx^3+ex^4+\ldots}$ find the coefficients of x^3 and x^4. (*Comp.* p. 125.)

(1) Ans. $\sqrt[3]{a}.\left\{\frac{d}{3a}-\frac{2bc}{9a^2}+\frac{5b^3}{81a^3}\right\}$.

(2) Ans. $\sqrt[3]{a}.\left\{\frac{e}{3a}-\frac{2bd}{9a^2}-\frac{c^2}{9a^2}+\frac{5b^2c}{27a^3}-\frac{10b^4}{243a^4}\right\}$

(55) In the expansion of $(1+x+2x^2+3x^3+\ldots)^2$ find the coefficient of x^n.
(*Comp.* p. 126.) Ans. $\frac{1}{6}(n^3+11n)$.

(56) Find the coefficient of x^r in the expansion of $(1+2x+3x^2+\ldots)^2$.
(*Comp.* p. 127.) Ans. $\frac{1}{6}(r+1)(r+2)(r+3)$.

(57) In the expansion of $(a_0+a_1x+a_2x^2+a_3x^3)^4$, find the number o terms. (*Comp.* p. 128.) Ans. 35.

(58) Find the number of terms in the expansion of $(a+b+c)^7$.
(*Comp.* p. 128.) Ans. 36.

(59) With n dice how many different throws can be made?

Ans. $\frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{\underline{|5}}$.

MISCELLANEOUS QUESTIONS ON SERIES.

Write down the n^{th} term of each of the following series which proceed according to the law indicated by the terms given :—

(1) $1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots\ldots$ Ans. $\frac{x^{n-1}}{\underline{|n-1}}$.

(2) $1+3+6+10+15+21+28+\ldots\ldots\ldots$ Ans. $\frac{1}{2}n(n+1)$.

(3) $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\ldots\ldots\ldots$ Ans. $\dfrac{1}{\frac{1}{2}n(n+1)}$.

(4) $1.2.4+2.3.5+3.4.6+\ldots\ldots$ Ans. $n(n+1)(n+3)$.

(5) $\frac{3\times5}{1^2\times3^2}+\frac{5\times7}{2^2\times4^2}+\frac{7\times9}{3^2\times5^2}+\ldots\ldots$ Ans. $\dfrac{(2n+1)(2n+3)}{n^2(n+2)^2}$.

(6) $1+\frac{1.6.11}{1.2.3}+\frac{1.6.11.16.21}{1.2.\,3\,.\,4\,.\,5}+\ldots\ldots$ Ans. $\dfrac{1.6.11.16\ldots(10n-9)}{1.2.\,3\,.\,4\,\ldots\,(2n-1)}$.

(7) $\frac{3}{5}+\frac{6}{8}+\frac{9}{11}+\frac{12}{14}+\ldots\ldots$ Ans. $\dfrac{3n}{3n+2}$.

(8) $\frac{1}{6.7.8}+\frac{1}{8.9.10}+\frac{1}{10.11.12}+\ldots\ldots$ Ans. $\dfrac{1}{(2n+4)(2n+5)(2n+6)}$.

(9) $\frac{19}{1.2.3}\times\frac{1}{4}+\frac{28}{2.3.4}\times\frac{1}{8}+\frac{39}{3.4.5}\times\frac{1}{16}+\ldots\ldots$ Ans. $\dfrac{(n+3)^2+3}{n(n+1)(n+2)}\times\dfrac{1}{2^{n+1}}$.

(10) $1^3=1$, $2^3=3+5$, $3^3=7+9+11$, &c. Write down n^3 after the same law, and verify the result. (*Comp.* p. 128.)

(11) Find the first three terms of the series whose $\overline{n+1}^{\text{th}}$ term is $\dfrac{(n+7)(n-5)}{(n+1)(n^2+1)}.2^n$. Ans. -35, -16, $-7\frac{1}{5}$.

(12) Find the $\overline{n+1}^{\text{th}}$ term of the series whose r^{th} term is $\dfrac{n(n-1)\ldots(n-r+2)}{1.2\ldots\ldots\ldots(r-1)}.\left(\dfrac{x}{1+2x}\right)^{r-1}$. Ans. $\left(\dfrac{x}{1+2x}\right)^n$.

(13) Find the first three terms of the series whose r^{th} term is $(-1)^{r+1}.\dfrac{2^{\frac{3r-1}{3}}}{3^{2(r-1)}}.\dfrac{1.4.7\ldots(3r-2)}{1.2.3\ldots\ldots r}$. Ans. $2^{\frac{2}{3}}$, $\dfrac{-2^{\frac{8}{3}}}{9}$, $\dfrac{7}{243}.2^{\frac{11}{3}}$.

EVOLUTION OF SURDS.

Extract the square root of each of the following surds :—

(1) $1\frac{7}{9}-2\frac{2}{3}\times\sqrt{\frac{1}{3}}$. Ans. $\frac{2}{3}(\sqrt{3}-1)$.

(2) $\sqrt{27}+2\sqrt{6}$. Ans. $\sqrt[4]{12}+\sqrt[4]{3}$.

(3) $\sqrt{32}-\sqrt{24}$. Ans. $\sqrt[4]{18}-\sqrt[4]{2}$.

(4) $3\sqrt{5}+\sqrt{40}$. Ans. $\sqrt[4]{20}+\sqrt[4]{5}$.

(5) $\left(a-\frac{b}{2}\right)^2+2\sqrt{a^3b-2a^2b^2+\frac{ab^3}{4}}$. Ans. $\sqrt{\left(a-\frac{b}{2}\right)^2-ab}+\sqrt{ab}$.

(6) $\frac{3a}{x}+\sqrt{\frac{12a^3b^2}{c^2x}-\frac{4a^4b^4}{c^4}}$. Ans. $\frac{ab}{c}+\sqrt{\frac{3a}{x}-\frac{a^2b^2}{c^2}}$

(7) $ab+4c^2-d^2+2\sqrt{4abc^2-abd^2}$. Ans. $\sqrt{ab}+\sqrt{4c^2-d}$

(8) Prove that $\sqrt{bc+2b\sqrt{bc-b^2}}+\sqrt{bc-2b\sqrt{bc-b^2}}$ is equal to $\pm 2b$.

(9) Extract the cube root of $7-5\sqrt{2}$. Ans. $1-\sqrt{2}$

(10) ... ... $45\pm 29\sqrt{2}$. Ans. $3\pm\sqrt{2}$

(11) ... ... $148+46\sqrt{11}$. Ans. $\sqrt[3]{2}\{2+\sqrt{11}\}$

(12) Extract the 4th root of $14+8\sqrt{3}$. Ans. $\frac{\sqrt{3}+1}{\sqrt[4]{2}}$

(13) Extract the 5th root of $41+29\sqrt{2}$. Ans. $1+\sqrt{2}$

(14) ... ... $228+132\sqrt{3}$. Ans. $(1+\sqrt{3})\sqrt[5]{3}$

(15) Extract the 6th root of $2889-1292\sqrt{5}$. Ans. $\sqrt{5}-2$

(16) Extract the 7th root of $239+169\sqrt{2}$. Ans. $1+\sqrt{2}$

(17) Extract the square root of $9+2\sqrt{3}+2\sqrt{5}+2\sqrt{15}$.
Ans. $1+\sqrt{3}+\sqrt{5}$

(18) Extract the square root of $6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$.
Ans. $1+\sqrt{2}+\sqrt{3}$

(19) Shew that $\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{\frac{1}{4}}$ is equal to $\frac{\sqrt{2}}{1+\sqrt{3}}$. (*Comp.* p. 129.)

(20) If $x=\sqrt[3]{r+\sqrt{r^2+q^3}}+\sqrt[3]{r-\sqrt{r^2+q^3}}$, shew that $x^3+3qx-2r=0$.

(21) Prove that $\sqrt{2+\sqrt{8}}$ may be expressed by $\sqrt{1+\sqrt{-1}}+\sqrt{1-\sqrt{-1}}$
Does it follow that $\sqrt{2+\sqrt{8}}$ is "impossible"?

(22) Prove that $\sqrt{\sqrt{b}\pm a}$ may be reduced to the form $\sqrt[4]{\alpha}\pm\sqrt[4]{\beta}$, wher $1-\frac{a^2}{b}$ is a complete square.

(23) Prove that $(20+\sqrt{392})^{\frac{1}{3}}+(20-\sqrt{392})^{\frac{1}{3}}$ is equal to 4.

(24) If $x=\sqrt[3]{-\frac{r}{2}+\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}+\sqrt[3]{-\frac{r}{2}-\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}$, and $r=0$, shew that the values of x are 0 and $\pm\sqrt{q}$. (*Comp.* p. 129.)

(25) If $\left(x-\frac{y}{p}\right).\{b-y-(a-x)p\}=c^2$, and $p=\sqrt{\frac{y^2-by}{x^2-ax}}$, prove that $\sqrt{(b-y)x}-\sqrt{(a-x)y}=c$. (*Comp.* p. 129.)

(26) Given $2\{x^2+y^2-x-y\}+1=0$; find the *real* values of x and y. (*Comp.* p. 130.)

Ans. $x=\frac{1}{2}$, $y=\frac{1}{2}$.

(27) Given $(x+y\sqrt{-1})^2=a+b\sqrt{-1}$; find the real values of x and y. (*Comp.* p. 130.)

Ans. $x=\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)}$, $y=\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}-a)}$.

(28) Given $\frac{3x+2y\sqrt{-1}}{5\sqrt{-1}-2}=\frac{15}{8x+3y\sqrt{-1}}$, find x and y. (*Comp.* p. 130.)

Ans. $x=1$, $y=3$.

(29) Find the relations subsisting between a, b, c, d, when the square root of $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ can be expressed in the form $\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}$. (*Comp.* p. 131.)

Ans. Each of the quantities $\sqrt{\frac{bc}{d}}$, $\sqrt{\frac{bd}{c}}$, $\sqrt{\frac{cd}{b}}$ rational, and the sum of them equal to $2a$.

INDETERMINATE COEFFICIENTS.

By the method of Indeterminate Coefficients shew that

(1) $$\frac{3+2x}{5+7x}=\frac{3}{5}-\frac{11}{5^2}x+\frac{7\times 11}{5^3}x^2-\frac{7^2\times 11}{5^4}x^3+\frac{7^3\times 11}{5^5}x^4-\ldots\ldots$$

(2) $$\frac{1+x}{(1-x)^3}=1^2+2^2.x+3^2.x^2+4^2.x^3+5^2.x^4+\ldots\ldots$$

(3) $$\frac{x+3}{(x-1)(x+2)}=\frac{4}{3(x-1)}-\frac{1}{3(x+2)}.$$

(4) $$\frac{x+1}{x(x-2)}=\frac{3}{2(x-2)}-\frac{1}{2x}.$$

(5) $$\frac{1}{x^2-(a+b)x+ab}=\frac{1}{(a-b)(x-a)}-\frac{1}{(a-b)(x-b)}.$$

(6) $$\frac{x+1}{x^2-7x+12}=\frac{5}{x-4}-\frac{4}{x-3}.$$

(7) $$\frac{3x-5}{x^2-6x+8}=\frac{7}{2}.\frac{1}{x-4}-\frac{1}{2}.\frac{1}{x-2}.$$

(8) $$\frac{x^2}{(x+1)(x+2)(x+3)}=\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}.$$

(9) $$\frac{3x^2-7x+6}{(x-1)^3}=\frac{2}{(x-1)^3}-\frac{1}{(x-1)^2}+\frac{3}{x-1}.$$

(10) $$\frac{1}{a^4-x^4}=\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}.$$

(11) $$\frac{x^2-x+1}{x^2(x+1)}=\frac{1}{x^2}-\frac{2}{x}+\frac{3}{x+1}.$$

(12) $$\frac{x-bx^3+dx^5}{1-ax^2+cx^4}=x+(a-b)x^3+(a^2-ab-c+d)x^5+\ldots\ldots$$

(13) $$\frac{1}{x^6-1}=\frac{1}{6}\left\{\frac{1}{x-1}-\frac{1}{x+1}+\frac{x-2}{x^2-x+1}-\frac{x+2}{x^2+x+1}\right\}.$$

(14) $$\frac{x^2+px+q}{(x-a)(x-b)(x-c)}=\frac{a^2+pa+q}{(a-b)(a-c)(x-a)}+\frac{b^2+pb+q}{(b-a)(b-c)(x-b)}$$
$$+\frac{c^2+pc+q}{(c-a)(c-b)(x-c)}.$$

(15) $$\sqrt{1+x+x^2+x^3\ldots\ldots}=1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots\ldots$$

(16) Resolve $7x^2-6x-1$ into two factors of the first degree. (*Comp.* p. 132.) Ans. $(x-1)(7x+1)$.

(17) Resolve $2x^2-21xy-11y^2-x+34y-3$ into factors of the first degree. (*Comp.* p. 132.) Ans. $(x-11y+1)(2x+y-3)$.

(18) Given $y=ax+bx^2+cx^3+dx^4+\ldots$, find x in a series of powers of y.

$$\text{Ans. } x=\frac{1}{a}y-\frac{b}{a^3}y^2+\frac{2b^2-ac}{a^5}y^3-\frac{5b^3-5abc+a^2d}{a^7}y^4+\ldots$$

(19) Given $y^3-axy-b^3=0$, find y in a series of powers of x.

$$\text{Ans. } y=b+\frac{ax}{3b}-\frac{a^3x^3}{3^4b^5}+\frac{a^4x^4}{3^5b^7}-\ldots$$

(20) Given $x=n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\ldots$, find n in a series of powers of x.

$$\text{Ans. } n=x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\ldots$$

(21) Given $x=z-\frac{1}{3}z^3+\frac{1}{5}z^5-\ldots$, and $y=z\sqrt{1-y^2}$, find y in a series of powers of x.

$$\text{Ans. } y=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\ldots$$

CONTINUED FRACTIONS.

(1) **Find** a series of fractions converging to $\frac{251}{764}$; also to $\frac{39}{139}$.

(1) Ans. $\frac{1}{3}, \frac{22}{67}, \frac{23}{70}, \frac{114}{347}$. (2) Ans. $\frac{1}{3}, \frac{1}{4}, \frac{2}{7}, \frac{7}{25}, \frac{16}{57}$

(2) Express $\frac{84}{227}$ in a continued fraction, and find the convergents.

Ans. The quotients are 2, 1, 2, 2, 1, 3, 2.

The convergents are $\frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{7}{19}, \frac{10}{27}, \frac{37}{100}, \frac{84}{227}$

(3) Find the convergents to the continued fraction whose quotient are, 1, 4, 9, 2, 1, 1, 4.

Ans. $\frac{1}{1}, \frac{5}{4}, \frac{46}{37}, \frac{97}{78}, \frac{143}{115}, \frac{240}{193}, \frac{1103}{887}$

(4) Find a series of fractions converging to $\sqrt{2}$; also to $\sqrt{45}$.

(1) Ans. $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$, &c. (2) Ans. $\frac{6}{1}, \frac{7}{1}, \frac{20}{3}, \frac{47}{7}, \frac{114}{17}$, &c.

(5) Express $\frac{3+\sqrt{7}}{2}$ in a continued fraction.

Ans. The quotients are 2, 1, 4, 1, 1, 1, 4, &c

(6) Find the convergents to 0·2422638......

Ans. $\frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{39}{161}, \frac{47}{194}$, &c

(7) From the last example deduce an explanation of the Julian an Gregorian corrections of the Calendar, having given the true length of th year to be 365·2422638......days. (*Comp.* p. 133.)

(8) Prove that $\frac{355}{113}$ differs from 3·14159 by a quantity less than 0·00001

(9) The lunar month, calculated on an average of 100 years, i 27·321661 days. Find a series of vulgar fractions approximately neare and nearer to this decimal fraction.

Ans. $\frac{27}{1}, \frac{82}{3}, \frac{765}{28}, \frac{3907}{143}$, &c.

(10) The sidereal revolution of Mercury is 87·969255 days; and that of Venus 224·700817 days. Represent these quantities approximately b less numbers.

Ans. For Mercury $\frac{87}{1}, \frac{88}{1}, \frac{2815}{32}$, &c.

For Venus $\frac{224}{1}, \frac{225}{1}, \frac{674}{3}, \frac{1573}{7}$, &c.

(11) Find the least fraction with only two figures in each term, approximating to $\frac{1947}{3359}$. Ans. $\frac{11}{19}$.

(12) Given $2^x = 6$, required x in the form of a continued fraction; and find the convergents.

Ans. $x = 2 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}+\dots}}$ Convergents are $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{13}{5}, \frac{31}{12}$, &c.

(13) If $\left(\frac{7}{12}\right)^x = \frac{3}{4}$, find x. (*Comp.* p. 134.) Ans. 0·53.

(14) If $3^x = 15$, find x. Ans. 2·465.

(15) If $\left(\frac{1}{2}\right)^x = \frac{3}{5}$, find x. Ans. 0·737.

(16) Approximate by continued fractions to the roots of the equations,

(1) $5x^2-3 = 0$, (2) $x^2-5x+3 = 0$. (*Comp.* p. 135.)

(1) Ans. $\frac{3}{5}, \frac{4}{5}, \frac{27}{35}, \frac{31}{40}, \frac{213}{275}, \frac{244}{315}$, &c.

(2) Ans. $\frac{4}{1}, \frac{13}{3}, \frac{43}{10}, \frac{142}{33}$, &c. and $\frac{1}{1}, \frac{2}{3}, \frac{7}{10}, \frac{23}{33}$, &c.

(17) Shew that $\sqrt{5}$ is greater than $\frac{682}{305}$ and less than $\frac{2889}{1292}$; and that it differs from the latter fraction by a quantity less than $\frac{1}{2\times 305\times 1292}$. (*Comp.* p. 136.)

INDETERMINATE EQUATIONS AND PROBLEMS.

(1) $14x-5y = 7$; find the least positive integral values of x and y. Ans. $x = 3$, $y = 7$.

(2) $27x+16y = 1600$; .... Ans. $x = 48$, $y = 19$.

(3) $19x-117y = 11$; .... Ans. $x = 56$, $y = 9$.

(4) $3x+5y = 26$; find *all* the values of x and y in positive integers. Ans. $x = 7, 2$. $y = 1, 4$.

(5) $11x+13y = 190$; ... ... Ans. $x = 9$, $y = 7$.

(6) $13x+16y = 97$; ... ... Ans. $x = 5$, $y = 2$.

(7) $11x+7y=108$; find *all* the values of x and y in positive integers.
Ans. $x=6$, $y=6$.

(8) Shew that there is no solution in whole numbers for the equation $49x-35y=11$.

(9) Given $x=4$, $y=9$, one solution of $2x+3y=35$, find all the solutions in positive integers.

$$\text{Ans. } \begin{cases} x = 1,\ 4,\ 7,\ 10,\ 13,\ 16. \\ y = 11,\ 9,\ 7,\ 5,\ 3,\ 1. \end{cases}$$

(10) Given $\left.\begin{array}{r} x+2y+3z=20 \\ \text{and } 4x+5y+6z=47 \end{array}\right\}$; find x, y, z.

$$\text{Ans. } \begin{cases} x = 1,\ 2,\ 3, \\ y = 5,\ 3,\ 1, \\ z = 3,\ 4,\ 5. \end{cases}$$

(11) Given $\left.\begin{array}{r} 6x+7y+4z=122 \\ \text{and } 11x+8y-6z=145 \end{array}\right\}$; find x, y, z.
Ans. $x=9$, $y=8$, $z=3$.

(12) Find all the positive integral solutions of $20x-21y=38$, and $3y+4z=34$.
Ans. $x=4$, $y=2$, $z=7$.

(13) Find all the positive integral solutions of $xy+x^2=2x+3y+29$. (*Comp.* p. 137.)
Ans. $x=4,\ 5.$ $y=21,\ 7$.

(14) Find all the positive integral solutions of $7xy-5x=3y+39$.

$$\text{Ans. } \begin{cases} x = 1,\ 3,\ 5,\ 21, \\ y = 11,\ 3,\ 2,\ 1. \end{cases}$$

(15) Find the number of solutions of $11x+15y=1031$ in positive integers.
Ans. 7.

(16) Find the number of solutions of $3x+7y+17z=100$ in positive integers. (*Comp.* p. 137.)
Ans. 12.

(17) Find the number of solutions of $20x+15y+6z=171$ in positive integers.
Ans. 6.

(18) Find two fractions having 7 and 9 for their denominators, and their sum $\frac{57}{63}$.
Ans. $\frac{4}{7}$, $\frac{3}{9}$.

(19) Find three fractions with denominators 3, 4, and 5, of which the sum is $\frac{133}{60}$.
Ans. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$.

(20) Find the three fractions whose sum is $\frac{401}{315}$.
Ans. $\frac{2}{5}$, $\frac{3}{7}$, $\frac{4}{9}$.

(21) Find a number which upon being divided by 39 gives a remainder 16, and by 56 a remainder 27.
Ans. 1147, or 3331, or &c.

(22) Find a number consisting of two digits which shall be equal to four times the sum of the digits. Ans. 12, or 24, or 36, or 48.

(23) Find the least number which, upon being divided by 11, 19, and 29, gives the remainders 3, 5, and 10, respectively. Ans. 4128.

(24) Find a number less than 400 which is a multiple of 7, and upon being divided by 2, 3, 4, 5, 6, always leaves 1 for a remainder. Ans. 301.

(25) Shew that the solution of $ax+by=c$ in positive integers is always possible, if a be prime to b, and $c > ab-(a+b)$. (*Comp.* p. 138.)

(26) In how many different ways is it possible to pay £20 in half-guineas *and* half-crowns? Ans. 7.

(27) A certain sum consists of £x. y shillings, and its half of £y. x shillings; find the sum. (*Comp.* p. 139.) Ans. £13. 6*s*.

(28) Find two numbers such that their sum shall be equal to the sum of their squares.

Ans. 1, 1, or $\frac{6}{5}$, $\frac{3}{5}$, or $\frac{2}{5}$, $\frac{6}{5}$, or $\frac{20}{17}$, $\frac{5}{17}$, or $\frac{15}{13}$, $\frac{3}{13}$, or &c.

(29) What value of x will make ax^2+bx+c^2 a complete square? (*Comp.* p. 139.) Ans. $x=\dfrac{bn^2-2cmn}{m^2-an^2}$.

(30) What integral values of x will make $2x^2+x+8$ a complete square? (*Comp.* p. 139.) Ans. 8, −4, −1, and 23.

(31) What value of b will make b^2-4ac a complete square? (*Comp.* p. 140.) Ans. $b=am+\dfrac{c}{m}$.

(32) Find three square numbers which are in Arithmetic Progression. (*Comp.* p. 140.) Ans. $(m^2-n^2-2mn)^2$, $(m^2+n^2)^2$, $(m^2-n^2+2mn)^2$.

SCALES OF NOTATION.

(1) 17486 is in the denary scale, find the equivalent number in the senary scale. Ans. 212542.

(2) 215855 is in the denary scale, find the same number in the duodenary scale. Ans. *t4tee*.

(3) *3t4e2* is in the duodenary scale, find the same number in the denary scale. Ans. 80198

(4) Transform 1534 from the senary to the denary scale. Ans. 418.

(5) Divide 14332216 by 6541 in the septenary scale. Ans. 1456

(6) Divide 95088918 by *tt*4 in the duodenary scale. Ans. *t*4*tee*.

(7) Multiply 64 ft. 6 in. by 8 ft. $9\frac{1}{4}$ in. Ans. 565 ft. 8′. 7″. 6‴.

(8) The difference between any number (in the denary scale) and that formed by reversing the order of the digits is divisible by 9. Prove it.

(9) Extract the square root of 25400544 in the senary scale. Ans. 4112.

(10) Extract the square root of 32*e*75721 in the duodenary scale; and then verify the result by squaring it. Ans. 62*te*.

(11) The number 124 in the denary is expressed by 147 in another scale, required the radix of the latter. Ans. 9.

(12) In what scale of notation will a number that is double of 145 be expressed by the same digits? (*Comp*. p. 141.) Ans. Radix = 15.

(13) Find the scale to which 24065 belongs, its equivalent in the denary scale being 6221. Ans. Radix = 7.

(14) The area of a rectangle is 29 ft. 4 in., and its length is 12 ft. 8 in.; find its breadth. Ans. 2 ft. 3 in. 6′.

(15) The area of a rectangle is 971 ft. 120 in., and breadth 24 ft. 9 in.; find its length. Ans. 39 ft. 3 in. 2′. 3″. &c.

(16) The area of a square is 17 ft. 54 in., what is the length of the side? Ans. 4 ft. 2 in. 0′. 2″. 10‴. &c.

(17) What is the cube of 6 ft. 6 in.? Ans. 274 ft. 1080 in.

(18) Prove that any number of 4 digits in the denary scale is divisible by 7, if the first and last digits be the same, and the digit in the place of hundreds be double that in the place of tens.

(19) Any number is divisible by 4, if the last two digits, taken in order to form a number, be divisible by 4.

(20) Any number is divisible by 8, if the number, consisting of the last three digits in order, be divisible by 8.

(21) There is a certain number consisting of two digits, which is equal to four times the sum of its digits; and if to the number 18 be added, the digits will be reversed. What is it? Ans. 24.

(22) There is a certain number, a multiple of 10, which exceeds the sum of its digits by 99; find the number. Ans. 100.

(23) Prove that the sum of all the numbers which are composed of the same digits is divisible by the sum of the digits, when the digits are all different.

(24) Find the greatest and the least numbers of 4 digits in the senar scale, as expressed in the denary scale. Ans. 1295, 216

(25) A certain number consists of two digits such that when the digit are reversed the number is divisible by 3, and is to the former number a 23 : 32. Required the number. (*Comp.* p. 141.) Ans. 96

(26) Any number consisting of an even number of digits, in a system whose radix is r, is divisible by $r+1$, if the digits equidistant from each end are the same.

(27) If N, N', be any two numbers in the denary scale composed o the same digits differently arranged, prove that $N \sim N'$ is divisible by 9. (*Comp.* p. 141.)

(28) The square of any number which has less than 10 digits, (in the denary scale) each of which is 1, will, when reckoned from either end, form the same Arithmetic Progression whose common difference is 1, and greatest term the number of digits in the root.

PROPERTIES OF NUMBERS.

(1) PROVE that n^3 divided by 4 cannot leave 2 for a remainder, n being any of the natural numbers. (*Comp.* p. 142.)

(2) No number can be a square which has any one of the numbers 2, 3 7, 8 for its last digit.

(3) Prove the following properties of a square number:—

(1) A square number cannot terminate with an odd number of ciphers

(2) If a square number terminate with 5, it must terminate with 25.

(3) If a square number terminate with an odd figure, the last figure but one will be even; and if it terminate with any even figure excep 4 or 0, the last figure but one will be odd.

(4) No square number can terminate with two figures the same, excep they be two ciphers, or two 4's.

(4) Any number divided by 6 leaves the same remainder as its cub divided by 6.

(5) If m be any odd square number greater than 1, prove that $(m+3)\times$ $(m+7)$ is divisible by 32.

(6) If each of the quantities a, b, n be a whole number, shew tha $\{2a+(n-1)b\}\frac{n}{2}$ is always a whole number. (*Comp.* p. 142.)

(7) Shew that every perfect cube number is of one of the forms $4n$, or $4n \pm 1$.

(8) Shew that x^5-5x^3+4x is divisible by 120, whatever positive whole number x may be. (*Comp.* p. 143.)

(9) Shew that $\frac{x+1}{12}(2x^2+x+3)$ is a whole number, if x be odd. (*Comp.* p. 143.)

(10) The difference of the squares of any two odd numbers is divisible by 8; and the difference of the squares of any two prime numbers, of which the less exceeds 5, is divisible by 24.

(11) If n be any whole number, one of the three n^2, n^2+1, n^2-1 is divisible by 5.

(12) If n be an odd number not divisible by 7, either n^3+1, or n^3-1 is divisible by 14.

(13) Shew that, when m is any even number greater than 2, $m^2(m^2-4)$ is divisible by 192; and, when m is any odd number greater than 3, $m(m^2-1)(m^2-9)$ is divisible by 1920.

(14) If n be a prime number greater than 3, $\frac{n^2-1}{24}$ is an integer. (*Comp.* p. 143.)

(15) If a and b be prime numbers, the number of numbers prime to ab and less than ab is equal to $(a-1)(b-1)-1$.

(16) If there be two binomials each of which is the sum of two squares, their product is the sum of two squares. (*Comp.* p. 144.)

(17) Neither the sum nor the difference of two irreducible fractions, whose denominators are different, can be an integer. (*Comp.* p. 144.)

(18) If n be any number, and a the difference between n and the next greater square number, and b the difference between n and the next less square number, shew that $n-ab$ is a square. (*Comp.* p. 144.)

(19) Prove that the product of two different primes cannot be a square. (*Comp.* p. 145.)

(20) Decompose 831600 into its prime factors; and find the multiplier which will make it a perfect cube.

(1) Ans. $11\times7\times5^2\times2^4\times3^3$. (2) Ans. 118580.

(21) Find the number of divisors of 1000. Ans. 16.

(22) Find the number of divisors of 30030. Ans. 64.

(23) If N is a number of the form $a^m b^n$, where a and b are prime numbers, shew that $N.\frac{a-1}{a}.\frac{b-1}{b}$ is the number of integers not greater than N and prime to it. (*Comp.* p. 145.)

(1) How many numbers are there not greater than 100 and prime to it? Ans. 40.

(2) How many less than 360 and prime to it? Ans. 96.

(24) If one number (A) have exactly as many places of figures as another (B), and also have more than the first half of its figures identical with the corresponding figures in B, shew that the difference between $\sqrt{A}$ and $\sqrt{B}$ will be less than $\frac{1}{n}$, if n be any whole number not less than 2. (*Comp.* p. 146.)

VANISHING FRACTIONS.

(1) FIND the value of $\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$ when $x=a$. Ans. $\frac{1}{\sqrt{2a}}$.

(2) Find the value of $\frac{a^3-b^3}{a^2-b^2}$ when $a=b$. Ans. $\frac{3a}{2}$.

(3) Find the value of $\frac{a^m-b^m}{a^n-b^n}$ when $a=b$. Ans. $\frac{m}{n}a^{m-n}$.

(4) Find the value of $\frac{a^{r+1}-b^{r+1}}{a^r b^r(a-b)}$ when $a=b$. Ans. $\frac{r+1}{a^r}$.

(5) Find the value of $\frac{a-(a^n-x^n)^{\frac{1}{n}}}{x^n}$ when $x=0$. Ans. $\frac{1}{na^{n-1}}$.

(6) Find the value of $\frac{\sqrt{2a^3x-x^4}-\sqrt{ax^3}}{a-\sqrt{ax}}$ when $x=a$. Ans. $5a$.

(7) Find the value of $\frac{x^3+5ax^2-4a^2x-2a^3}{x^2-a^2}$ when $x=a$. Ans. $\frac{9a}{2}$.

(8) Find the value of $\frac{\sqrt{a^2+ax+x^2}-\sqrt{a^2-ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$ when $x=0$.

Ans. $\sqrt{a}$.

(9) Find the value of $\frac{\sqrt{x^2+a}-\sqrt{a^2+x}+\sqrt{a^2-x^2}}{\sqrt{a^3-x^3}}$ when $x=a$.

Ans. $\sqrt{\frac{2}{3a}}$.

CONVERGING AND DIVERGING SERIES.

(1) PROVE that $1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots\ldots$ will begin to converge at some point for any value of x, however great.

(2) Shew that the series for $(1+x)^n$, given by the Binomial Theorem, will always be "convergent" when $x<1$; and determine after how many terms the convergency will begin. (*Comp.* p. 147.)

Ans. After $\overline{r+1}$ terms, r being the next whole number which $>\frac{nx-1}{1+x}$.

(3) Shew that both the following series are convergent;

(1) $1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-\frac{x^6}{\underline{|6}}+\ldots\ldots$ (2) $x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\frac{x^7}{\underline{|7}}+\ldots\ldots$

and if $x=264$, at which term will the former series begin to converge? (*Comp.* p. 148.)

Ans. $\frac{x^{266}}{\underline{|266}}$.

(4) In the expression $ax^n+bx^{n-1}+cx^{n-2}+\ldots\ldots$ if a be any fixed quantity however small, and b, c, d, &c. any fixed quantities however great, shew that x may be taken so great that ax^n shall contain $bx^{n-1}+cx^{n-2}+\ldots\ldots$ as many times as we please. (*Comp.* p. 148.)

(5) How small must x be taken, so that the third term of the infinite series $1+3x+5x^2+7x^3+\ldots\ldots$ may contain the sum of all that follows 500 times at least? (*Comp.* p. 149.)

Ans. $x<\frac{1}{702}$.

LOGARITHMS.

Apply the Logarithmic Tables to find,

(1) 2^{64}. Ans. 18446750000000000000.

(2) $\frac{239\times827\times543}{76\times17}$. Ans. 83069·32.

(3) $\sqrt[6]{235{\cdot}78}$. Ans. 2·485522...

(4) $\left(\frac{9}{8}\right)^{21}$. Ans. 11·86322...

(5) $\sqrt[5]{\frac{7}{3}\sqrt[4]{6}}$. Ans. 1·295695...

(6) $\sqrt[4]{\frac{132\times(7{\cdot}356)^9}{\sqrt{(3{\cdot}25)^5}}}$. Ans. 144·5972...

(7) $\frac{(1{\cdot}05)^7-1}{(1{\cdot}05)^7\times0{\cdot}05}$. Ans. 5·79...

(8) $\sqrt[5]{\left(\frac{23}{417}\right)^3}$. Ans. 0·17577...

(9) £15. 7*s*. 3½*d*.$\times(1{\cdot}03)^{50}$. Ans. £67. 7*s*. 1*d*.

(10) Given $20^x=100$, find x. Ans. $x=1{\cdot}537244$.

(11) Given $c^{mx}=a.b^{nx-1}$, find x. Ans. $x=\frac{\log a-\log b}{m\log c-n\log b}$.

(12) Given $a^{4x}+a^{2x}=a^{6x}$, find x. Ans. $x=\dfrac{\log\frac{1}{2}(1+\sqrt{5})}{2\log a}$.

(13) Given $a^{x}+\dfrac{1}{a^{x}}=b$, find x. Ans. $x=\dfrac{\log\frac{1}{2}(b\pm\sqrt{b^{2}-4})}{\log a}$.

(14) Given $(\sqrt[r]{a^{c}})^{x}=b^{cx-4a}$, find x. Ans. $x=\dfrac{4ar.\log b}{c(r\log b-\log a)}$.

(15) Given $x^{y}=y^{x}$, and $x^{p}=y^{q}$, find x and y. (*Comp.* p. 150.)

Ans. $x=\left(\dfrac{p}{q}\right)^{\frac{q}{p-q}}$, $y=\left(\dfrac{p}{q}\right)^{\frac{p}{p-q}}$.

(16) Given $3^{2x}.5^{3x-4}=7^{x-1}.11^{2-x}$, find x. (*Comp.* p. 151.)

Ans. $x=1{\cdot}242073\ldots$

(17) Given $(a^{4}-2a^{2}b^{2}+b^{4})^{x-1}=\dfrac{(a-b)^{2x}}{(a+b)^{2}}$, find x. (*Comp.* p. 151.)

Ans. $x=\dfrac{\log\overline{a-b}}{\log\overline{a+b}}$.

(18) Given $5^{x+1}-5^{x-3}+14\frac{1}{9}=4739-5^{x-4}+\frac{2}{7}-5^{x-2}$, find x. (*Comp.* p. 151.)

Ans. 4·25.

(19) Given $5\times3^{x}-3\times2^{y}=30000,$ and $3\times3^{x}+6\times2^{y}=20000,$ find x and y. (*Comp.* p. 152.)

Ans. $\begin{cases}x=7{\cdot}94,\\ y=8{\cdot}002.\end{cases}$

(20) Given $3^{x^{2}-4x+5}=1200$, find x. (*Comp.* p. 152.)

Ans. $x=4{\cdot}33$, or $-0{\cdot}33$.

(21) Given $a^{1}.a^{3}.a^{5}.a^{7}.\&c.=p$, find the number of factors a^{1}, a^{3}, a^{5}, &c. (*Comp.* p. 152.)

Ans. $\sqrt{\dfrac{\log p}{\log a}}$.

INTEREST AND ANNUITIES.

(1) WHAT principal put out at simple interest for 3 years, at the rate of 5 per cent. per annum, will amount to £828? Ans. £720.

(2) A person borrows £450 at 5 per cent. simple interest, and returns for it £517. 10*s.*; for what time was it lent? Ans. 3 years.

(3) A person returns £610 for the loan of £600 for one month, what is the rate of interest allowed? Ans. 20 per cent.

(4) What will a capital of £12000 amount to in 10 years, at 6 per cent. per annum compound interest, the interest being paid half-yearly?

Ans. £21673. *6s.* *2d.*

(5) Prove that the amount of £1 in *n* years at compound interest is given within less than a farthing by the first four terms of the expansion of $(1+r)^n$£, the rate of interest being not greater than 4 per cent., and *n* not greater than 10. (*Comp.* p. 153.)

(6) Find in what time a sum of money will double itself, put out at 5 per cent. per annum, reckoning compound interest. Ans. 14·2 years.

(7) Find in what time at compound interest, reckoning 5 per cent. per annum, £100 will amount to £1000. Ans. 47·194 years.

(8) What is the amount of one farthing, for 500 years, at 3 per cent. per annum, compound interest? Ans. £2731. *2s.* $5\frac{1}{4}$*d.*

(9) Shew that the common rule for determining the equated time of payment of several sums due at different times is in favour of the payer.

(10) Required the discount on £160 for a quarter of a year, reckoning interest at the rate of 5 per cent. per annum. Ans. £1. *19s.* $6\frac{2}{27}$*d.*

(11) What will be the amount of £1212 per annum left unpaid for 76 years, reckoning 4 per cent. per annum compound interest?

Ans. £566702. *14s.* *4d.*

(12) What will an annuity of £250 amount to in 7 years, paid half-yearly, allowing 6 per cent. per annum simple interest? Ans. £2091. *5s.*

(13) What annuity improved at the rate of 8 per cent. per annum, compound interest, will at the end of 10 years amount to £3000?

Ans. £207. *1s.* *9d.*

(14) What is the present value of an annuity of £20 to continue for 40 years, reckoning interest at the rate of 6 per cent. per annum?

Ans. £300. *18s.* *6d.*

(15) An annuity of £20 for 21 years is sold for £220; required the rate of interest allowed to the purchaser. (*Comp.* p. 154.)

Ans. £6. *16s.* *5d.* per cent. nearly.

(16) If a lease of $55\frac{1}{4}$ years be purchased for £100, what rent ought to be received, that the purchaser may make $5\frac{1}{4}$ per cent. per annum for his money? Ans. £5. *16s.*

(17) An annuity A is to commence at the end of p years, and to continue q years; find the equivalent annuity to commence immediately and to continue q years. (*Comp.* p. 154.) Ans. $\frac{A}{(1+r)^p}$.

(18) The discount on a promissory note of £100 amounted to £7. 10*s.* and the interest made by the banker was £5·405405... per cent.; find the interval at the end of which the note was payable. Ans. $1\frac{1}{2}$ years.

(19) A person puts out P£ at interest, and adds to his capital at the end of every year $\frac{1}{m}$th part of the interest for that year; find the amount at the end of n years. (*Comp.* p. 154.) Ans. $P\left\{\frac{mr+m+r}{m}\right\}^n$.

(20) The lease of an estate is granted for 7 years at a pepper-corn rent, with the condition that the tenant at the expiration of the lease may renew the same on paying a fine of £100. What is the value of the landlord's interest in the estate immediately after any such renewal, allowing compound interest at the rate of 5 per cent. per annum?

Ans. £245. 12*s.* 10*d.*

(21) A person spends in the first year m times the interest of his property; in the second $2m$ times that of the remainder; in the third $3m$ times that of what remained at the end of the second; and so on. At the end of $2p$ years he has nothing left. Shew that in the p^{th} year he spends as much as he has left at the end of that year. (*Comp.* p. 155.)

(22) The reversion of an estate in fee simple producing £60 a year is made over for the discharge of a debt of £577. 4*s.* 5*d.* How soon ought the creditor to take possession, if he be allowed 5 per cent. per annum interest for his debt? Ans. 15 years.

(23) A person puts his whole fortune P£ out at interest, at the rate of r£ per 1£ per annum, and requires for his annual expences p£ more than the whole interest of P£. In how many years, at this rate, will he have spent the whole? and if, when he has spent half his capital, he diminishes his expenditure one half, how much longer on this than on the former supposition will he continue solvent? (*Comp.* p. 155.)

(1) Ans. $\frac{\log\overline{Pr+p}-\log p}{\log\overline{1+r}}$. (2) Ans. $\log\frac{\left(\frac{Pr}{2}+p\right)-\log p}{\log\overline{1+r}}$.

(24) Find the present value of an annuity of £20 a year, to commence in 10 years, and then to continue 11 years, reckoning 4 per cent. per annum compound interest. Ans. £118. 7*s.* $3\frac{1}{2}$*d.*

(25) What sum ought to be paid for the Reversion of an annuity of £50 for 7 years after the next 14, that the purchaser may make 6 per cent. per annum of his money? Ans. £123. 9*s.* 1*d.*

(26) If a person purchase the Reversion of an estate after 20 years for £500, what rent ought it to produce that he may make 6 per cent. per annum of his money? Ans. £96. 4*s.* 3*d.*

(27) A debt of a£ accumulating at compound interest is discharged in n years by annual payments of $\frac{a}{m}$£; prove that $n = -\frac{\log(1-mr)}{\log(1+r)}$.

(28) If M_s, M_c represent the sums to which an annuity would amount in n years at simple and compound interest respectively, prove that

$$\frac{M_s}{M_c} = \frac{nr}{2} \cdot \frac{(n-1)r+2}{(1+r)^n-1}.$$

(29) If two joint-proprietors have an equal interest in a freehold estate worth p£ per annum, but one of them purchase the whole to himsel by allowing the other an equivalent annuity of q£ for n years, shew that

$$\frac{p}{q} = 2\left(1 - \frac{1}{(1+r)^n}\right). \quad (Comp.\ \text{p. } 157.)$$

(30) If P represent the population of any place at a certain time, and every year the number of deaths is $\frac{1}{p}$th, and the number of births $\frac{1}{q}$th, of the whole population at the beginning of that year; required the amoun of the population at the end of n years from that time. (*Comp.* p. 157.)

Ans. $P.\left\{1 + \frac{p-q}{pq}\right\}^n$

(31) In the last problem, if $p = 60$, and $q = 45$, shew that the population will be doubled in 125 years, nearly. (*Comp.* p. 157.)

(32) What must be the annual increase in the population of any country, that it may double itself in a century?

Ans. Between $\frac{1}{143}$rd and $\frac{1}{144}$th

CHANCES, AND LIFE ANNUITIES.

(1) What is the chance of drawing the four aces from a pack of cards in four successive trials? (*Comp.* p. 158.) Ans. $\frac{1}{270725}$.

(2) There are 4 white balls and 3 black placed at random in a line find the chance of the extreme balls being both black. (*Comp.* p. 158.)

Ans. $\frac{1}{7}$

(3) Of two bags one contains 9 balls, and the other 6, and in each bag the balls are marked *a*, *b*, *c*, *d*, &c. If one be drawn from each bag, wha is the chance that the two will have the same letter-mark? (*Comp.* p. 158.

Ans. $\frac{1}{9}$

(4) *A* plays one game with *B*, and another with *C*; the odds that he does not win both games are 4 to 1; the odds that he beats *B* are 3 to 2. What are the odds in his game with *C*? (*Comp.* p. 159.)

Ans. Odds against him 2 to 1.

(5) From a common pack of cards 12 are dealt to as many persons, one to each, the cards collected, shuffled, and the same repeated. What is the chance that a given person will on both occasions have the same card dealt to him? What is the chance that the two cards dealt to him will have the sum of their numbers equal to 3? (*Comp.* p. 159.)

(1) Ans. $\frac{1}{52}$. (2) Ans. $\frac{2}{169}$.

(6) Two dice are placed together at random so as to form a parallelopiped: determine the chance that two or more adjacent faces will have the same marks. (*Comp.* p. 160.)

Ans. $\frac{7}{24}$.

(7) From a bag containing 2 guineas, 3 sovereigns, and 5 shillings, a person is allowed to draw 3 of them indiscriminately: what is the value of his expectation? (*Comp.* p. 160.) Ans. $32\frac{1}{10}$ shillings.

(8) A shilling is thrown upon a chess-board, a square of which will just include 4 shillings, find the chance of its falling clear of a division. (*Comp.* p. 161.)

Ans. $\frac{1}{4}$.

(9) Three men, *A*, *B*, *C*, in succession throw a die, on condition that he who first throws an ace shall receive £1; what are the values of their several expectations? (*Comp.* p. 161.)

Ans. *A*'s = $7s.\ 10\frac{86}{91}d.$ *B*'s = $6s.\ 7\frac{11}{91}d.$ *C*'s = $5s.\ 5\frac{85}{91}d.$

(10) There are 5 persons, out of which 4 are going to play at whist. They all cut, and the lowest sits out. What is the chance that two specified individuals will be partners? (*Comp.* p. 161.)

Ans. $\frac{1}{5}$.

(11) There is a lottery containing black and white balls, from each drawing of which it is as likely a black ball shall arise as a white one, what is the chance of drawing 11 balls all white? (*Comp.* p. 162.)

Ans. $\frac{1}{2048}$.

(12) There is a lottery of 10 green, 12 white, and 14 red balls. Let 2 have been drawn, what is the probability that they will be green and white? (*Comp.* p. 162.) Ans. 17 to 4 against it.

(13) A die is thrown time after time: in how many times have we an even chance of throwing an ace? (*Comp.* p. 162.)

Ans. Not quite an even chance in 3, but more than even in 4, times

(14) Two witnesses, on each of whom it is 3 to 1 that he speaks truth, agree in affirming that a certain event did happen, which of itself is equally likely to have happened or not. What is the chance that the event did happen? (*Comp.* p. 163.) Ans. 9 to 1.

(15) *A* speaks truth 3 times out of 4, *B* 4 times out of 5, *C* 6 times out of 7; what is the probability of the truth of what *A* and *B* agree in asserting, but which *C* denies? (*Comp.* p. 163.) Ans. 2 to 1.

(16) Thirteen persons are required to take their places at a round table by lot; shew that it is 5 to 1 that two particular persons do not occupy contiguous seats. (*Comp.* p. 163.)

(17) *P* bets *Q* £10 to £590 that three races will be won by the three horses *A*, *B*, *C*, against which the betting is 4 to 1, 3 to 1, and 2 to 1, respectively. The first race having been won by *A*, and it being known that the second race was won either by *B*, or by a horse *F* against which the betting was 6 to 1, find the value of *P*'s expectation. (*Comp.* p. 164.)

Ans. £117. $5s.\ 5\frac{5}{11}d.$

(18) Supposing the House of Commons composed of *m* Tories and *n* Whigs, find the probability that a Committee of $p+q$ selected by ballot will consist of *p* Tories and *q* Whigs. (*Comp.* p. 164.)

Ans. $\dfrac{{}_mC_p \times {}_nC_q}{{}_{m+n}C_{p+q}}$.*

(19) There are 3 balls in a bag, of which one is white and one black, and the third white or black; determine the chance of drawing 2 black ones, if 2 be taken. (*Comp.* p. 165.) Ans. $\frac{1}{6}$.

(20) In a lottery all the tickets are blanks but one: each person draws a ticket, and retains it. Shew that each has an equal chance of drawing the prize. (*Comp.* p. 165.)

(21) A collection is made of ten letters taken at random from an alphabet consisting of 20 consonants and 5 vowels; what is the probability that it will contain 3 vowels and no more? (*Comp.* p. 165.) Ans. $\frac{60}{253}$.

(22) At the game of whist, what is the chance of dealing one ace and no more to a specified person? And what is the chance of dealing one ace to each person? (*Comp.* p. 166.)

(1) Ans. $\frac{9139}{20825}$, or $\frac{32}{73}$ nearly. (2) Ans. $\frac{2197}{20825}$, or $\frac{1}{9}$ nearly.

(23) There are two bags each containing 4 white and 4 black balls. Four are taken at a venture from one of them and transferred to the other. Then 8 being drawn from the latter, 6 of them prove white and two black; what is the chance that, if another be drawn, it will be white? (*Comp.* p. 167.) Ans. $\frac{91}{457}$.

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* ${}_nC_r$ signifies the Number of Combinations of *n* things taken *r* together.

(24) At the game of whist what is the chance of the dealer and his partner holding the four honours? (*Comp.* p. 167.)

Ans. $\frac{115}{1666}$, or $\frac{2}{29}$ nearly.

(25) In dealing a pack of cards, what is the chance that all the hearts will be found in the first 20 cards dealt, first without regard to order, and secondly in the order of their value? (*Comp.* p. 168.)

(1) Ans. $\frac{\underline{|39} \times \underline{|20}}{\underline{|7} \times \underline{|52}}$. (2) Ans. $\frac{\underline{|20} \times \underline{|39}}{\underline{|7} \times \underline{|13} \times \underline{|52}}$.

(26) A person puts his hand into a bag containing 12 balls, draws out a certain number at random, and transfers them without examination to a second empty bag. He then puts his hand into this second bag, and draws out a certain number in the same way as before. Shew that the odds in favour of his drawing out an odd number from the second bag are nearly 341 to 340. (*Comp.* p. 169.)

(27) Supposing it an even chance that, on *A* aged 46 marrying *B* aged 36, they will both be alive at the end of x years; find x, when, according to De Moivre's hypothesis, of 86 persons born together one dies annually until all are extinct. (*Comp.* p. 169.)

Ans. $x = 13$, nearly.

(28)* A person 35 years of age wishes to buy an annuity for what may happen to remain of his life after 50 years of age. What is the Present Value of the annuity reckoning interest at 4 per cent. and using Dr. Halley's Table†? (*Comp.* p. 170.) Ans. 4½ years' purchase nearly.

(29) An annuity of £10, for the life of a person now 30 years old, is to commence at the end of 11 years, if another person now 40 should be then dead. Required the Present Value of the annuity, reckoning interest at 4 per cent., and using Dr. Halley's Table. (*Comp.* p. 170.)

Ans. £17. 16*s*.

(30) An estate or annuity of £10 for ever will be lost to the heirs of a person now 34, if his life should fail in 11 years. What ought he to give for the Assurance of it for this term according to Dr. Halley's Table, reckoning interest at 4 per cent.? (*Comp.* p. 171.) Ans. £43·8.

(31) A person now 40 is willing to pay £200 down, besides an annual payment for 10 years, to entitle him to a life-annuity of £44 after he attains the age of 50. What ought the annual payment to be, reckoning interest at 4 per cent., and using Dr. Halley's Table? (*Comp.* p. 171.)

Ans. £8·55.

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* This and the three following questions are taken from *Price's Annuities.*

† See Page 299.

MISCELLANEOUS EXAMPLES.

1st Series.

(1) Find the G.C.M. of

$$(ax+by)^2-(a-b)(x+z)(ax+by)+(a-b)^2xz,$$

and $(ax-by)^2-(a+b)(x+z)(ax-by)+(a+b)^2xz$. (*Comp.* p. 172.)

Ans. $b(x+y)$.

(2) There are four numbers in Arith^c. Prog^n. The sum of the two extremes is 8, and the product of the means is 15. What are the numbers? (*App.* p. 353.) Ans. 1, 3, 5, 7.

(3) There are three numbers in Geom^c. Prog^n., whose product is 64, and sum 14. What are the numbers? (*App.* p. 354.)

Ans. 2, 4, 8, or 8, 4, 2.

(4) In what proportion must substances of "specific gravity" a and b be mixed, so that the "specific gravity" of the mixture may be c? (*App.* p. 348.) Ans. $\frac{a-c}{c-b}$ cubic feet of the latter to 1 of the former.

(5) From a vessel of wine containing a gallons b gallons are drawn off, and the vessel is filled up with water. Find the quantity of wine remaining in the vessel, when this has been repeated n times. (*App.* p. 349.)

Ans. $\frac{(a-b)^n}{{}^{n-1}}$.

(6) The advance of the hour-hand of a watch before the minute-hand is measured by $15\frac{2}{3}$ of the minute divisions; and it is between 9 and 10 o'clock. Find the exact time indicated by the watch. (*App.* p. 349.)

Ans. 28 min. before 10 o'clock.

(7) In comparing the rates of a watch and a clock, it was observed on one morning, when it was 12^h. by the clock, that the watch was at 11^h. 59^m. 49^s.; and two mornings after, when it was 9^h. by the clock, the watch was at 8^h. 59^m. 58^s. The clock is known to gain $0{\cdot}1^s$. in 24 hours, find the gaining rate of the watch. (*App.* p. 350.)

Ans. $4{\cdot}9000055^s$... in 24 hours.

(8) The product of two numbers is p, and the difference of their cubes is equal to m times the cube of their difference. What are the numbers? (*App.* p. 352.) Ans. $\frac{1}{2}\cdot\frac{\sqrt{(4m-1)p}+\sqrt{3p}}{\sqrt{m-1}}$, and $\frac{1}{2}\cdot\frac{\sqrt{(4m-1)p}-\sqrt{3p}}{\sqrt{m-1}}$.

(9) Shew that $x^n-na^{n-1}x+(n-1)a^n$ is divisible by $(x-a)^2$, if n be a whole number. (*Comp.* p. 172.)

(10) Shew that x^p-y^n is divisible by $x+y^{\frac{n}{p}}$, when p is an even number. (*Comp.* p. 173.)

(11) If N and n be nearly equal to each other, shew that

$$\sqrt{\frac{N}{n}} = \frac{N}{N+n} + \frac{1}{4}\cdot\frac{N+n}{n}, \text{ very nearly.}$$

Also if $\frac{N}{N+n}$, and $\frac{1}{4}\cdot\frac{N+n}{n}$, have their first p decimal places the same, this approximation for $\sqrt{\frac{N}{n}}$ may be relied on up to $2p$ decimal places at least. (*Comp.* p. 173.)

(12) Find the value of x which, when z is indefinitely increased, makes $(4x+1)(2z+1)^2 = 5(3x+1)(z+3)^2$; also find the values of x and y which make $\frac{2z^2+(z-a)z+2b(x-2c)}{3z^2+(y-b)z+3a(y-3c)}$ independent of z. (*Comp.* p. 174.)

(1) Ans. $x = 1$. (2) Ans. $x = a+2c$, $y = b+3c$.

(13) Shew that $x^4+px^3+qx^2+rx+s$ can be resolved into two rational quadratic factors, if $-s$ be a perfect square, and equal to $\frac{r^2}{p^2-4q}$. (*Comp.* p. 175.)

Hence solve the equation $x^4-6x^3+5x^2+8x-4 = 0$. (*Comp.* p. 175.)

Ans. $x = 2$, or -1, or $\frac{1}{2}(5\pm\sqrt{17})$.

(14) Prove that the square root of $\left(x-\frac{1}{x}\right)^2+\left(y-\frac{1}{y}\right)^2$ is a *rational* function of a and b, if $x = \frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}\right)$, and $y = \frac{ax-b}{a-bx}$. (*Comp.* p. 175.)

(15) In what cases can $ay^2+bxy+cx^2+dy+ex+f$ be resolved into rational factors of the first degree? (*Comp.* p. 176.)

Ans. When $ae^2+cd^2+fb^2 = bde+4acf$.

(16) A number consists of n figures; what is the number of figures in its r^{th} root? (*Comp.* p. 176.)

Ans. Not less than $\frac{n}{r}$, and not more than $\frac{n+r-1}{r}$.

(17) In the year 1843 January 1 was a Sunday; when will this happen again? (*Comp.* p. 177.) Ans. 1854, 1860, 1865, 1871, &c.

(18) What day of the week was Sept. 14, 1752? given that the same day of the month A.D. 1846 was a Monday. (*Comp.* p. 177.)

Ans. Thursday.

(19) How often are there five Sundays in February; and when does this happen in the present century after the year 1844, the first day of that year being a Monday? (*Comp.* p. 177.)

(1) Ans. Once in 28 years. (2) Ans. A.D. 1852, and A.D. 1880.

(20) If q be the integral part of $\frac{p}{4}$, and r the remainder when $p+q+4$ is divided by 7, shew that in the p^{th} year of this century Advent Sunday is generally $7-r$ days *after* Nov. 27; but, when $r=0$, it falls *on* Nov. 27. (*Comp.* p. 178.)

(21) It was calculated that, if the gross revenue of a state were increased in the proportion of $2\frac{1}{4} : 1$, after deducting the interest of the national debt and the cost of collection (the latter of which varies as the square root of the sum collected), the available income would be increased in the proportion of $3\frac{16}{31} : 1$. If, on the other hand, the gross revenue were diminished in the proportion of $1\frac{7}{9} : 1$, the available income would be reduced in the proportion of $7\frac{3}{4} : 1$, and would in fact amount to only 4 millions. Find the amount of the revenue, and the interest of the debt. (*App.* p. 355.)

(1) Ans. 64 millions £. (2) Ans. 29 millions £.

(22) Fine gold chains are manufactured at Venice, and are sold at so much per braccio, a braccio being a measure containing about two feet English. When there are 90 links in an inch, the value of the workmanship of a braccio is equal to the whole value of a braccio when there are but 30 links in an inch; and the whole value of the braccio in the former case is equal to three times the difference between the cost of the material and workmanship of a bracchio in the latter, together with $4\frac{4}{9}$ francs. Supposing that the workmanship in each braccio varies as the number of links in an inch, and the weight of metal varies inversely as the square of that number, find the values of the material and workmanship in a braccio of each of the chains. (*App.* p. 358.)

$$(1)\ \text{Ans.}\ \left.\begin{array}{l}40\ \text{francs}\\ 20\ \ldots\ldots\end{array}\right\}. \qquad (2)\ \text{Ans.}\ \left.\begin{array}{l}4\frac{4}{9}\ \text{francs}\\ 60\ \ldots\ldots\end{array}\right\}.$$

(23) A steam-vessel leaves Oban for Staffa with a supply of whisky b above proof (which is assumed to mean that $a+b$ gallons of spirit are mixed with c gallons of water), sufficient for two days' consumption, provided it receive no addition to its crew. On arriving at Tobermory m of its passengers remain behind, but by reason of contrary winds its progress to Staffa the following morning is retarded, so that on its return to Oban it is with difficulty enabled to reach Iona by midnight. It here receives n additional passengers, and also p gallons of whisky d above proof. On an average each passenger dilutes his whisky with water till it is e below proof, and consumes q pints of the mixture daily. The vessel arrives at Oban on the evening of the third day after its departure, by which time the supplies of whisky are both exhausted. Required the number of passengers on board when it left Oban, and the number of gallons of whisky in the first supply. (*Comp.* p. 178.)

$$(1)\ \text{Ans.}\ 2m-n+\frac{8p}{q}.\frac{a+d}{a-e}.\frac{a-e+c}{a+d+c}.$$

$$(2)\ \text{Ans.}\ (2m-n)\frac{q}{4}.\frac{a-e}{a+b}.\frac{a+b+c}{a-e+c}+2p.\frac{a+d}{a+b}.\frac{a+b+c}{a+d+c}.$$

(24) A pack of np cards is dealt regularly round to p persons with their faces uppermost, every card dealt to each person being placed upon that previously dealt to him; the hands are then taken up, turned so as to have their backs uppermost, and placed upon one another; that hand which contains a particular card (A) being always placed below r other hands. The cards are then dealt again, the hands taken up, turned, and placed upon one another as before; and so on:—Shew that, if m and q be the whole numbers next greater than $\frac{\log 2np}{\log p}$ and $\frac{rnp}{p-1}$ respectively, the card A will, at the end of the m^{th} and every succeeding operation, occupy the q^{th} place, or be restricted to the q^{th} and $\overline{q-1}^{\text{th}}$ places from the top, according as rn is indivisible or divisible by $p-1$. (*App.* p. 359.)

MISCELLANEOUS EXAMPLES.

2nd SERIES.

[The *Solutions* of the whole series will be found in the *Companion.*]

SOLVE the following equations:

(1) $2^{x+1}+4^x = 80.$ Ans. $x = 3.$

(2) $x^3+3x = a^3-\frac{1}{a^3}.$ Ans. $x = a-\frac{1}{a}$, or &c.

(3) $\left.\begin{aligned} 7^{\left(\frac{x}{2}+\frac{y}{3}\right)} &= 2401, \\ 6^{\left(\frac{x}{4}+\frac{2}{y}\right)} &= 1296. \end{aligned}\right\}$ Ans. $\left.\begin{aligned} x &= 4, \\ y &= 6. \end{aligned}\right\}$

(4) $(x-3)(x-4)(x-5)(x-6) = 1\times2\times3\times4.$

Ans. $x = 2$, or 7, or $\frac{1}{2}(9\pm\sqrt{-15})$.

(5) $x^4+ax^3+bx^2+cx+\frac{c^2}{a^2} = 0.$

(6) Reduce to simplest form $\frac{(1-a^2)(1-b^2)(1-c^2)-(c+ab)(b+ac)(a+bc)}{1-a^2-b^2-c^2-2abc}$.

Ans. $1+abc$.

(7) Find x, when $(x+a)(x+b)(x+c)(x+d)-(x-a)(x-b)(x-c)(x-d) = (a+b+c+d)\{(x+a)(x+b)(x+c)+(x-a)(x-b)(x-c)\}$ is an '*Equation*'; and shew that there are three, and only three, relations between a, b, c, any one of which will cause it to be an '*Identity*'.

(1) Ans. $x = 0$. (2) Ans. $a+b = 0$, or $a+c = 0$, or $b+c = 0$.

(8) If $(a^2+bc)^2(b^2+ac)^2(c^2+ab)^2 = (a^2-bc)^2(b^2-ac)^2(c^2-ab)^2$, prove that either $a^3+b^3+c^3+abc = 0$, or $a^{-3}+b^{-3}+c^{-3}+a^{-1}b^{-1}c^{-1} = 0$.

(9) Four bells commence tolling together, and toll at intervals of 18, 45, 81, 105, seconds, respectively; what time will elapse before they again toll simultaneously? Ans. 1hr. 34½min.

(10) A number of wheels commence rolling at the same moment from the same place in the same direction, travelling at the same rate, the number of feet in their circumferences being a_1, a_2, a_3,...a_n, which are n different prime numbers. How far will they have travelled, when they have all first completed simultaneously numbers of entire revolutions, and how many revolutions will *each* have completed?

(1) Ans. $a_1a_2a_3\ldots a_n$. (2) Ans. $a_2a_3\ldots a_n$ for the 1st; $a_1a_3a_4\ldots a_n$ for 2nd; &c.

(11) Find the sum of n such fractions as

$$\frac{1}{1+x}, \quad \frac{2x}{1+x^2}, \quad \frac{4x^3}{1+x^4}, \quad \frac{8x^7}{1+x^8}, \quad \&c.$$

Ans. $\dfrac{1}{1-x} - \dfrac{mx^{m-1}}{1-x^m}$, where $m = 2^n$.

(12) Find the product of n such binomials as

$$x+\frac{1}{x}, \quad x^2+\frac{1}{x^2}, \quad x^4+\frac{1}{x^4}, \quad x^8+\frac{1}{x^8}, \quad \&c.$$

Ans. $\dfrac{1-x^{2m}}{(1-x^2)x^{m-1}}$, where $m = 2^n$.

(13) If a number contain n digits, prove that its square root contains $\frac{1}{2}n+\frac{1}{4}-\frac{1}{4}(-1)^n$ digits.

(14) On June 21, A.D. 1851 the Duke of Wellington had lived exactly 30,000 days. Find the day and year of his birth.

Ans. May 1, A.D. 1769.

(15) Apply the method of proof called "demonstrative induction" to prove that

$$1^3+2^3+3^3+\ldots+n^3 = (1+2+3+\ldots+n)^2.$$

(16) If x be real, prove that $x^2-8x+22$ can never be less than 6.

(17) Prove that the volume of a sphere whose radius is 6 inches is equal to the sum of the volumes of three spheres whose radii are 3, 4, and 5 inches; given that the volume of a sphere varies as the cube of its radius.

(18) Standard gold being coined at the rate of £3. 17*s*. 10½*d* per oz. what is the least integral number of ounces that can be coined into an integral number of sovereigns? Ans. 160.

(19) Find m and n in terms of a and b, so that $\dfrac{ma+nb}{m+n}$ may be the Arith^c. Mean between m and n, and the Geom^c. Mean between a and b.

Ans. $\dfrac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, $\dfrac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

(20) A vessel contains a gallons of wine, and another b gallons of water; c gallons are taken out of each and transferred to the other; and this operation is repeated any number of times. Shew that, if $c = ab \div (a+b)$, the quantity of wine in each vessel will always remain the same after the first operation.

(21) In the last problem, shew that, if c be general in value, the quantity of wine in the second vessel after r operations will be

$$\frac{ab}{a+b}(1-p^r), \text{ where } c = \frac{ab}{a+b}(1-p).$$

(22) Divide an odd number, $2n+1$, into two whole numbers, so that their product may be a maximum. Ans. n, and $n+1$.

(23) If x be real, prove that $\dfrac{2x-7}{2x^2-2x-5}$ can have no value between $\dfrac{1}{11}$ and 1.

(24) Eliminate a and b from the equations

$$\frac{a^3-x^3}{b^3-y^3} = \frac{2x+3y}{3x+2y}, \quad a^3-b^3 = (x-y)^3, \quad a^{\frac{3}{2}}+b^{\frac{3}{2}} = z^{\frac{3}{2}}.$$

Ans. $x^{\frac{1}{2}}+y^{\frac{1}{2}} = z^{\frac{1}{2}}$.

(25) Eliminate m, n, p, q, from the equations

$$\frac{x-p}{m}+\frac{y+q}{n} = \frac{pm}{a^2}+\frac{qn}{b^2} = \frac{m^2}{a^2}-\frac{n^2}{b^2} = \frac{p^2}{a^2}+\frac{q^2}{b^2}-1 = 0.$$

Ans. $\dfrac{x}{a}+\dfrac{y}{b} = \sqrt{2}$.

(26) If the roots of the equation $ax^2+bx+c = 0$ are in the ratio of m to n, shew that $\dfrac{b^2}{ac} = \dfrac{(m+n)^2}{mn}$.

(27) Find the sum of all the numbers of the form 121, 12321, 1234321, &c. in the scale whose radix is r.

$$\text{Ans. } \frac{1}{(r-1)^2}\left\{\frac{r^4(r^{18}-1)}{r^2-1} - 2r^2 \cdot \frac{r^{11}-1}{r-1} + 9\right\}.$$

(28) Shew that 12345654321 is divisible by 12321 in any scale of which the radix > 6.

(29) Find the series in A.P. of which the sum of the first n terms is equal to n^2, whatever be the value of n. Ans. 1, 3, 5, 7, &c.

(30) From the last Ex. deduce the integral solutions of the equation $x^2 = y^2+z^2$.

$$\text{Ans. } \left.\begin{array}{r} x = 5, \\ y = 4, \\ z = 3. \end{array}\right\} \left.\begin{array}{r} x = 13, \\ y = 12, \\ z = 5. \end{array}\right\} \left.\begin{array}{r} x = 25, \\ y = 24, \\ z = 7. \end{array}\right\} \text{\&c.}$$

(31) Shew that $\sqrt[m+n+p+q]{abcd}$ lies between the greatest and least of the quantities $\sqrt[m]{a}$, $\sqrt[n]{b}$, $\sqrt[p]{c}$, $\sqrt[q]{d}$.

(32) The sum of two numbers is 45, and their L.C.M. is 168; what are the numbers? Ans. 21, and 24.

(33) The G.C.M. of two numbers is 16, and their L.C.M. is 192; what are the numbers? Ans. 64 and 48.

(34) Find x when $18x-3x^2>24$. Ans. $x = 3$.

(35) Find the values of x which satisfy the *inequality* $x^2<10x-16$.

Ans. $x = 3, 4, 5, 6, 7$.

(36) Find the greatest value of $\dfrac{x^2+14x+9}{x^2+2x+3}$. Ans. 4.

(37) If x be real, prove that $\dfrac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9.

(38) Prove that $1.2.3\ldots n<\left(\dfrac{n+1}{2}\right)^n$.

(39) Two smiths begin to strike their anvils together. The one (A) gives 12 strokes in 7 minutes, the other (B) 17 strokes in 9 minutes. What strokes of each most nearly coincide in the first half hour?

Ans. The 11th of A, and the 10th of B.

(40) There are n points in a plane, no three of which are in the same straight line, with the exception of p which are all in the same straight line: find the number of triangles which result from joining them.

Ans. $\frac{1}{6}\{n(n-1)(n-2)-p(p-1)(p-2)\}$.

(41) Prove that $\dfrac{\underline{|n_1+n_2}}{\underline{|n_1}\cdot\underline{|n_2}}$ is an integer; and deduce that

$$\frac{\underline{|n_1+n_2+n_3+\ldots+n_r}}{\underline{|n_1}\cdot\underline{|n_2}\cdot\underline{|n_3}\cdots\cdots\underline{|n_r}} \text{ is an integer.}$$

(42) Prove, without the aid of the Binomial Theorem, that if ${}_nC_r$ represent the number of combinations of n things r together,

$${}_nC_1+{}_nC_2+{}_nC_3+\ldots+{}_nC_n = 2^n-1.$$

(43) Along a straight line are placed n points. The distance between the first two points is one inch; and in general the distance between the r^{th} and $(r+1)^{\text{th}}$ points exceeds one inch by $\frac{1}{m}$th of the distance between the r^{th} and $(r-1)^{\text{th}}$ points. Find the distance between the last two points, and between the first and last points.

Ex. If $m = n = 10$, shew that the distance between the extreme points is 9·87654321 inches.

(1) Ans. $\dfrac{1-m^{-(n-1)}}{1-m^{-1}}$. (2) Ans. $\dfrac{1}{1-m^{-1}}\left\{n-1-m^{-1}.\dfrac{1-m^{-(n-1)}}{1-m^{-1}}\right\}$.

(44) Two straight rods, each c inches long, and divided into m and n equal parts respectively, where m and n are prime to one another, are placed in longitudinal contact with their ends coincident. Prove that no two divisions are at a less distance than $\frac{c}{mn}$ inches; and that two pairs of divisions are *at* this distance.

Ex. If $m = 250$, and $n = 243$, find those divisions which are at the least distance. Ans. The 107th, and 104th; or the 143th, and 139th.

(45) Two bells toll together for an hour: one tolls 244 times, and the other 251 times, the first and last tolls of each taking place at the beginning and end of the hour respectively. Of the strokes (excluding the first and last) find those which are most nearly simultaneous; and determine a person's station in the straight line joining the bells, that those which are most nearly simultaneous of all may appear to him absolutely coincident; given that sound travels at the rate of 1080 feet per second, and the bells being a miles asunder.

(1) Ans. $\left\{\begin{array}{l}\text{The 105th of I,}\\ \text{and 108th of II;}\end{array}\right.\left\{\begin{array}{l}\text{also the 140th of I,}\\ \text{and 144th of II.}\end{array}\right.$

(2) Ans. See *Companion.*

(46) Three bells commenced tolling simultaneously, and tolled at intervals of 25, 29, 33, seconds respectively. In less than half an hour the first ceased, and the second and third tolled 18 seconds, and 21 seconds, respectively, after this, and then ceased. How many times did each bell toll? Ans. 49, 43, and 38.

(47) Two particles a, b, start simultaneously from the same point, and move in the same direction, along the same straight line; a moves with the uniform speed of 2 feet per second, whilst the speed of b is such that it moves over 1 foot the first second, and the number of feet described by it in any time *varies* as the square of the time. Prove that, during the motion, the particles will be three times at any given distance, less than a foot, from each other: and interpret the fourth result obtained in the algebraical solution of the problem.

(48) Supposing the receipts on a railway to vary as the increase of speed above 20 miles an hour, whilst the cost of working the trains varies as the square of that increase, and that at 40 miles an hour the expenses are just paid; find the speed at which the profits will be the greatest.

Ans. 30 miles.

(49) Determine whether the infinite series $\frac{a}{m+p} + \frac{a^2}{m+2p} + \frac{a^3}{m+3p} + \ldots$ is convergent or divergent. Ans. Convergent or divergent as $a <$ or > 1.

(50) Prove that every quadratic surd, supposing the unit to be a line, may have an exact geometrical representation.

(51) A person continually walks at an uniform speed, and always in the same direction, round the boundary of a square field: and another continually walks, at the same uniform speed, from one end to the other of a diagonal of the field. Prove that, according to their relative initial positions, they will either (1) never meet at all, or (2) meet once; but they can never meet more than once however long they continue to walk.

(52) A sum of money in £. *s.* *d.* is multiplied by a certain number; the pence are now half what they were before, and the shillings and pounds each what the shillings were at first. What is the sum, and the multiplier? (1) Ans. £9. 19*s.* 8*d.* (2) Ans. 2.

(53) If $\frac{N_1}{D_1}, \frac{N_2}{D_2}$, are any two consecutive convergents to $\frac{a}{b}$, shew that the error in taking $\frac{N_1}{D_1}$ for $\frac{a}{b} < \frac{1}{D_1 D_2}$ but $> \frac{1}{D_1(D_2+D_1)}$.

(54) If $\rho = \sqrt{\alpha^2+\beta^2}$ be defined to be the '*modulus*' of a binomial of the form $\alpha+\beta\sqrt{-1}$, where α, β, are rational; and $\rho_1, \rho_2, \rho_3, \ldots \rho_n$, be the moduli of n such binomials; prove that the modulus of the symbolical product of these n binomials is $\rho_1 \rho_2 \rho_3 \ldots \rho_n$.

(55) Explain the notation of '*functions*'; and shew that, if $F(x) = a^x$, $F(x) \times F(y) = F(x+y)$.

(56) If $F(n, m) = \frac{1}{m} - n . \frac{1}{m+p} + \frac{n(n-1)}{1\,.\,2} . \frac{1}{m+2p} - \&c.$, shew that

$$F(n, m) = \frac{np}{m} . F\{(n-1), (m+p)\};$$

and thence deduce the sum of the series for $F(n, m)$, when n is a positive integer. Ans. $\frac{1.2.3\ldots n}{m(m+p)\ldots(m+np)} . p^n$.

(57) The Julian Period consists of 7980 years; the cycles of the sun, moon, and the Indictions, of 28, 19, and 15, years respectively. If any particular year be the n^{th} of the Julian period, the remainders after the division of n by 28, 19, 15, are the years of the Solar and Lunar cycles, and of the Indictions, respectively. Prove the following Rule for determining the year of the Julian period when the years of the Solar and Lunar cycles and of the Indictions are given.

Multiply the number of the year in the Solar Cycle by 4845, in the Lunar by 4200, and in the cycle of Indiction by 6916; divide the sum of these products by 7980, and the *remainder* is the year of the Julian period sought.

(58) If an import duty of r shillings a quarter be laid on foreign corn when the price of corn in the English market is p shillings a quarter, and e, f, are the numbers of quarters of English and Foreign corn consumed in a year; and if the imposition of such a duty causes the price to rise to $p+\frac{r}{n}$ shillings a quarter, and the consumption to sink, so that the same sum of money is still expended, the English produce remaining constant; find the value of r most productive of revenue. Ans. $np\left\{\sqrt{1+\frac{f}{e}}-1\right\}$.

(59) A square signboard is divided into 16 equal squares by vertical and horizontal lines. In how many ways can 4 of these squares be painted white, 4 black, 4 red, and 4 blue, without repeating the same colour in the same vertical or horizontal row?

Ans. 576.

(60) Shew that the greatest coefficient, formed from the index n, in the expansion of $(a_1+a_2+a_3\ldots+a_m)^n$ is $\frac{\lfloor n}{(\lfloor q)^m.(q+1)^r}$, where q is the quotient and r the remainder, when n is divided by m.

MISCELLANEOUS EXAMPLES.

3rd Series.

[Solutions will be found in the *Companion*.]

Solve the following equations:—

(1) $x^3-\sqrt[3]{6}.x=1$. Ans. $x=\sqrt[3]{\frac{1}{\sqrt[3]{2}-1}}$.

(2) $x^3-3x=a^3+\frac{1}{a^3}$. Ans. $x=a+\frac{1}{a}$, or &c.

(3) $(x+a)(x+2a)(x+3a)(x+4a) = c^4$. Ans. $x = -\frac{5a}{2} \pm \sqrt{\frac{5a^2}{4} \pm \sqrt{c^4+a^4}}$.

(4) $x^3+px^2+\left(p-1+\frac{1}{p-1}\right)x+1 = 0$. Ans. $x = 1-p$, or &c.

(5) $(p-1)^2x^3+px^2+\left(p-1+\frac{1}{p-1}\right)x+1 = 0$. Ans. $x = \frac{1}{1-p}$, or &c.

(6) $\left.\begin{aligned} x^2(y-z) &= a^3, \\ y^2(z-x) &= b^3, \\ z^2(x-y) &= c^3. \end{aligned}\right\}$

$$\text{Ans. } x^3 = \frac{R^2-a^6}{(R+b^3)(R-c^3)} . R, \quad y^3 = \frac{R^2-b^6}{(R+c^3)(R-a^3)} . R, \quad z^3 = \frac{R^2-c^6}{(R+a^3)(R-b^3)} . R,$$

where $R^2 = -\frac{a^3b^3c^3}{a^3+b^3+c^3}$.

(7) $\left.\begin{aligned} x^2(y+z) &= a^3, \\ y^2(z+x) &= b^3, \\ xyz &= c^3. \end{aligned}\right\}$ Ans. $x^3 = \frac{(a^3+c^3)(a^3b^3-c^6)}{(b^3+c^3)^2}$, $y^3 = \frac{(b^3+c^3)(a^3b^3-c^6)}{(a^3+c^3)^2}$,

$$z^3 = \frac{(a^3+c^3)(b^3+c^3)c^9}{(a^3b^3-c^6)^2}.$$

(8) $xyz = a^2(y+z) = b^2(z+x) = c^2(x+y)$.

$$\text{Ans. } x^2 = 2 . \frac{\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2}}{\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)\left(\frac{1}{a^2}+\frac{1}{c^2}-\frac{1}{b^2}\right)},$$

and similar expressions for y^2 and z^2.

(9) $xyz = a(y^2+z^2) = b(z^2+x^2) = c(x^2+y^2)$.

Ans. $x^2 = \frac{4}{\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)}$, and similarly y^2 and z^2.

(10) $\left.\begin{aligned} (x+y)(x+z) &= a^2, \\ (y+z)(y+x) &= b^2, \\ (z+x)(z+y) &= c^2. \end{aligned}\right\}$

Ans. $x = \pm \frac{abc}{2}\left(-\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$, and similarly y and z.

(11) $\left.\begin{aligned} x^3+y^3+z^3 &= 3xyz, \\ 3a-x+z = 3b-y+x &= 3c-z+y. \end{aligned}\right\}$ Ans. $x = a-b$, $y = b-c$, $z = c-a$.

(12) Obtain in the simplest form the value of x which satisfies the equation

$$(x+a-b-c-d)(x-a+b-c-d)(x-a-b+c-d)(x-a-b-c+d)$$
$$=(x-2a)(x-2b)(x-2c)(x-2d);$$

and investigate three equations, independent of x, which include all those relations between a, b, c, d, for any one of which the equation becomes an *identity*.

(1) Ans. $x=\frac{1}{2}(a+b+c+d)$.

(2) Ans. $\left.\begin{aligned} a+b&=c+d,\\ a+c&=b+d,\\ a+d&=b+c.\end{aligned}\right\}$

(13) Two numbers are as 3 : 5, and their G.C.M. is 555: what are the numbers? Ans. 1665, and 2775.

(14) Each of two points moves uniformly in the circumference of a circle: one goes round 5 times while the other goes round twice; and at the end of 21590 days they return for the first time to the place from which they started. How long does each take to go round?

Ans. 4318 days, and 10795 days.

(15) Four chronometers, which gain 6, 15, 27, 35, seconds a day respectively, shew true time. After how many days will this happen again? Ans. 43200 days.

(16) Two points revolve in the circumference of two circles. At starting they are in a common diameter of the circles; and before arriving again simultaneously in their initial positions, they have been in common diameters n times or m times, according as they have revolved in the same or in opposite directions. Prove that $m+n$ and $m-n$ must have a common divisor 4, and no higher common divisor; and that the ratio of the periods of revolution of the points is $m+n : m-n$.

(17) Three particles start at the same point to move uniformly in the circumference of a circle, and the period of the first : that of the second : that of the third :: a prime number : a second prime number : a third prime number. The first and second arrive together at the starting-point 6 min. before the first and third, and 28 min. before the second and third arrive there together; also when the first and second arrive there together, they have been together as many times as the second and third when they arrive there together; and all three arrive there together 1 h. 55 min. 30 sec. after starting. Determine the periods of revolution. Ans. $1\frac{1}{2}$min., $3\frac{1}{2}$min., $5\frac{1}{2}$min., respectively.

(18) If the G.C.M. of $m+n$ and $m-n$ be 4: prove that the G.C.M. of m and n will be either 2, or 4; and that when it is 2, each of $\frac{m}{2}$, $\frac{n}{2}$ is odd, and when it is 4, one of $\frac{m}{4}$, $\frac{n}{4}$ is odd, and the other even.

(19) From the longer of two rods a piece equal to the shorter is cut off: with the shorter and the remainder of the longer a similar operation is performed; and so on, until the rods are of equal length. Prove that if this last length be taken for the unit of measurement, the lengths of the original rods will be represented by numbers prime to each other. Also,

Explain that there are many relative lengths of the original rods for which the cut rods will never become equal, however long the operation be continued.

(20) Define commensurable magnitudes; and prove that magnitudes which are commensurable with the same magnitude are commensurable with one another.

(21) The adjacent sides of a rectangular parallelogram are respectively equal to the hypothenuses of two right-angled triangles whose sides are commensurable with the unit of linear measurement. Prove that its area will be commensurable or incommensurable with the corresponding unit of square measurement, according as its sides are or are not commensurable with each other.

(22) Shew how to find *geometrically* lines equal to the G.C.M. and the L.C.M. of two given commensurable straight lines.

(23) Prove that

$$\frac{a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}=(a+b)(b+c)(c+a).$$

(24) Obtain the continued product of

$$a+b+c+d,\quad a+b-c-d,\quad a-b-c+d,\quad a-b+c-d;$$

also of

$$-a+b+c+d,\quad a-b+c+d,\quad a+b-c+d,\quad a+b+c-d;$$

and shew that the sum of these products $= 16abcd$.

(25) Find the continued product of n such trinomials as

$$x^2-ax+a^2,\quad x^4-a^2x^2+a^4,\quad x^8-a^4x^4+a^8,\quad x^{16}-a^8x^8+a^{16},\ \&c.$$

Ans. $\dfrac{x^{2m}+a^mx^m+a^{2m}}{x^2+ax+a^2}$, where $m=2^n$.

(26) Find the sum of n such fractions as

$$\frac{2x-a}{x^2-ax+a^2},\quad \frac{4x^3-2a^2x}{x^4-a^2x^2+a^4},\quad \frac{8x^7-4a^4x^3}{x^8-a^4x^4+a^8},\ \&c.$$

Ans. $\dfrac{2mx^{2m-1}+ma^mx^{m-1}}{x^{2m}+a^mx^m+a^{2m}}-\dfrac{2x+a}{x^2+ax+a^2}$, where $m=2^n$.

(27) Prove that $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a)$.

(28) For what unit of time will the durations 11574^s. and 360360^s. be represented by numbers prime to each other? Resolve the same question for 3·6 hours and 2·76 hours.

(1) Ans. 18 seconds. (2) Ans. 432 seconds.

(29) Divide $1+2x^{2n+1}+x^{4n+2}$ by $1+2x+x^2$; writing down the $(2n-m+1)^{\text{th}}$ and $(2n+m+1)^{\text{th}}$ terms in the quotient; the $(2n-m)^{\text{th}}$ and $(2n+m)^{\text{th}}$ remainders; and the complete quotient.

(1) Ans. $(-1)^m.(2n-m+1).x^{2n-m}$.

(2) ... $(-1)^m.(2n-m+1).x^{2n+m}$.

(3) ... $(-1)^m\{(2n-m+1)x^{2n-m}+(2n-m).x^{2n-m+1}\}+2x^{2n+1}+x^{4n+2}$.

(4) ... $(-1)^m\{(2n-m+1)^{2n+m}+(2n-m+2)x^{2n+m+1}\}+x^{4n+2}$.

(5) ... $1-2x+3x^2-\ldots+(2n+1)x^{2n}-2nx^{2n+1}+\ldots+x^{4n}$.

(30) Eliminate x, y, z from the equations

$$\left.\begin{aligned}(x-y)(y-z)(z-x) &= a^3,\\ (x+y)(y+z)(z+x) &= b^3,\\ (x^2+y^2)(y^2+z^2)(z^2+x^2) &= c^6,\\ (x^4+y^4)(y^4+z^4)(z^4+x^4) &= p^{12}.\end{aligned}\right\}$$

Ans. $4c^6\pm4\sqrt{2p^{12}-a^6b^6}=\{b^3\pm\sqrt{2c^6-a^6}\}^2$.

(31) Eliminate x from the equations

$$\left.\begin{aligned}32\frac{c}{a} &= \left(\frac{x}{a}\right)^5+10\frac{x}{a}+5\left(\frac{a}{x}\right)^3,\\ 32\frac{a}{c} &= \left(\frac{a}{x}\right)^5+10\frac{a}{x}+5\left(\frac{x}{a}\right)^3.\end{aligned}\right\}$$

Ans. $1=\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{2}{5}}-\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{2}{5}}$.

(32) Eliminate x, y, z from the equations

$$\left.\begin{aligned}\frac{x}{y}+\frac{y}{z}+\frac{z}{x} &= \alpha,\\ \frac{x}{z}+\frac{y}{x}+\frac{z}{y} &= \beta,\\ \left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right) &= \gamma.\end{aligned}\right\}$$

Ans. $\alpha\beta=1+\gamma$.

(33) A symmetrical form of the condition, that the equations

$$ax+a'=bx+b'=cx+c'$$

may be simultaneous, is $a'(b-c)+b'(c-a)+c'(a-b)=0$;

and a symmetrical form of the value of x is $\dfrac{aa'(b-c)+bb'(c-a)+cc'(a-b)}{(a-b)(b-c)(c-a)}$.

(34) If $\dfrac{a-a'}{a'-a''}=\dfrac{b-b'}{b'-b''}=\dfrac{c-c'}{c'-c''}$, prove that

$$\frac{ab'-a'b}{b'b''-a''b'}=\frac{bc'-b'c}{b'c''-b''c'}=\frac{ca'-c'a}{c'a''-c''a'},$$

and = each of the former; and each of these six fractions

$$=\frac{(a+b+c)-(a'+b'+c')}{(a'+b'+c')-(a''+b''+c'')}.$$

(35) Prove that the six equations

$$\begin{aligned}
a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)&=0,\\
b_1(c_2a_3-c_3a_2)+b_2(c_3a_1-c_1a_3)+b_3(c_1a_2-c_2a_1)&=0,\\
c_1(a_2b_3-a_3b_2)+c_2(a_3b_1-a_1b_3)+c_3(a_1b_2-a_2b_1)&=0,\\
d_1(b_2c_3-b_3c_2)+d_2(b_3c_1-b_1c_3)+d_3(b_1c_2-b_2c_1)&=0,\\
d_1(c_2a_3-c_3a_2)+d_2(c_3a_1-c_1a_3)+d_3(c_1a_2-c_2a_1)&=0,\\
d_1(a_2b_3-a_3b_2)+d_2(a_3b_1-a_1b_3)+d_3(a_1b_2-a_2b_1)&=0,
\end{aligned}$$

are equivalent to only two independent equations: and find them in the most simple symmetrical shape.

$$\text{Ans. } \frac{\dfrac{a_1}{d_1}-\dfrac{a_2}{d_2}}{\dfrac{a_2}{d_2}-\dfrac{a_3}{d_3}}=\frac{\dfrac{b_1}{d_1}-\dfrac{b_2}{d_2}}{\dfrac{b_2}{d_2}-\dfrac{b_3}{d_3}}=\frac{\dfrac{c_1}{d_1}-\dfrac{c_2}{d_2}}{\dfrac{c_2}{d_2}-\dfrac{c_3}{d_3}}.$$

(36) There are p coins $c_1, c_2, c_3 \dots c_p$. If m_1 of the coins $c_1 = n_2$ of c_2; m_2 of $c_2 = n_3$ of c_3; m_3 of $c_3 = n_4$ of c_4, &c.; m_{p-1} of $c_{p-1} = n_p$ of c_p: how many of $c_p = n_1$ of c_1?

Ans. $\dfrac{n_1.n_2.n_3\dots n_p}{m_1.m_2.m_3\dots m_{p-1}}$.

Establish the "chain rule".

(37) There are two amalgams of the same bulk, each composed of mercury and gold, in the ratios of 2 : 9, and 3 : 19, respectively. If they were fused together, what would be the ratio of mercury to gold in the resulting amalgam? Ans. 7 : 37.

(38) At noon on a certain day a clock and a watch, each of which goes uniformly, are set to true time. It was calculated, that the clock would be as much wrong when it should shew any time as the watch would be when it should come to shew the same time; and that it would be midnight by the clock one second before it would be midnight by the watch. Find, in fractions of a second, the daily gaining and losing rates of the clock and the watch.

Ans. $\dfrac{86400}{86400-1}$ secs., and $\dfrac{86400}{86400+1}$ secs.

(39) A shilling's worth of Bavarian kreuzers is more numerous by 6 than a shilling's worth of Austrian kreuzers; and 15 Austrian kreuzers

are worth a penny more than 15 Bavarian kreuzers. How many of them respectively make a shilling? Ans. 30 Austrian, 36 Bavarian.

Enunciate the problem indicated by the negative result.

(40) From a vessel which will contain a gallons, filled with a fluid a_1, a gallon being drawn off, the vessel is filled up with a fluid a_2; a gallon being drawn off from the mixture, the vessel is filled up with a fluid a_3; and so on, until the vessel contains a portion of each of the fluids $a_1, a_2, a_3, \dots a_n$. How much of each fluid does it contain?

Ans. $a\left(1-\frac{1}{a}\right)^{n-1}$, $\left(1-\frac{1}{a}\right)^{n-2}$, $\left(1-\frac{1}{a}\right)^{n-3}$, ... $\left(1-\frac{1}{a}\right)^{2}$, $\left(1-\frac{1}{a}\right)$, 1, gallons, respectively.

(41) Prove that $(a+b)(b+c)(c+a) > 8abc$, and $< \frac{8}{3}(a^3+b^3+c^3)$, unless $a = b = c$.

(42) Which is greater, $mx+\frac{n}{x}$, or $m+n$? $mx^n+\frac{n}{x^m}$, or $m+n$?

$$(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d), \text{ or } 16abcd?$$

(1) Ans. When $x>1$, the former or the latter, according as $x \gtrless \frac{n}{m}$: when $x<1$, the former or the latter, according as $x \lessgtr \frac{n}{m}$. (2) Ans. The former. (3) Ans. The former or the latter, according as the greatest and least of a, b, c, d, together is $\lessgtr$ the other two together.

(43) If a, b, c be in Harmonical Progression, prove that

$$\frac{a}{b+c}, \quad \frac{b}{c+a}, \quad \frac{c}{a+b},$$

are also in Harmonical Progression.

(44) The relative powers of the instruments $a_1, a_2, a_3, \dots a_n$, are expressed by saying that generally $a_1, a_2, a_3, \dots a_{r-1}$, working together can do as much work in one day as a_r can do in p days. If a_n can do a piece of work in one hour, how long will it take a_1 and a_r, where $r>1$, to do it?

(1) Ans. $\frac{1}{p}\left(1+\frac{1}{p}\right)^{n-2}$ hours. (2) Ans. $\left(1+\frac{1}{p}\right)^{n-r}$ hours.

(45) If m and n be any two positive integers, of which m is the greater, and $p = 2^m$, $q = 2^n$; prove that $x^{2p}+a^px^p+a^{2p}$ is exactly divisible by both $x^{2q}+a^qx^q+a^{2q}$ and $x^{2q}-a^qx^q+a^{2q}$.

(46) If $a_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$, prove that

$$a_1 - \frac{a_2}{2} + \frac{a_3}{3} - \dots + \frac{(-1)^{n-1}}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

(47) If $p_r=\dfrac{1.3.5\dots(2r-1)}{2.4.6\dots\ 2r}$, prove that

$$p_{2n+1}+p_1.p_{2n}+p_2.p_{2n-1}+\dots+p_{n-1}.p_{n+2}+p_n.p_{n+1}=\frac{1}{2}.$$

(48) If t_0, t_1, t_2, t_3, &c., represent the terms of the binomial expansion of $(a+x)^n$, prove that

$$(t_0-t_2+t_4-\&c.)^2+(t_1-t_3+t_5-\&c.)^2=(a^2+x^2)^n.$$

(49) If a_0, a_1, a_2, a_3, &c. be the coefficients in order of the expansion of $(1+x+x^2+\dots+x^p)^n$, prove that

(1) their sum $=(p+1)^n$;

(2) $a_1+2a_2+3a_3+\dots+np.a_{np}=\frac{1}{2}np(p+1)^n$.

(50) Apply the general term in the Multinomial Theorem to prove that the coefficient of x^{2p+1} in the expansion of $(a_0+a_1x+a_2x^2+\dots)^2$ is

$$2.(a_0.a_{2p+1}+a_1.a_{2p}+a_2.a_{2p-1}+\dots+a_p.a_{p+1}).$$

(51) In the expansion of $(1+x+x^2+\dots+x^r)^{2n}$, where n is a positive integer, prove that (1) the coefficients of terms equidistant from the beginning and the end are equal; (2) the coefficient of the middle term is the greatest; (3) the coefficients continually increase from the first up to the greatest.

(52) If a_0, a_1, a_2, a_3, &c. be the coefficients in order of the expansion of $(1+x+x^2)^n$, prove that

$$a_0^2-a_1^2+a_2^2-a_3^2+\dots+(-1)^{n-1}a_{n-1}^2=\tfrac{1}{2}a_n\{1-(-1)^n a_n\}.$$

(53) Shew that if m be not less than n, the greatest coefficient in the expansion of $(a_1+a_2+a_3+\dots+a_m)^n$ is $\underline{|n}$; and that the number of terms which have this coefficient is $\dfrac{\underline{|m}}{\underline{|n}\,\underline{|m-n}}$.

(54) Prove by the method of *Demonstrative Induction* that the number of different throws which can be made with n dice, on the supposition that the sum of the numbers turned up is r, is equal to the coefficient of x^r in the expansion of $(x+x^2+x^3+x^4+x^5+x^6)^n$.

Enunciate the corresponding proposition when each die has p faces, and the numbers marked on the p faces of each die are α_1, α_2, $\alpha_3\dots\alpha_p$.

Ans. Number of throws = coefficient of x^r in the expansion of

$$(x^{\alpha_1}+x^{\alpha_2}+\dots+x^{\alpha_p})^n.$$

(55) Prove that the numerators of any two consecutive convergents to a continued fraction are prime to each other, as also their denominators.

(56) The Cambridge Lent Term always ends on a Friday, suppose at midnight. Prove that if it commence on the $2p^{\text{th}}$ day of the week, it will divide either on the $(p-1)^{\text{th}}$ day of the week at midnight, or on the $(3+p)^{\text{th}}$ day of the week at noon: but if it commence on the $(2p+1)^{\text{th}}$ day of the week, it will divide either on the p^{th} day of the week at noon, or on the $(3+p)^{\text{th}}$ day of the week at midnight.

(57) Prove that (1) $1+3+5+\ldots\ldots+(2n-1)=n^2$;

(2) $1^3+2^3+3^3+\ldots\ldots+n^3=(1+2+3+\ldots\ldots+n)^2$.

Apply (1) to find solutions in positive integers of $x^n=y^2-z^2$, where n is a positive integer; and apply (2) to find solutions in positive integers of

(58) Prove that the product of any r consecutive integers is divisible by $\underline{|a_1}\cdot\underline{|a_2}\cdot\underline{|a_3}\ldots\ldots\underline{|a_p}$, if $a_1+a_2+a_3+\ldots\ldots+a_p=r$.

(59) In how many ways can a line 100800 inches long be divided into equal parts, each some multiple of an inch? Ans. 124.

(60) How many different rectangular parallelopipeds are there satisfying the condition that each edge of each parallelopiped shall be equal to some one of n given lines all of different lengths?

Ans. $\dfrac{n(n+1)(n+2)}{1\,.\,2\,.\,3}$.

(61) Prove that if, in any scale of notation, the sum of two numbers is a multiple of the radix, then (1) the digits in which the squares of the numbers terminate are the same; (2) the sum of this digit and the digit in which the product of the numbers terminates is equal to the radix.

(62) A certain number, when represented in the scale of 2, has each of its last three digits (counting from left to right) zero, and the next digit towards the left different from zero; when represented in either of the scales of 3 or 5, the last digit is zero, and the last but one different from zero; and in every other scale (twelve scales excepted) the last digit is different from zero. What is the number? Ans. 120.

(63) Prove that every even power of every odd number when divided by 8 leaves 1 for a remainder.

(64) If n be a prime number, prove that $1^n+2^n+3^n+\ldots\ldots+(rn)^n$ is a multiple of n.

(65) If n be any prime number excepting 2, and N any odd number prime to n, prove that $N^{n-1}-1$ is divisible by $8n$.

(66) A gentleman being asked the size of his paddock replied:— Between one and two roods: also were it smaller by 3 square yards it

would be a square number of square yards; and if my brother's paddock, which is a square number of square yards, were larger by one square yard, it would be exactly half as large as mine. What was the size of his paddock? Ans. 1684 square yards.

(67) A walks at a uniform speed, known to be greater than 3 and less than 4 miles an hour, between two places 20 miles apart. An hour having elapsed since A's departure, B starts after him from the same place, walking at the uniform speed of 4 miles an hour. Shew that the odds are 2 to 1 against B's overtaking A.

(68) When $2n$ dice are thrown, prove that the sum of the numbers turned up is more likely to be $7n$ than any other number; and when $2n+1$ dice are thrown, prove that the sum of the numbers turned up is more likely to be $7n+3$, or $7n+4$, than any other number, these being equally probable.

(69) A handful of shot is taken at random out of a bag: what is the chance that the number of shot in the handful is prime to the number of shot in the bag? Ex. Let the number of shot in the bag be 105.

(1) Ans. $\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots\ldots$ where the number of shot in the bag $= a^p b^q c^r \ldots\ldots$ (2) Ans. $\frac{16}{35}$.

(70) A coin is to be tossed twice: what is the chance that head will turn up at least once? Ans. $\frac{3}{4}$.

Point out the error in the following solution by D'Alembert:—Only three different events are possible; (1) head the first time, which makes it unnecessary to toss again; (2) tail the first time and head the second; (3) tail both times: of these three events two are favourable; therefore the required chance is $\frac{2}{3}$.

(71) In each of n caskets are p jewels worth 1, 2, 3, ...p, guineas respectively: a person being allowed to draw a jewel from each casket, find (1) the most probable collective value of the jewels he will draw: (2) the value of his expectation. (3) What would be the value of his expectation, if he were allowed to draw r jewels from each casket? If $p = 2$, and one jewel be drawn from each casket, what is the chance (4) that the value of the jewels drawn is $\overline{n+r}$ guineas? (5) that the collective value of the jewels drawn is that collective value which is most probable?

(1) Ans. $\frac{n}{2}(p+1)$ guineas, if either n be even or p odd;

and $\frac{n}{2}(p+1)\pm\frac{1}{2}$ guineas, n being odd and p even, each of these being equally probable.

(2) Ans. $\frac{n}{2}(p+1)$ guineas. (3) Ans. $\frac{rn}{2}(p+1)$ guineas.

(4) Ans. $\frac{1}{2^n}\cdot\frac{\lfloor\underline{n}}{\lfloor\underline{r}\cdot\lfloor\underline{n-r}}$. (5) Ans. $\frac{\lfloor\underline{n}}{(2.4.6\ldots\ldots n)^2}$,

or $\frac{\lfloor\underline{n}}{2.4.6\ldots\ldots\overline{n-1}\times 2.4.6\ldots\ldots\overline{n+1}}$, according as n is even or odd.

(72) A person borrows £c on the following terms. It is to be paid off in n years: and at the end of each year is to be paid interest at a given rate r on the sum remaining unpaid at the beginning of the year, together with such a portion of the principal that the whole sum paid on account of principal and interest together shall be the same for every year. Investigate a formula for the sum to be paid every year.

Ans. $c.\frac{r(1+r)^n}{(1+r)^n-1}$.

(73) From the gold fields are brought $2n$ specimens of gold dust, no two of which are of the same degree of fineness. Each specimen is divided into as many equal portions as is necessary to the following operation, viz. to form as many different mixtures as possible by taking a portion of dust from some specimen and mixing it with a portion from some other specimen. Each of these mixtures is now divided into as many equal portions as is necessary to the following operation, viz. to form as many new mixtures as possible by mixing together portions from any n of the former mixtures. Prove that $1.3.5.7\ldots\ldots(2n-1)$ of the latter mixtures will be of the same degree of fineness as a mixture formed by mixing together all the dust of $2n$ specimens exactly like the original specimens.

COLLEGE EXAMINATION PAPERS.

[*Solutions* of all the following Equations and Problems will be found in the *Companion.*]

I.

St John's College. *June,* 1848.

(1) $\dfrac{\dfrac{3}{x}-1}{2}-\dfrac{9\left(\dfrac{1}{2x}-1\right)-\dfrac{2}{5}\left(\dfrac{9}{2x}-4\right)}{\dfrac{3}{x}-4}=\dfrac{\dfrac{9}{x}+19}{6}$; find x. Ans. $x=\dfrac{411}{662}$.

(2) $\sqrt{a+x}+\sqrt{a-x}=\sqrt{\dfrac{3b^2+x^2}{a+b}}$; find x. Ans. $x=\pm\sqrt{2ab-b^2}$.

(3) $\left.\begin{array}{r} x^3=31x^2-4y^2, \\ y^3=31y^2-4x^2; \end{array}\right\}$ find x and y.

Ans. $\begin{cases} x=30, \text{ or } 15, \text{ or } 27, \text{ or } \dfrac{7}{2}(3\pm\sqrt{33}). \\ y=15, \text{ or } 30, \text{ or } 27, \text{ or } \dfrac{7}{2}(3\mp\sqrt{33}). \end{cases}$

(4) $\left.\begin{array}{r} (x+y)^{\frac{1}{3}}+(x-y)^{\frac{1}{3}}=a^{\frac{1}{3}}, \\ (x^2+y^2)^{\frac{1}{3}}+(x^2-y^2)^{\frac{1}{3}}=a^{\frac{2}{3}}; \end{array}\right\}$ find x and y.

Ans. $\begin{cases} x=\dfrac{1}{2}(1-\sqrt{3})a, \\ y=\sqrt{\left(1-\dfrac{11}{18}\sqrt{3}\right)}a. \end{cases}$

(5) A railway train travels from A to C passing through B, where it stops 7 minutes; two minutes after leaving B it meets an express train, which started from C, when the former was 28 miles on the other side of B: the express travels at double the rate of the other, and performs the journey from C to B in $1\frac{1}{2}$ hours; and if on reaching A it returned at once to C, it would arrive 3 minutes after the first train. Find the distances between A, B, and C, and the speed of each train.

Ans. $AB=31\frac{1}{2}$ miles, $BC=63$ miles: speed of the ordinary train 21 miles,
.... express .... 42 ....

(6) To meet a deficiency of m millions in the revenue of a country, an *additional* tax of a per cent. was laid upon articles exported, and the tax upon imports was diminished c per cent.: in consequence of these alterations the value of the imports was increased so as to be n times as great

as the exports, and the deficiency was made up. It was afterwards found that if the additional tax upon the exports had been a' per cent., and the tax upon imports diminished c' per cent., the values of the articles being altered as before, the deficiency would not have been made up by m' millions. Find the values of the exports and imports after the alteration of the tax.

$$\text{Ans. Val. of exports} = \frac{m'}{(a-a')+n(c'-c)}\times 100£;$$

$$\text{Val. of imports} = \frac{m'n}{(a-a')+n(c'-c)}\times 100£.$$

(7) Fifty thousand voters, who have to return a member to an assembly, are divided into sections of equal size, and each section chooses an elector, the member being returned by the majority of such electors. There are two candidates, *A* and *B*. In those sections which return electors favourable to *A*, the majority is double the minority, while in those favourable to *B*, the minority forms only a tenth of the whole. After the primary elections a third candidate *C* comes forward, and is joined by so many electors of each party, that he is returned by a majority of 3 over *A*, and 14 over *B*. If *C* had not come forward, *A* would have been returned by a majority 19 less than the whole number of votes actually polled by *C*; and if the elections had been by the 50,000 voters *directly* between *A* and *B*, *B* would have had a majority of 6000. Find the number of sections.

Ans. 100.

II.

St John's College. *June*, 1849.

(1) $2b\{\sqrt{x+a}-b\}+2c\{\sqrt{x-a}+c\}=a$; find x.

$$\text{Ans. } x=(b\pm c)^2+\frac{1}{4}.\frac{a^2}{(b\pm c)^2}.$$

(2) $\sqrt{2x-1}+\sqrt{3x-2}=\sqrt{4x-3}+\sqrt{5x-4}$; find x. Ans. $x=1$.

(3) $8x^{\frac{3}{4}}+81=18x^{\frac{1}{2}}+45x^{\frac{1}{4}}$; find x. Ans. $x=3^4$, or $\left(\frac{3}{2}\right)^4$, or $\left(-\frac{9}{4}\right)^4$.

(4)
$$\left.\begin{aligned}\frac{y-x+\sqrt{2xy-3x^2}}{y-2x}&=3.\frac{(2y-3x)^{\frac{3}{2}}+x^{\frac{3}{2}}}{(2y-3x)^{\frac{3}{2}}-x^{\frac{3}{2}}},\\ \frac{y}{x^2}+\frac{16}{81}\left(x-\sqrt{x}-\frac{3}{4}\right)&=\frac{4}{9x}(2\sqrt{xy}-\sqrt{y});\end{aligned}\right\}\text{ find } x \text{ and } y.$$

$$\text{Ans. }\left\{\begin{aligned}x&=\frac{9}{16}(1\pm\sqrt{1\pm4\sqrt{2}})^2,\\ y&=\frac{9}{8}(1\pm\sqrt{1\pm4\sqrt{2}})^2.\end{aligned}\right.$$

(5) The distance between the two termini *A* and *Z* of a railway is 100 miles. A train starting from *A* runs up-hill during the first 30 miles of its

journey, the next 50 miles are on a level, and the remaining 20 are up-hill. The train may be supposed to travel 5 miles an hour faster on the horizontal road than when it is ascending a hill. There are to be stoppages at stations B, C, D, and E, at distances 20, $42\frac{1}{2}$, $67\frac{1}{2}$, and 90, miles respectively from A, and each stoppage may be supposed to cause a detention of 3 minutes. Find the time of arrival at B, C, D, and E, of the train which starts from A at $8^h. 0^m.$ and arrives at Z at $12^h. 42^m.$

Ans. At B 9^h, C $10^h. 3^m$, D $11^h. 6^m$, E $12^h. 9^m$.

(6) A number of vessels A_1, A_2, A_3,...A_r, A_{r+1},...A_m are arranged in a row. A_1 contains a quantity of wine, A_2 a quantity of water, and the remaining vessels A_3,...A_r, A_{r+1},...A_m, contain any quantity of any other fluids. $\frac{1}{n}$ of the wine in A_1 is taken from A_1 and added to the contents of A_2, $\frac{1}{n}$ of the mixture is taken from A_2 and poured into A_3, $\frac{1}{n}$ of the contents of A_3 is poured into A_4, and so on to the end of the series of vessels. Again, $\frac{1}{n}$ of wine remaining in A_1 is poured into A_2, $\frac{1}{n}$ of the contents of A_2 into A_3, and so on. A_1 is supposed never to receive any addition. It is found that 60 times the quantity of wine in the vessel A_r after $r-1$ abstractions of fluid from that vessel $=31$ times the quantity of wine in the same vessel after r abstractions. Also 59 times the quantity of water in A_r after $r-1$ abstractions of fluid from that vessel $= 31$ times the quantity of water in the same vessel after r abstractions. Find the numerical values of r and n.

Ans. $r = n = 31$.

(7) Two points P and Q are connected by a wire (A), $\frac{1}{8}$th of an inch in diameter and 50 miles in length, which is used for transmitting a galvanic current. It is required to replace the wire (A) by three others (a), (b), and (c), composed of different metals, and of lengths 50, 60, and 70, miles respectively. These new wires must be of such diameters that the current, which previously passed along (A) may be divided, so that the quantities which pass along (a), (b), and (c), may be as 3, 4, and 5. The quantity of galvanic fluid that will pass along a wire is supposed to vary inversely as the resistance, and the resistance to vary directly as the length of wire to be traversed, inversely as the sectional area of the wire, and inversely as the conductibility. Also the sectional area of a wire varies as the square of its diameter. It is found by experiment, that the quantity of galvanic fluid which will pass along a portion of the wire (A), $\frac{1}{8}$th of an inch in diameter and 15 yds. long, may be denoted by 1000 k. Also portions of wire $\frac{1}{10}$th of an inch in diameter, composed of the same metals as (a), (b), and (c), of lengths 20, 10, and 40 yds. respectively, are capable of transmitting quantities of galvanic fluid 750 k, 5400 k, and 3500 k, respectively. Find the least possible diameters of wires (a), (b), and (c), in order that the above conditions may be satisfied.

Ans. $\frac{1}{20}$, $\frac{1}{30}$, $\frac{1}{40}$.

III.

ST JOHN'S COLLEGE. *June*, 1850.

(1) $(x+a)\left(1+\frac{1}{x^2+a^2}\right)+\sqrt{2ax}\left(1-\frac{1}{x^2+a^2}\right)=2$; find x.

Ans. $x=1\pm\sqrt{2a-a^2}$.

(2) $\frac{1}{6x^2-7x+2}+\frac{1}{12x^2-17x+6}=8x^2-6x+1$; find x.

Ans. $x=\frac{1}{2}\left\{1\pm\sqrt{\frac{1\pm\sqrt{33}}{8}}\right\}$.

(3) $$\left.\begin{aligned}(x^2+y^2+c^2)^{\frac{1}{3}}+(x-y+c)^{\frac{2}{3}}&=2(4xy)^{\frac{1}{3}},\\ \frac{1}{y}&=\frac{1}{x}+\frac{1}{c};\end{aligned}\right\}\text{ find } x \text{ and } y.$$

Ans. $\begin{cases} x=\frac{1}{2}(1\pm\sqrt{5}).c, \\ y=\frac{1}{2}(-1\pm\sqrt{5}).c. \end{cases}$

(4) $$\left.\begin{aligned}2(x^2+xy+y^2-a^2)+\sqrt{3}(x^2-y^2)&=0,\\ 2(x^2-xz+z^2-b^2)+\sqrt{3}(x^2-z^2)&=0,\\ y^3-c^3+3(yz^2-c^3)&=0;\end{aligned}\right\}\text{ find } x \text{ and } y.$$

Ans. $\begin{cases} y=\frac{1}{2}(m+\sqrt[3]{8c^3-m^3}), \quad z=\frac{1}{2}(m-\sqrt[3]{8c^3-m^3}), \\ x=\frac{1}{\sqrt{3}+1}\{a+b-\frac{1}{2}(\sqrt{3}-1)\sqrt[3]{8c^3-m^3}\}, \text{ where } m=(\sqrt{3}+1)(a-b). \end{cases}$

(5) *A* lends one-half of his money to *B* at 5 per cent. per annum simple interest, and the remaining half he invests in the three per cents. at 90. *B* pays the interest regularly during the first five years, but afterwards neglects to do so till other five years' interest is due, when *A* calls in all his money, and *B* becomes a bankrupt paying 10*s*. in the pound. *A* sells out when the funds are at 81, and then he finds that the whole sum he has received as principal and interest in the ten years exceeds the sum that he originally possessed by £34. 13*s*. 4*d*. How much did he lend *B*?

Ans. £320.

(6) *A*, *B*, and *C* are three villages. The road from *A* to *B* is level, and *C* is on a hill above *A* and *B*. The distances *AB*, *BC*, and *CA*, are respectively 24, 14·4, and 28·8, miles. *P* walks up hill $\frac{1}{4}$th slower, and down hill $\frac{1}{3}$rd faster, than when the road is level. *Q* walks up hill $\frac{1}{5}$th slower, and down hill $\frac{1}{5}$th faster, than when the road is level. *P* travels round in the direction *ACB* in 30 minutes less than *Q* requires to go round in the opposite direction. *Q* travels round in the direction *ACB* in 1 hour

48 minutes more than P takes to make the circuit in the opposite direction; find the rates of each on a level road. Also supposing them both to start from A in opposite directions, find their points of meeting.

(1) Ans. P's rate 12 miles, Q's 10 miles, per hour. (2) Ans. At C, or between B and C at a dist. $5\frac{1}{3}$ miles from C, according as P or Q takes the direction ABC.

(7) Suppose that in the course of any one year the number of births in Ireland is on an average 32 for 1000 living in the island at the commencement of that year, the number of deaths and emigrations to the colonies 21 in 1000, and of migrations to England 1 in 100. In England suppose the number of births in the course of any one year to be 3 for every 100 inhabitants living at the beginning of the year, the number of deaths and emigrations to the colonies 289 in 10,000, and of migrations to Ireland 1 in 10,000. If the number of inhabitants in England was twice the number in Ireland at the beginning of 1850, in what year will the population of the former be three times that of the latter, according to the law above stated? Ans. A.D. 1957.

IV.

ST JOHN'S COLLEGE. *June*, 1851.

(1) $\dfrac{x^2+2x+2}{x+1}+\dfrac{x^2+8x+20}{x+4}=\dfrac{x^2+4x+6}{x+2}+\dfrac{x^2+6x+12}{x+3}$; find x.

Ans. $x=0$, or $-2\frac{1}{2}$.

(2) $(5x^2+x+10)^2+(x^2+7x+1)^2=(3x^2-x+5)^2+(4x^2+5x+8)^2$; find x.

Ans. $x=2\pm\sqrt{2}$, or $3\pm\sqrt{3}$.

(3) $(x^2+4x-2)^2+3=4x(3x^2+4)$; find x.

Ans. $x=\dfrac{1}{2}\{2+\sqrt{2}\pm\sqrt{16-7\sqrt{2}}\}$.

(4)
$$\left.\begin{aligned}&3x+3y-z=3,\\&x^2+y^2-z^2=\frac{14-9z}{2},\\&x^3+y^3+z^3=3xyz+\frac{17z+44}{4};\end{aligned}\right\}\ \text{find } x,\ y, \text{ and } z.\qquad \text{Ans.}\ \left\{\begin{aligned}&x=1\tfrac{1}{2},\ \text{or}\ \frac{1}{2},\\&y=\frac{1}{2},\ \text{or}\ 1\tfrac{1}{2},\\&z=3.\end{aligned}\right.$$

(5) A derives his income from a fixed rental, B from his profession, C from both. In the first year A pays as much income-tax as B and C together, but in the second year, B's and C's professional incomes being doubled, B pays as much as C, which is $\frac{4}{5}$ths of what A pays; also the total amount of their incomes in the two years is £5500. Assuming that the income-tax

is higher for a fixed rental than for professional income in the ratio 3 : 2, find the incomes of *A*, *B*, *C*, in the first year.

Ans. *A*'s £1000, *B*'s £600, *C*'s £700.

(6) *A* and *B* start at the same time in a boat-race; *A* has 100 yards start and has to row to a post *D*, *B* to a post *C*. At first *A*'s rate : *B*'s :: 40 : 39, but when the distance between *A* and *B* is $\frac{1}{6}$th of the remaining distance *A* has to row, *A*'s speed is diminished in the ratio of 79 : 80, so that two minutes afterwards the distance between them is three yards more than half the remaining distance *B* has to row. At this point of *B*'s course, his rate which has hitherto been uniform is increased eight yards a minute, while *A*'s is still further diminished six yards a minute; and in one minute more *B* arrives at the post *C*, *A* being then three yards from the post *D*. Find the distance between *C* and *D*. Ans. 116 yards.

(7) Three men, *A*, *B*, *C*, walk in the same direction in the circumferences of three concentric circles, starting simultaneously from points where they are at their least distances from each other. *A* walks his circuit in an even number of hours, (greater than four), *B* and *C* their circuits in one hour and two hours less respectively. Whenever *A* and *B* are at their greatest distance from each other, they alter their rates in such a manner, that the times they would take to walk their circuits at the rates they are *then* going are interchanged; and whenever *A* and *C* are *again* at their least distance their times are interchanged in a similar manner. When *A* and *B* are at their greatest distance the first time, *A* has walked a distance equal to twenty-two times *C*'s circuit; and when they are at their greatest distance the third time, *B* has walked a distance equal to forty-two times *A*'s circuit, and *C* has then walked ten miles less than forty times *B*'s circuit, and is at his least distance from *B*. Required the rates of *A*, *B*, *C*, at first.

Ans. 3, 4, and 5, miles per hour respectively.

V.

ST JOHN'S COLLEGE. *June*, 1852.

(1) $(a+b)^2x+\left(\dfrac{ab}{a-b}\right)^2\cdot\left(4-\dfrac{3}{x}\right)=2ab$; find x.

Ans. $x=\dfrac{3ab}{(a+b)^2}$, or $\dfrac{-ab}{(a-b)^2}$.

(2) $(x+2\sqrt{x})^{\frac{1}{2}}-(x-2\sqrt{x})^{\frac{1}{2}}=2(x^2-4x)^{\frac{1}{8}}$; find x.

Ans. $x=0$, or $2(1\pm\sqrt{2})$.

(3) $(12x-1)(6x-1)(4x-1)(3x-1)=\dfrac{1}{96}$; find x.

Ans. $x=\dfrac{1}{24}(5\pm\sqrt{5\pm2\sqrt{5}})$.

(4) $$\left.\begin{aligned}\frac{x+y}{4}&=\frac{xy}{x+y}+\frac{1}{2},\\ x^2+y^2&=\frac{8x^2y^2}{(x+y)^2}+22;\end{aligned}\right\}$$ find x and y. Ans. $\begin{cases} x=6, \text{ or } 2,\\ y=2, \text{ or } 6.\end{cases}$

(5) A and B set out at the same time from the same place, and walk in the same direction. Whenever the distance between them is an even number of miles, A increases his speed $\frac{1}{4}$ mile per hour, and when it is an odd number of miles, B increases his speed $\frac{1}{2}$ mile per hour. When A is 4 miles in advance, B has walked $30\frac{2}{3}$ miles, and A has walked $1\frac{1}{3}$ miles more than he would have done in the same time, had he walked uniformly at his first rate. Required the rates of A and B at starting.

Ans. A 4 miles, B 3 miles, per hour.

(6) Three vessels, A, B, C, contain liquids in the proportion $3 : 2 : 1$, and their prices per gallon are as $1 : 2 : 3$. A certain quantity is poured from A into B, and the same quantity from the mixture in B into C; the same quantity is now poured from the mixture in C into B, and from the new mixture in B into A. The value of the liquid in A is thereby increased in the ratio of $35 : 27$; but had the quantity poured out each time been one gallon more than it was, its value would have been increased in the ratio of $3 : 2$. Find the quantity of liquid in each vessel.

Ans. Quantity in A, B, C, 3, 2, 1, gallons respectively.

(7) Reckoning meat by the stone, wheat by the quarter, and hay by the load, the price of hay at first was equal to that of wheat together with 4 times that of meat. The rise or fall of meat is $\frac{1}{6}$th of the rise or fall of wheat, together with $\frac{1}{16}$th that of hay, except during the first month, when from scarcity of fodder, hay rose 8*s.*, and the effect on meat from this cause alone was a fall of 6*d.* The *variation* in hay was 8*s.* each month, and wheat rose 2*s.* a month from the first, until, after a certain number of months, there was the same relation between the prices as at first; wheat being now 50*s.* and remaining stationary. After as many months more, hay was 12 times the price of meat. Required the prices of wheat and hay at first, that of meat being 6*s.* Ans. Wheat 44*s.* Hay 68*s.*

VI.

St John's College. *May*, 1853.

(1) $$\frac{1}{5}\cdot\frac{(x+1)(x-3)}{(x+2)(x-4)}+\frac{1}{9}\cdot\frac{(x+3)(x-5)}{(x+4)(x-6)}-\frac{2}{13}\cdot\frac{(x+5)(x-7)}{(x+6)(x-8)}=\frac{92}{585};$$ find x.

Ans. $x=1\pm\sqrt{19}$.

(2) $\left(\frac{x+6}{x-6}\right)\left(\frac{x-4}{x+4}\right)^2+\left(\frac{x-6}{x+6}\right)\left(\frac{x+9}{x-9}\right)^2=2.\frac{x^2+36}{x^2-36}$; find x.

Ans. $x=0$, or $\frac{6}{5}(1\pm\sqrt{26}$

(3) $(x-1)^2+(a-1)^2=2(ax+1)+\sqrt{3(x+a)^2+4ax}$; find x.

Ans. $x=(a+2)\pm\sqrt{8a+3}$

(4) $\left.\begin{aligned}x^2+a^2+y^2+b^2&=\sqrt{2}.\{x(a+y)-b(a-y)\},\\ x^2-a^2-y^2+b^2&=\sqrt{2}.\{x(a-y)+b(a+y)\};\end{aligned}\right\}$ find x and y.

Ans. $\begin{cases}x=a\sqrt{2}+b\\ y=b\sqrt{2}+a\end{cases}$

(5) A rectangular field is divided by two lines parallel to two of its adjacent sides into four parts, of which the least has its longer side in the shorter side of the field. The ratio of the perimeters of the greates and least parts is 4 : 1, and that of the other two is 3 : 2. Had howeve the sides of the least part been double their present lengths, the ratio o the areas of those parts which would *then* have been greatest and leas would have been 4 : 1, and of the other two, one would have contained an acre more than the other. Find the number of acres in the field.

Ans. $5\frac{3}{8}$.

(6) Four points, A, B, C, D, move uniformly with velocities in Geometrical Progression in four equidistant and equal parallel lines, whose extremities are situated in two parallel lines, from one of which they start together, and when they reach the other extremities, they return, and so on continually. After a certain interval B, C, D, are in a straight line for the first time; after twice that interval A, B, C, are in a straight line for the first time; 36 seconds after this, A, C, D, are in a straight line for the second time, and the space that has been passed over by B is 14 inches more than that passed over by A when A, B, C, D, are again in a straight line. Required the velocities of A, B, C, D.

Ans. 1, 2, 4, 8, inches per second respectively.

(7) A town is supplied with gas at a stated price per thousand cubic feet, whenever the annual consumption and price of coals are certain fixed quantities; but it is agreed, that in any year when they differ from these values, a variation in the former at the rate of m per cent. shall cause an *opposite* variation in the stated price of gas at the rate of n per cent. $(n<m)$, and a variation in the latter at the rate of p per cent. shall cause a *like* variation in the stated price of gas at the rate of q per cent. After r years the *average* annual consumption and *average* price of coals have accorded with their assigned values, and yet the town has paid £P more than it would have done, had there been no variation from the assigned values of the annual consumption and price of coals. But if the above-

mentioned relative variations in the consumption and price of gas had bee p and q per cent., and those of the prices of coals and gas m and n pe cent., the town would have paid £Q less. Supposing that in any year (th x^{th}) there is a variation in the consumption and price of coals from the fixed values of x per cent.; find the greatest sum the town could pay fo the gas it consumed in any given year.

Ans. The greatest cost of gas in the t^{th} year

$$=\frac{100^2}{S}\cdot\frac{P.\frac{n}{m}+Q.\frac{q}{p}}{\frac{q^2}{p^2}-\frac{n^2}{m^2}}\cdot\left(1+\frac{t}{100}\right)\cdot\left\{1-\left(\frac{n}{m}-\frac{q}{p}\right)\frac{t}{100}\right\},$$

where $S = 1^2+2^2+3^2+\ldots\ldots+r^2$.

VII.

ST JOHN'S COLLEGE. *June*, 1854.

(1) $\dfrac{x+1}{x-1}+\dfrac{x-2}{x+2}+\dfrac{x-3}{x+3}+\dfrac{x+4}{x-4}=4$; find x. Ans. $x=-\frac{1}{2}\left(1\pm\sqrt{\frac{69}{5}}\right)$.

(2) $(a+x)^8+(a^2+x^2)^4=c^4x^4$; find x.

Ans. $x=\dfrac{a}{2}\{m-1\pm\sqrt{(m+1)(m-3)}\}$, where $m=\left(\sqrt{8+\dfrac{c^4}{2a^4}}-3\right)^{\frac{1}{2}}$.

(3) $(x+1)(x^2+1)(x^6-1)=px^4(x-1)$; find x.

Ans. $x=1$, or $\frac{1}{2}(m\pm\sqrt{m^2-4})$, where $m=\frac{1}{2}(-1\pm\sqrt{5\pm4\sqrt{p+1}})$.

(4) $\dfrac{a^3}{x}+3yz=\dfrac{b^3}{y}+3zx=\dfrac{c^3}{z}+3xy=xy+yz+zx$; find x, y, and z.

Ans. $\dfrac{x}{m+a^3}=\dfrac{y}{m+b^3}=\dfrac{z}{m+c^3}=\dfrac{1}{\sqrt[3]{3m^2+2mp_1+p_2}}$, where

$$m=\frac{\pm\sqrt{p_2^{\,2}-3p_1p_3-p_2}}{p_1},\ p_1=a^3+b^3+c^3,\ p_2=b^3c^3+c^3a^3+a^3b^3,\ p_3=a^3b^3c^3.$$

(5) A person starts to walk, at an uniform speed, without stopping, from Cambridge to Madingley and back, at the same time that another starts to walk, at an uniform speed, without stopping, from Madingley to Cambridge and back. They meet a mile and a half from Madingley; and

again, an hour after, a mile from Cambridge. Find their rates of walking and the distance between Cambridge and Madingley.

Ans. Their rates are 4, and 3, miles an hour: the distance is $3\frac{1}{2}$ miles

(6) Sovereigns in number equal to four hundred times the numbe representing the ratio of the weight of pure gold to the weight of dross in a Sovereign contain as much pure gold as 133 Napoleons together with Napoleons in number equal to six hundred times the number representing the ratio of the weight of pure gold to the weight of dross in a Napoleon also there is as much pure gold in 400 Sovereigns as in 503 Napoleons and 11000 Sovereigns weigh as much as 13581 Napoleons. How much o (1) a Sovereign, (2) a Napoleon, is pure gold?

(1) Ans. $\frac{11}{12}$ths. (2) Ans. $\frac{9}{10}$ths.

(7) Suppose that each of the University Presses at Cambridge and Oxford has a fixed demand, every new year's day, for 5000 copies of an edition of the Bible containing 30 sheets, the price of the paper for which is 12*s.* 6*d.* per ream of 500 sheets, the expense of setting up the type £140, and the cost of press work 5*s.* for every thousand sheets printed. The custom is, at Oxford to keep the type continually standing, at Cambridge to take it down as soon as the printing of what is thought proper at any particular time is completed: in consequence it is necessary to purchase (suppose) twenty times as much type at Oxford as at Cambridge. Assuming that at Oxford the 5000 copies are always printed just before they are wanted, and that at Cambridge a supply is printed at one time for the most advantageous possible number of years (simple interest being reckoned at the rate of 5 per cent. per annum on all money laid out in the material and workmanship o any stock kept on hand one or more years), the capital required for the above purpose at Oxford is £417. 10*s.* 0*d.* less than the *average* capital required for the same purpose at Cambridge. What is the cost of the type for this edition of the Bible at Oxford and at Cambridge?

Ans. Cost of Type at Cambridge = £50.
.... Oxford = £1000.

VIII.

ST JOHN'S COLLEGE. *June*, 1855.

(1) $\sqrt{\frac{x^2-2x+3}{x^2+2x+4}}+\sqrt{\frac{x^2+2x+4}{x^2-2x+3}}=2\frac{1}{2}$; find x. Ans. $x=2$, or $\frac{4}{3}$, or &c.

(2) $\left.\begin{array}{r} x^3+3axy+y^3=b^3, \\ x+y=c; \end{array}\right\}$ find x and y. Ans. $\left\{\begin{array}{l} x=\frac{1}{2}\left\{c\pm\left(c^2-\frac{4}{3}\cdot\frac{b^3-c^3}{a-c}\right)^{\frac{1}{2}}\right\} \\ y=\frac{1}{2}\left\{c\mp\left(c^2-\frac{4}{3}\cdot\frac{b^3-c^3}{a-c}\right)^{\frac{1}{2}}\right\} \end{array}\right.$

(3) $x^{\frac{1}{3}}+\frac{1}{x^{\frac{1}{3}}}=\sqrt{\frac{1}{3}\cdot\frac{x^{\frac{3}{2}}+\frac{1}{x^{\frac{3}{2}}}}{x^{\frac{1}{2}}+\frac{1}{x^{\frac{1}{2}}}}}$; find x.

Ans. $x=\frac{a^3-3a}{2}\left\{1\pm\sqrt{1-\frac{4}{(a^3-3a)^2}}\right\}$, where $a=\frac{1}{\sqrt[3]{2}-1}$.

(4) $$\left.\begin{aligned} x^2+y^2+z^2 &= 38, \\ 2x+3y+5z &= 29, \\ 15x^2+10y^2+6z^2 &= 12xz+12yz+297; \end{aligned}\right\}$$ find x, y, and z. Ans. $\begin{cases} x=5, \\ y=3, \\ z=2. \end{cases}$

(5) A person who copied a manuscript in a regular manner found that the number of lines copied in the first half hour was less by ten than the square root of the whole number of lines in the manuscript: and that the square of the number of lines copied in the first 49 minutes was equal to twice the number of lines then remaining to be copied. How many lines were there in the manuscript? Ans. 4900.

(6) A, B, C, walk uniformly in three roads, starting at the same moment from their point of intersection. The roads in which A and B walk are at right angles; and the road in which C walks lies somewhere between them. A line joining the positions of A and C at any time is parallel to the road in which B walks; and a line joining the positions of B and C at any time is parallel to the road in which A walks. When C has walked 8 miles more than A, he begins to return; and when he has walked $5\frac{1}{3}$ miles more than B, all three are in the same straight line. Find the *relative* speeds of A, B, and C.

Ans. 3 : 4 : 5.

(7) A farmer has between 270 and 280 quarters of wheat, which he agrees to sell to a miller on the following terms. Not less than 20, and not more than 30, quarters (the exact number being at the option of the farmer) are to be delivered at the beginning of each month of the year, for which the farmer is to be paid according to the market price of wheat at the time of each delivery. The price of wheat at the beginning of June is the same as the price at the beginning of July; and the price increases uniformly a certain number of shillings per quarter each month during the first six months of the year, and decreases uniformly the same number of shillings per quarter each month during the last six months of the year. The farmer, by making the most of his wheat under this contract, receives £7. 8*s*. more for it than he would have done had he delivered the same average quantity each month. If, however, the number of quarters to be delivered each month had been not less than 20, and not greater than 25 (the exact number being at the option of the miller), the miller, by arranging the delivery in the way most advantageous to himself, would have paid £11. 12*s*. less than he actually paid: and the

amount paid in April and September together in the latter case would have been £33. 12*s.* less than the amount paid in May and August together in the former case. How much wheat had the farmer, and what did he receive for it?

Ans. 276 quarters, for which he received £766. 8*s.*

IX.

ST JOHN'S COLLEGE. *May*, 1856.

(1) $\frac{5}{x-1}+\frac{4}{x+2}+\frac{21}{x-3}=\frac{5}{x+1}+\frac{4}{x-2}+\frac{21}{x+3}$; find x.

Ans. $x=\pm\sqrt{2}$, or $\pm\sqrt{3}$.

(2) $\frac{x^2}{3}+\frac{48}{x^2}=10\left(\frac{x}{3}-\frac{4}{x}\right)$; find x. Ans. $x=6$, or -2, or $3\pm\sqrt{21}$.

(3) $\frac{(x+a+b)^5+(x+c+d)^5}{(x+a+c)^5+(x+b+d)^5}=\frac{(a+b-c-d)^2}{(a-b+c-d)^2}$; find x.

Ans. $x=-\frac{1}{2}(a+b+c+d)$, or $-\frac{1}{2}(a+b+c+d)\pm\frac{1}{2}\sqrt[4]{5(a+b-c-d)^2(a-b+c-d)^2}$.

(4) $$\left.\begin{aligned}\frac{(ac+1)(x^2+1)}{x+1}&=\frac{(a^2+1)(xy+1)}{y+1},\\ \frac{(ac+1)(y^2+1)}{y+1}&=\frac{(c^2+1)(xy+1)}{x+1}.\end{aligned}\right\}\text{Ans.}\left\{\begin{aligned}x&=\pm\sqrt{-1},\text{ or }\frac{1+a}{1-a},\text{ or }\frac{1-a}{1+a},\\ y&=\pm\sqrt{-1},\text{ or }\frac{1+c}{1-c},\text{ or }\frac{1-c}{1+c}.\end{aligned}\right.$$

(5) Each of three cubical vessels *A*, *B*, *C*, whose capacities are as 1 : 8 : 27, respectively, is partially filled with water, the quantities of water in them being as 1 : 2 : 3 respectively. So much water is now poured from *A* into *B*, and so much from *B* into *C*, as to make the depth of water the same in each vessel. After this, $128\frac{4}{7}$ cubic feet of water is poured from *C* into *B*, and then so much from *B* into *A*, as to leave the depth of water in *A* twice as great as the depth of water in *B*. The quantity of water in *A* is now less by 100 cubic feet than it was originally. How much water did each of the vessels originally contain?

Ans. 500, 1000, and 1500, cubic feet respectively.

(6) A fraudulent tradesman contrives to employ his *false* balance both in buying and selling a certain article: thereby gaining at the rate of 11 per cent. more on his outlay than he would gain were the balance *true*. If however the scale-pans, in which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the article. Determine the legitimate gain per cent. on the article.

Ans. 10 per cent.

(7) A metallic lump m_1 is compounded of the metals a and b, another lump m_1 of the metals b and c. The ratio of the weight of a in m_1 to that of c in m_2 is three times the ratio of the weights of b in m_1, m_2, respectively; and three times the weight of m_1 is ten times the weight of m_2. In two other lumps μ_1, μ_2, of the same volumes and compounded of the same metals as m_1, m_2, respectively, the ratio of the weights of a and b in μ_1 is equal to the ratio of the weights of b and c in m_2; the ratio of the weights of b and c in μ_2 is equal to the ratio of the weights of a and b in m_1; and five times the weight of μ_1 is eighteen times the weight of μ_2. Having given that the weights of equal volumes of a, b, c, are as $3 : 2 : 1$ respectively, determine the proportions in which each of the lumps is compounded.

Ans. m_1 and m_2 are compounded of equal volumes of the metals: μ_1 is compounded of volumes of a and b in ratio $4 : 3$, and μ_2 of volumes of b and c in ratio $3 : 4$. In m_1, weight of a : weight of b :: $3 : 2$; in m_2, weight of b : weight of c :: $2 : 1$; in μ_1, weight of a : weight of b :: $2 : 1$; and in μ_2, weight of b : weight of c :: $3 : 2$.

X.

ST JOHN'S COLLEGE. *June*, 1857.

(1) $(x+1)^5+(x-1)^5 = 19\{(x+1)^3+(x-1)^3\}$; find x.

Ans. $x = 0$, or $\pm\sqrt{13}$, or $\pm 2\sqrt{-1}$.

(2) $(x+1)(x+2)(x+3)(x+4) = (x+1)^2+(x+2)^2+(x+3)^2+(x+4)^2$; find x.

Ans. $x = -\dfrac{5}{2} \pm \sqrt{\dfrac{13}{4} \pm \sqrt{15}}$.

(3) $\left.\begin{aligned}(x^2+1)(y^2+1) &= 10,\\ (x+y)(xy-1) &= 3;\end{aligned}\right\}$ find x and y.

Ans. $\begin{cases} x = 0,\ 1,\ \pm 2,\ -3,\ \dfrac{1}{2}(1 \pm \sqrt{-15}).\\ y = -3,\ \pm 2,\ 1,\ 0,\ \dfrac{1}{2}(1 \mp \sqrt{-15}).\end{cases}$

(4) $\left.\begin{aligned}x^2+y^2-z^2 &= (x+y-z)^2+2,\\ x^3+y^3-z^3 &= (x+y-z)^3+9,\\ x^4+y^4-z^4 &= (x+y-z)^4+29;\end{aligned}\right\}$ find x, y, and z. Ans. $\begin{cases} x = \dfrac{5}{2}, \text{ or } \dfrac{1}{2},\\ y = \dfrac{1}{2}, \text{ or } \dfrac{5}{2},\\ z = \dfrac{3}{2}.\end{cases}$

(5) Alfred, Edward, and Herbert come each with his pail to a well; when a question arises about the quantity of water in the well: but none of them knowing how much his pail will hold they cannot settle the dispute. Luckily Mary comes up with a pint measure, by aid of which they discover that Alfred's pail holds half a gallon more than Edward's, and a gallon more than Herbert's: but before the precise content of any pail is found out an accident happens, and Mary's measure is broken. They are

now however in a position to ascertain the quantity of water in the well: for they find that it fills each pail an exact number of times; and that the number of times it fills Edward's is greater by eight than the number of times it fills Alfred's, and less by forty than the number of times it fills Herbert's. How much water was there in the well?

Ans. 15 gallons.

(6) Seven-ninths of the stronger of two glasses of wine and water of equal size is mixed with two-ninths of the weaker, and the remainder of the weaker with the remainder of the stronger. The stronger of these two new glasses is a certain number of times stronger than the weaker; and the stronger of the two original glasses was twice the same number of times stronger than the weaker. Compare the strengths of the two original glasses: the strength of a glass of wine and water being defined to be the ratio of the quantity of wine to the quantity of the whole mixture in the glass.

Ans. 4 : 1.

(7) Two Tyrolese Jäger agree to shoot at a mark on the following terms: each is to shoot an even number of times fixed for each beforehand, and for every time that either hits he is to receive from the other a number of kreuzers equal to the whole number of times that he misses. They have two matches without varying the conditions. In the first match, the second Jager misses as often as the first hits in the second match, and the first Jager misses twice as often as the second hits in the second match; and the second Jager has to pay to the first a balance of 4 kreuzers. In the second match, each hits exactly the number of times most favourable to him, and the second has to pay to the first a balance of 36 kreuzers. How many times did each hit and miss in each match?

Ans. In the first match, the first Jager hit 4 times, and missed 16;
and the second Jager hit 6 times, and missed 10.
In the second match, the first Jager hit, and missed, 10 times;
and the second Jager hit, and missed, 8 times.

XI.

ST JOHN'S COLLEGE. *June*, 1858.

(1) $\left(\frac{x+a}{x+b}\right)^2+\left(\frac{x-a}{x-b}\right)^2=\left(\frac{a}{b}+\frac{b}{a}\right)\cdot\frac{x^2-a^2}{x^2-b^2}$; find x.

Ans. $x=\frac{1}{2}\{\pm(a+b)\pm\sqrt{a^2+6ab+b^2}\}$.

(2) $\left.\begin{aligned} mx^2+2ny&=m,\\ ny^2+2mx&=n;\end{aligned}\right\}$ find x and y.

Ans. $\begin{cases} x=-1\pm\sqrt{\frac{2n}{m}}, \text{ or } 1\pm\sqrt{-\frac{2n}{m}},\\ y=-1\pm\sqrt{\frac{2m}{n}}, \text{ or } 1\mp\sqrt{-\frac{2m}{n}}.\end{cases}$

(3) $$\left.\begin{array}{l}\dfrac{x}{a+y+z}=\dfrac{y}{b+x+z}=\dfrac{z}{c+x+y},\\ (x+y+z)^2=ax+by+cz;\end{array}\right\}\text{ find } x,\ y, \text{ and } z.$$

Ans. $x=\dfrac{1}{3}\cdot\dfrac{a^2+b^2+c^2+a\sqrt{3(a^2+b^2+c^2)}}{a+b+c+\sqrt{3(a^2+b^2+c^2)}}$; and similarly for y and z.

(4) $$\left.\begin{array}{c}a(x+b)^3-b(x+a)^3=b(y+b)^3-a(y+a)^3,\\ x^2-2y(a+b)=y^2+a^2+b^2;\end{array}\right\}\text{ find } x \text{ and } y.$$

Ans. See *Companion.*

(5) Lady Sadler's lecturer, being in haste, had to ascend his staircase of three flights, of which the number of steps in the lower flight : that in the upper as 8 : 9; and half that in the lower flight is greater by 2 than $\frac{1}{7}$th of the whole staircase. He begins running at 3 times his usual pace of ascent, which would take him up the upper flight in $13\frac{1}{2}$ seconds; but for the middle and upper flights his pace becomes diminished in the ratio 6 : 5 : 4. When he had ascended $\frac{2}{3}$rds of the upper flight he had to wait 31 seconds to recover his breath; walks the remainder at the rate of 1 step less than usual in 5 seconds; and discovers that he has been longer about it, than if he had walked up, in the ratio of 69 : 50. What was the number of steps in the staircase? Ans. 70.

(6) A room is to be papered with paper 2 feet in width at 8*d.* a yard, and carpeted with carpet at 3*s.* 9*d.* a square yard; but $\frac{1}{6}$th of the surface of the walls is taken up by a door and windows. Now the dimensions of the room are taken by an inaccurate foot-rule; hence the estimated cost is less than the actual cost in the proportion of 1 : 1·0201. Had however the Rule been 4 times as much *less* than a foot as it was *more* than a foot, the estimated cost in this case would have exceeded the actual cost by 17*s.* $0\frac{1}{6}$*d.* Having given that the cost of the carpet is double that of the paper, and that a square whose side is equal to the diagonal of the room contains 84 yards, find the dimensions of the room.

Ans. 20, 16, and 10, feet.

(7) n men A_1, A_2, A_3,...A_n, enter into partnership for nm years, the capital of the firm at the beginning of the $(m+1)$th year being £P; and they agree, after deducting their respective expenditures at the end of each year from their gain in that year to employ the remainder in the trade. Now the gain of the firm and the sum of the expenditures of A_1, A_2,....A_n, in any year are proportional to the capital employed in that year; and the separate expenditures of A_1, A_2,...in any year are proportional to their respective gains in that year. Supposing A_1's capital at beginning of the $(m+1)$th year : A_2's at beginning of the $(2m+1)$th : A_3's at beginning of the $(3m+1)$th... as $a_1 : a_2 : a_3$...; and that the ratio of the sum of the expenditures of A_1, A_2... during the $(r-1)$th and $(r+1)$th periods of m years to that

of their expenditure during the r^{th} period of m years is $a+\frac{1}{a} : 1$; find the original capitals of the several partners.

$$\text{Ans. Capital of } A_1 = \frac{Pa_1a^{n-2}}{a_1a^{n-1}+a_2a^{n-2}+\ldots+a_n},$$

$$\ldots\ldots\ldots\ldots\ A_2 = \frac{Pa_2a^{n-3}}{a_1a^{n-1}+a_2a^{n-2}+\ldots+a_n},$$

$$\ldots\ldots\ldots\ldots\ A_n = \frac{Pa_na^{-1}}{a_1a^{n-1}+a_2a^{n-2}+\ldots+a_n}.$$

XII.

ST JOHN'S COLLEGE. *June*, 1859.

(1) $x^2(m^2-n^2)+x\{m(2am-p)-n(2bn-p)\}$
$= (am-bn)(p-am-bn)$; find x.

$$\text{Ans. } x = \frac{bn-am}{m-n}, \text{ or } \frac{p-am-bn}{m+n}.$$

(2) $\dfrac{(x+a)(x+b)}{yz} = \dfrac{(y+a)(y+b)}{xz} = \dfrac{(z+a)(z+b)}{xy} = m$; find x, y, and z.

$$\text{Ans. } \left.\begin{aligned} x &= 0,\ \ 0,\ -b,\ -a, \\ y &= -b,\ -a,\ \ 0,\ \ 0, \\ z &= -a,\ -b,\ -a,\ -b. \end{aligned}\right\}$$

(3) $\left.\begin{aligned} x^2+xy+y^2 &= m(1-2\sqrt{3})x+5my, \\ x^2-xy &= 5mx+m(1-2\sqrt{3})y; \end{aligned}\right\}$ find x and y.

$$\text{Ans. } x = \frac{m}{2}\left(5+\frac{4}{\sqrt{3}}\right) \pm m\sqrt{\frac{1}{12}(43-40\sqrt{3})},$$

$$y = \frac{m}{2}\left(1-\frac{5}{\sqrt{3}}\right) \pm m\sqrt{-\frac{1}{6}(34+37\sqrt{3})}.$$

(4) $\left.\begin{aligned} (\sqrt{5}-1)(x+yz)(y+zx)(x+xy) &= \sqrt{5}+1, \\ (\sqrt{5}+1)(1-x^2)(1-y^2)(1-z^2) &= \sqrt{5}-1, \\ x^2+y^2+z^2+2xyz &= \sqrt{5}+1; \end{aligned}\right\}$ find x, y, and z.

$$\text{Ans. } x = 0,\ y = \pm z = \pm\sqrt{\frac{1}{2}(\sqrt{5}+1)},$$

$$\text{or } y = 0,\ x = \pm z = \pm\sqrt{\frac{1}{2}(\sqrt{5}+1)},$$

$$\text{or } z = 0,\ x = \pm y = \pm\sqrt{\frac{1}{2}(\sqrt{5}+1)}.$$

(5) The Cambridge University Musical Society consists of half and full (or performing) members, who have respectively 1 and 2 tickets each to the concerts: a permanent list of persons invited is kept, which may be considered as arising from the full members (known to be even in number) assigning 1 and 2 names alternately. On the occasion of a concert each member applies for an invitation for a stranger; the last 20 are rejected, and still, from the smallness of the room, 9 had to stand for every 20 seated. When the new Town-Hall is built, (to hold 3 times as many people as the present one), if we assume the Society to receive an addition of 40 half, and 20 full, members, and that each member shall be allowed 1 more ticket, and shall obtain an invitation for 1 and 2 strangers respectively, and that the Committee moreover will increase the permanent list in the ratio of 10 : 9, and will invite 50 strangers, the Hall will be filled, and there will be present 30 more strangers than twice the number on the permanent list. 50 full members are to be supposed to perform at the first concert, and to sit amongst the audience at the second. How many will the Hall hold? Ans. 400.

(6) The perimeters of three rectangles *A*, *B*, *C*, are as 2 : 3 : 5, and their areas as 2 : 3 : 7. Now a rectangle, equal in area to *A*, *B*, *C*, together, and *similar* to *B*, contains as many linear inches in its perimeter, as *A* and *B* together contain square inches; also the perimeter of a rectangle, equal in area to the excess of *B* and *C* together over *A*, and *similar* to *A*, is 8 inches less than the perimeter of a square equal in area to *A*, *B*, *C*, together. Find the sides of the rectangles *A*, *B*, *C*.

Ans. Of *A*, 6 in. and 4 in.; of *B*, 12 in. and 3 in.; of *C*, 21 in. and 4 in.

(7) A railway crosses obliquely a river flowing due East and West, and at a certain signal-post a branch turns off, making an angle with the main line equal to that made with the same by a road running due South from the junction to a wharf 1 mile from the bridge. A barge starts from the wharf at the same time as a train passes a station before it crosses the river; 45 min. afterwards the train is observed from the barge, and found to be on the branch, 9 miles from the junction, and in a straight line with the signal-post, having passed the junction when the barge was under the bridge. On the return of the barge, which was towed at half its original rate (the stream being contrary), when it was $\frac{1}{\sqrt{6}}$ of a mile from the wharf, a horseman set out from the junction, and arrived at the wharf simultaneously with the barge, having travelled at a rate which is the mean proportional between that of the train and the original rate of the barge. What was the rate of the train, and the distance of the station from the bridge?

(1) Ans. 18 miles per hour. (2) Ans. $2\frac{1}{2}$ miles.

EASY EXERCISES.

EXERCISES. A.

If a stand for 10, b for 3, and x for 7, what is the value of each of the following quantities?

(1) $a+b+x$.

(2) $a+b-x$.

(3) $a-b+x$.

(4) $a-b-x$.

(5) $2a-x$.

(6) $4a+3b-2x$.

(7) $7a+2b-2x$.

(8) $5a-4b-4x$.

(9) $2ab-3x$.

(10) $2a+5-3bx+100$

(11) $7ab-abx$.

(12) $3a+bx-xx$.

(13) What is the *coefficient* of x in $3ax$?

(14) What is the *coefficient* of x in $6abx$?

(15) What is the *coefficient* of bx in $6abx$?

(16) What is the *coefficient* of a in each of the quantities $2a$, $2ab$, abx, $3abx$, ma, axx, pax, $abxy$?

(17) What is the *coefficient* of 25 in 125?

(18) What is the difference between $3+x$, and $3x$, when x stands for 7?

(19) What is the difference between $3a+x$, and $3a-x$, when a stands for 10, and x for 6?

(20) What is the difference between $3a+x$, and $3ax$, when a stands for 3, and x for 2?

Find the value of each of the following quantities, when a stands for 10, b for 3, and x for 7:—

(21) $3ax \div 7$.

(22) $3ax \div 7b$.

(23) $\frac{2a+x}{b}$.

(24) $\frac{3b+3x}{a}$.

(25) $\frac{a-x}{b}$.

(26) $\frac{3a}{b}+2x-\frac{abx}{21a}$.

(27) $\frac{5a+x}{b}-\frac{5b+a}{2x-3b}$.

(28) $\frac{3x}{4a+2}+\frac{4bx}{10a-16}$.

(29) $\frac{2a+4b}{3x-a-b}-\frac{a-2b}{x-b}$.

(30) $\frac{ma}{b+x}+\frac{nb}{a-x}-\frac{px}{a-b}$.

If a stand for 1, b for 9, and c for 8, find the value of each of the following quantities :

(31) $a^2+b^2-c^2$.

(32) $13a^2+3b^2-4c^2$.

(33) $5abc-22b^2+3c^3$.

(34) a^2b+b^2c.

(35) $12ab^2+20a^2b-2bc^2$.

(36) $\frac{b^2}{a}+\frac{a^2}{b}-\frac{c^2}{2a}$.

(37) $\frac{8ab^2}{3c}-\frac{9ac^2}{2bc}$.

(38) $ma^2+nb^2-pc^2$.

(39) $2\sqrt{b}-\sqrt{2c}$.

(40) $\sqrt{ab}+\sqrt{a^2b}$.

(41) $a+\sqrt{b}-\sqrt{ab}+2\sqrt{2bc}$.

(42) $\sqrt{2c+b}-\sqrt{2b-2a}$.

(43) $m\sqrt{\frac{b}{a}}+n\sqrt{\frac{bc}{2}}-p\sqrt{2ac}$.

(44) $\sqrt[3]{ac}+\sqrt{4b}-2\sqrt[3]{c}$.

(45) $\sqrt{b+c-a}-\sqrt[3]{3b-2c-3a}$.

(46) $\sqrt[3]{a}+\sqrt{2c}.\sqrt[3]{\frac{bc}{9}}-4\sqrt[3]{b-a}$.

(47) What is the difference between $3a$ and a^3, when a stands for 2 ?

(48) What is the difference between $2\sqrt{x}$ and $2+\sqrt{x}$, when x is 100 ?

(49) What is the difference between $3\sqrt{x}$ and $\sqrt[3]{x}$, when x is 64 ?

(50) What is the difference between $\sqrt{a+b}$ and $\sqrt{a}+b$, when a stands for 1, and b for 8 ?

(51) What is the difference between $\sqrt{\frac{a}{b}}$ and $\frac{\sqrt{a}}{b}$, when a stands for 16, and b for 4 ?

EXERCISES. B.

Add together

(1) $a+b$, and $a+b$.

(2) $a+b$, and $a-b$.

(3) $a-b$, and $a-b$.

(4) $a-b+c$, and $a+b-c$.

(5) $a-b+c$, and $a+b+c$.

(6) $1-2m+3n$, and $3m-2n+1$.

(7) $5m+3$, and $2m-4$.

(8) $3xy-2x$, and $xy+6x$.

(9) $4p-2q+1$, and $7-3p+q$.

(10) $5ab-2bc$, and $ab+bc$.

(11) $2ax+3by$, and $ax-by$.

(12) $3a-2b+4c$, and $2a-3b+c$.

(13) $xy+x-7$, and $3xy-2x+3$.

(14) $p+q-pq$, and $2pq-3p+2q$.

(15) $p^2+2pq+q^2$, and $p^2-2pq+q^2$.

(16) $7ab-5ac+1$, and $ab+6ac-2$.

(17) $7x-6y$, $-x-3y$, $-x+y$, $-2x+3y$, and $x+8y$.

(18) $3-a$, $-8-a$, $7a-1$, $-a-1$, and $9+a$.

(19) $2a-5b+3c-d$, and $a+5b-c+2d$.

(20) $a^2+2ab+b^2$, and $2a^2-ab-3b^2$.

(21) $3x^2-6x+5$, $2x-3-x^2$, and $4-x-2x^2$.

(22) $ac+bd$, $bd-cd$, and $ac+cd$.

(23) $3x^2-4y^2$, x^2+y^2, and $\frac{1}{2}x^2-\frac{1}{2}y^2$.

(24) $x^3-3ax^2+3a^2x-a^3$, and $3a^3-2a^2x+4ax^2-x^3$.

(25) $8mn+m$, and $1-n-7mn$.

(26) $9x-8y-7$, and $3z-9x+6y+7$.

(27) $a^3-2ab^2+a^2b$, $3ab^2-2a^2b$, and b^3+c^3.

(28) $a^3-\frac{3}{2}ab^2$, $b^3-\frac{1}{2}a^2b$, and $ab^2-\frac{1}{2}a^2b$.

(29) $\frac{1}{4}x^2+2xy$, $\frac{3}{4}x^2-xy+y^2$, and $mx+ny$.

(30) $ad+2bd-3cd$, $\frac{1}{2}ad-\frac{1}{2}bd$, and $\frac{1}{2}ab+2cd-ac$.

(31) I received m shillings from my father, the same from my mother, and n shillings from each of three friends, express the whole sum.

(32) A certain sum is divided between A, B, and C; B receives a pounds more than A, and C receives b pounds more than B; if A receives x pounds, find an expression for the whole sum divided.

EXERCISES. C.

(1) From a take $b-x$.

(2) From $a+b-c-d$ take $a-b+c-d$.

(3) From $6a-2b-3c$ take $a-2b+2c$.

(4) From $a+x-b-5c$ take $x+8b-5c$.

(5) From $3x+2y-5z$ take $2x+3y+4z$.

(6) From $2ax+by-c$ take $ax-by+c$.

(7) From $3bc-ab+a$ take $2bc+ab-a$.

(8) From $xy+x^2+y^2$ take $xy-x^2+y^2$.

(9) From $2xy+3x^2+4y^2$ take $xy-2x^2-y^2$.

(10) From $2mn+5m-3n$ take $mn+m+n$.

(11) From $-2xy+mx-py$ take $-3xy-2mx-py$.

(12) From $5abc-2ab-3ac$ take $2abc+ab-ac+1$.

(13) From $a^2-b^2+c^2$ take $a^2-2b^2-2c^2$.

(14) From $4ax-3a^2+2x^2$ take $2ax-a^2+4x^2$.

(15) From $3a^2b+2a^2c-5c^2$ take $a^2b-a^2c-7c^2$.

(16) From $2xy+3a-a^2b+5$ take $2a-a^2b+6$.

(17) From $\dfrac{2}{3}ax-\dfrac{1}{3}xy+\dfrac{2}{3}$ take $\dfrac{1}{3}ax+\dfrac{2}{3}xy-\dfrac{1}{3}$.

(18) From $a+b-c$ take $\dfrac{1}{2}a-\dfrac{1}{2}b-\dfrac{1}{2}c$.

(19) The united ages of a father and his son make 60 years, and the father was 30 years old when the son was born, what is the age of each?

(26) Divide 1 into two fractional parts, so that one part shall exceed the other by $\frac{1}{2}$.

EXERCISES. C*.

Simplify each of the following quahtities:

(1) $ab+a(c-b)$.

(2) $4(1-x)+3x$.

(3) $2(a+x)-2(a-x)$.

(4) $2(a+b)(a-b)$.

(5) $5(1-x)+(1+5x)\times 2$.

(6) $\frac{a-x}{2}-\overline{x-2a}$.

(7) $\frac{1}{2}(a+b)-\frac{1}{2}(a-b)$.

(8) $(a+7)x^2+(b-7)x^2$.

(9) $2-(-4+5x)$.

(10) $1-(1-\overline{1-x})$.

(11) $(6a-\overline{b+c})-(a-\overline{b-2c})$.

(12) $\frac{1}{2}(a-x)(2a+x)+\frac{1}{2}x(a+x)$.

(13) $(1+x)(1-x)(1+x^2)$.

(14) $2\left(x^2-\frac{1}{4}\right)\div(2x+1)+\frac{1}{2}$.

(15) $\frac{1}{2}\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{1}{2}\left(\frac{a}{b}-\frac{c}{d}\right)$.

(16) $\left\{\frac{a(a+b)+b^2}{a}\right\}\div\left\{\frac{b(a+b)+a^2}{b}\right\}$.

(17) $4\times\left\{\frac{3}{8(1-x)}+\frac{1}{8(1+x)}\right\}$.

(18) $\frac{2x(2x-a)}{(a-2x)^2}+\frac{a}{a-2x}$.

(19) $\frac{2}{3}x(x+1)\{x+2-\frac{1}{2}(2x+1)\}$.

(20) $\{1-\overline{1-x}\,]^2\}x(2+x)$.

EXERCISES. D.

Multiply

(1) axy by b.

(2) $3mn$ by $-p$.

(3) $-2xy$ by $4a$.

(4) $-2xy$ by $-4a$.

(5) $\frac{1}{2}ab$ by $2c$.

(6) $3mn$ by mp.

(7) $m+n-p$ by 3.
(8) $ax+bx^2$ by p.
(9) $ad+2bd$ by $2a$.
(10) $4a^2-2axy$ by ax.
(11) $3x-2xy+6$ by $-xy$.
(12) $1-2ax+3bx^2$ by $-3n$.
(13) $2ab-3ac+5bd$ by $-2x$.
(14) $2xy-3$ by $7x$.
(15) $2ax+by-cz$ by $2xyz$.
(16) $2a^2-bx+d$ by by.
(17) $a+x$ by $b+y$.
(18) $6x+4$ by $x-1$.
(19) $x-4$ by $x+3$.
(20) $2x-5$ by $3x-2$.
(21) $1-x$ by $x+1$.
(22) $1-x$ by $x-2x^2$.
(23) $ax+by$ by $2x-y$.
(24) $a+2x$ by $a-3x$.
(25) $7x-1$ by $5x-4$.
(26) $2ax-3by$ by $4y-3x$.
(27) $1-2mn$ by $2m+n$.
(28) a^2-bc by $ac-b^2$.
(29) $1+2x+3y$ by $x-y$.
(30) $a+x-y$ by $b-y$.
(31) $ac-bc+ad$ by $2a-b$.
(32) a^3+a^2+a+1 by $a-1$.
(33) $x^3+ax^2+a^2x+a^3$ by $x-a$.
(34) $4x^2-6x+9$ by $2x+3$.
(35) $4+2x+x^2$ by $4-2x+x^2$.
(36) a^3-2x^2 by a^3-x^2.
(37) $x^3+3x^2+9x+27$ by $x-3$.
(38) $2a^4x^2+3b^2y$ by $2a^4x^2-3b^2y$.
(39) $2a^2-3ab+b^2$ by $2a^2+3ab-b^2$.
(40) a^6+a^5-a-1 by $1-a+a^2-a^3+a^4$.

EXERCISES. E.

Divide

(1) $7x$ by 7.
(2) $7x$ by x.
(3) $7ax$ by a.
(4) $7ax$ by $7x$.
(5) $3abx$ by ab.
(6) $3abc$ by $3bc$.
(7) $-axy$ by x.
(8) axy by $-x$.
(9) $6a^2mn$ by $-2mna$.
(10) $14a^3xy^2$ by $7a^2y$.
(11) $-7mn^2px$ by $\frac{1}{2}mnp$.
(12) $-\frac{2}{3}abx^2y$ by $-\frac{1}{3}axy$.
(13) $3ac-2abd$ by a.
(14) $4ac-2abd$ by $2a$.
(15) $8x^2-6xy$ by $-2x$.
(16) $3bc+24abc^2-6b^2c^2$ by $3bc$.
(17) $4a^2x^2-8abx-2ax$ by $-2ax$.
(18) $a^3x^2-5abx^3+6ax^4$ by ax^2.
(19) x^2+3x+2 by $x+2$.
(20) $ac-bc+ad-bd$ by $a-b$.
(21) $6+3a-2b-ab$ by $2+a$.
(22) $4a^2-15x^2-4ax$ by $2a+3x$.
(23) $a^2-ax-6x^2$ by $a-3x$.
(24) $2ab+6abc-8abcd$ by $1+3c-4cd$.
(25) $3x^2+16x-35$ by $x+7$.
(26) $3x^4+14x^3+9x+2$ by x^2+5x+1.

(27) $ab+2a^2-3b^2-4bc-ac-c^2$ by $2a+3b+c$.
(28) $15a^4+10a^3x+4a^2x^2+6ax^3-3x^4$ by $3a^2-x^2+2ax$.
(29) $qp^3+3p^2q^2-2pq^3-2q^4$ by $p-q$.
(30) $a^2x^3+a^5-2abx^3+b^2x^3+a^3b^2-2a^4b$ by $ax-bx+a^2-ab$.
(31) $32x^5+243$ by $2x+3$.

EXERCISES. F.

Find the G.C.M. of

(1) 128, and 84.
(2) 125, and 900.
(3) 80, 100, and 140.
(4) ax, and bx.
(5) bx^2, and b^2x^3.
(6) apx^2, and a^2px.
(7) $5a^2bx$, and $20abxy$.
(8) $15a^6b^2$, and $3a^2b^6$.
(9) $9a^4b^2c^6$, and $27a^6b^3c^9$.
(10) $14m^2np^3$, and $7mnp$.
(11) $abxy$, and $2acxy$.
(12) $\frac{4}{5}a^2$, and $\frac{2}{5}ab$.
(13) abd, acd, and bcd.
(14) pxy, x^2y^2, and apx.
(15) x^2-y^2, and $x^2+2xy+y^2$.
(16) $ax+bx$, and $ay+by$.
(17) a^3-ab^2, and ab^2+b^3.
(18) $2a^3-2ab$, and $5a^3-5ab$.
(19) $14a^2-7ab$, and $10ac-5bc$.
(20) $(x+y)^3$, and $(x^2-y^2)^2$.
(21) x^3-2x-1, and x^2+2x+1.
(22) $x^6-3x^4+3x^2-2$, and x^4-3x^2+2.
(23) $2x^3-3x^2-9x+5$, and $2x^2-7x+3$.
(24) $4x^2+3x-10$, and $4x^3+7x^2-3x-15$.
(25) $x^2-7x+10$, and $4x^3-25x^2+20x+25$.
(26) x^2+2x-3, and x^2+5x+6.
(27) x^3-x^2-2x, and $2x^3+3x^2+x$.
(28) x^3-3x-2, and x^4-x^3+x-10.
(29) $2x^3+x^2-x+3$, and $2x^3+5x^2+x-3$.
(30) $x^2+xy-12y^2$, and $x^2-5xy+6y^2$.
(31) $6x^2-12x+6$, and $3x^2-3$.

EXERCISES. G.

Find the G.C.M. of

(1) $4+12x+9x^2$, and $2+13x+15x^2$.
(2) $a^3+a^2b-ab^2-b^3$, and $a^3-a^2b-ab^2+b^3$.
(3) $3x^2+8x+5$, and $2x^3-x^2-3x$.

(4) $x^3-11x^2+39x-45$, and $3x^2-22x+39$.

(5) $6x^2-7x-20$, and $4x^2-27x+5$.

(6) $3x^3-13x^2+23x-21$, and $6x^3+x^2-44x+21$.

(7) $3x^2-16x-12$, and $2x^3-16x^2-24x+288$.

(8) $8x^3+6x^2-4x-3$, and $12x^3+5x^2+x+3$.

(9) $3x-1-2x^2$, and $1-x-2x^2$.

(10) x^4-41x^2+16, and $x^4-7x^3+28x-16$.

(11) x^3-1, x^2-2x+1, and x^2-1.

(12) $a^2+2ab+b^2$, a^2-b^2, and $a^3+2a^2b+2ab^2+b^3$.

Reduce the following fractions to lowest terms:—

(13) $\frac{2ax}{3x}$.

(14) $\frac{4abc}{2ac}$.

(15) $\frac{20abx}{15a^2}$.

(16) $\frac{3abx^2}{6ax}$.

(17) $\frac{75ax^2y^3}{15a^2y^3}$.

(18) $\frac{ab^3x}{2ab^4x^2}$.

(19) $\frac{mx-nx}{mnx}$.

(20) $\frac{2x^2-3x}{5x}$.

(21) $\frac{14a^3+21a^2}{7a^2b}$.

(22) $\frac{4bc+2c}{2ac}$.

(23) $\frac{3ax-2x^2}{2ax-3x^2}$.

(24) $\frac{mnp-m^2p+mp^2}{m^2p-mnp+mp^2}$.

(25) $\frac{a^2-x^2}{a+x}$.

(26) $\frac{a^2-4x^2}{a-2x}$.

(27) $\frac{a^2+ab}{b^2+ab}$.

(28) $\frac{a^2+b^2+2ab}{a^2-b^2}$.

(29) $\frac{9x^2-12xy+4y^2}{6x-4y}$.

(30) $\frac{3ax^2-15a^2x}{2ax-10a^2}$.

(31) $\frac{3bx+cx}{6ab+2ac}$.

(32) $\frac{a^2x+a^3}{ax^2-a^3}$.

(33) $\frac{\frac{1}{2}mnx^2-2mnp^2}{\frac{1}{2}nx+np}$.

(34) $\frac{x^2+2x-3}{x^2+5x+6}$.

(35) $\frac{3x^2+x-2}{3x^2+4x-4}$.

(36) $\frac{2a^2-3a+1}{a^2+a-2}$.

(37) $\frac{x^3+a^3}{x^2+2ax+a^2}$.

(38) $\frac{x^5-2x^4-15x^3}{x^4-8x^3+15x^2}$.

(39) $\frac{4+12x+9x^2}{2+13x+15x^2}$.

(40) $\frac{1+x^3}{1+2x+2x^2+x^3}$.

(41) $\frac{x^4-x^2-2x+2}{2x^3-x-1}$.

(42) $\frac{56x^3-28x^2-42x+1}{42x^2-28x-14}$

EXERCISES. H.

Find the L.C.M. of

(1) 21, and 24.

(2) 12, 16, and 20.

(3) 4, 7, 8, and 14.

(4) 4, 7, 14, 21, and 24.

(5) 1, 2, 3, 4, 5, 6, 7, 8, 9.

(6) 21, 22, 23, and 24.

(7) ax, and bx.

(8) ax, and $2xy$.

(9) $2x$, $6x$, and $8x$.

(10) ab, ac, and bc.

(11) x^2, y^2, and $2xy$.

(12) bd, c^2d, cd^2, and bc

(13) $2a$, $a+x$, and $a-x$.

(14) $2a$, $4b$, $1+x$, and $1-x$.

(15) $3xy$, x^2+xy, and y^2+xy.

(16) a^3b^2, a^2b^3, $ab-bx$, and a^2+ax.

(17) $1+a$, $1-a$, $1+a+a^2$, and $1-a+a^2$.

(18) $1-x$, $1+x$, $1-x^2$, and $1-2x+x^2$.

(19) x^2+3x+2, $15(x+1)$, and $20(x+2)$.

(20) x^3-7x-6, x^2-2x-3, and x^2-x-6.

(21) $8x^3-14x+6$, $4x^2+4x-3$, and $4x^2+2x-6$.

EXERCISES. I.

Add together

(1) $\frac{x}{5}$, $\frac{2x}{5}$, and $\frac{3x}{5}$.

(2) $\frac{2ab}{3}$, and $\frac{ab}{6}$.

(3) $\frac{2a}{3}$, $\frac{a}{3}$, and $\frac{1}{3}$.

(4) $\frac{a+x}{5}$, and $\frac{a-x}{5}$.

(5) $\frac{2x+1}{7}$, and $\frac{4x-5}{7}$.

(6) $\frac{2x+1}{7}$, and $\frac{4x-5}{21}$.

(7) $\frac{1}{a}$, $\frac{2}{a}$, and $\frac{3}{a}$.

(8) $\frac{1}{ab}$, $\frac{2}{ab}$, and $\frac{3}{ab}$.

(9) $\frac{2}{xy}$, $\frac{1}{x^2}$, and $\frac{1}{y^2}$.

(10) $\frac{1}{a}$, $\frac{2}{ab}$, $\frac{3}{abc}$.

(11) x, $\frac{3x-5}{2}$, and $\frac{2x-4}{3}$.

(12) $\frac{x}{6}$, $\frac{7x-6}{3}$, and $\frac{4x+1}{12}$.

(13) $\frac{4x-5}{10}$, $\frac{2x}{5}$, and $\frac{7x+6}{25}$.

(14) $\frac{3}{x}$, $\frac{1}{3x}$, and $\frac{4}{5x}$.

(15) $\frac{4}{5y}$, $\frac{1}{2y}$, and $\frac{3}{4y}$.

(16) $\frac{x}{a}$, $\frac{y}{b}$, and $\frac{z}{c}$.

(17) $\frac{xy-ab}{ab}$, $\frac{xy-bc}{bc}$, and 2.

(18) $\frac{a-b}{ab}$, $\frac{b-c}{bc}$, and $\frac{c-a}{ac}$.

Subtract

(19) $\frac{4x}{5}$ from $\frac{9x}{10}$.

(20) $\frac{7x}{8}$ from x.

(21) $\frac{5x+4}{9}$ from $\frac{10x+17}{18}$.

(22) $\frac{2x-3}{4}$ from $\frac{5x-1}{8}$.

(23) $\frac{3y+x+13}{10}$ from $\frac{3x+y}{5}+1$.

(24) $\frac{15+3x}{x+1}$ from $7+\frac{24}{x+1}$.

(25) $\frac{2}{x}+\frac{4}{x}$ from $\frac{3}{x}+\frac{5}{x}$.

(26) $\frac{x}{x+1}$ from $\frac{3x}{x+2}$.

(27) $\frac{2x-7}{21}$ from $\frac{3x+7}{14}$.

(28) $\frac{x}{10}+\frac{4}{25}$ from $\frac{11x-13}{25}$.

(29) $\frac{a}{b+cx}$ from $\frac{a}{b}$.

(30) $\frac{2x}{x+y}$ from $\frac{x+y}{y}$.

(31) $\frac{2}{1+x}$ from $\frac{3+2x}{1+x^2+2x}$.

(32) $\frac{x-y}{x+y}$ from $\frac{x+y}{x-y}$.

Add together

(33) $\frac{x}{x+2}$, and $\frac{x}{x-2}$.

(34) $\frac{x}{a+x}$, $\frac{a}{a-x}$, and 1.

(35) $\frac{x}{x+y}$, $\frac{y}{x-y}$, and $\frac{y}{x+y}$.

(36) $\frac{x-1}{x+2}$, and $\frac{x+2}{x-3}$.

(37) $\frac{2x}{(2x-1)^3}$, $\frac{1}{(2x-1)^2}$, and $\frac{1}{2x-1}$.

(38) $\frac{a+x}{2(a-x)^2}$, and $\frac{1}{6(a-x)}$.

(39) $\frac{x^2+y^2}{x^2-y^2}$, $\frac{x}{x+y}$, and $\frac{y}{y-x}$.

(40) $\frac{3}{1+x^3}$, and $\frac{x-2}{1-x+x^2}$.

(41) $\frac{1}{2x^2-4x+2}$, $\frac{1}{2x^2+4x+2}$, and $\frac{1}{1-x^2}$.

(42) $\frac{1}{20(x^2+3x+2)}$, $\frac{1}{15(x+1)}$, and $\frac{1}{20(x+2)}$.

$(x+1)(x+2)$

EXERCISES. J.

(1) Multiply $\frac{x}{2}$ by 8.

(2) .... $\frac{3x}{2}$ by 2.

(3) .... $\frac{5x}{4}$ by 2.

(4) .... $\frac{x}{3}$ by 6.

(5) .... $\frac{a-x}{2}$ by 4.

(6) .... $\frac{7x}{15}$ by 60.

(7) .... $\frac{2x}{21}$ by 84.

(8) .... $\frac{3x-5}{2}$ by 6.

(9) .... $\frac{12+9x}{16}$ by 80.

(10) .... $\frac{8-7x}{4\frac{1}{2}}$ by 9.

(11) Multiply $\frac{6x+13}{1\frac{1}{4}}$ by 15.

(12) .... $\frac{2x-1}{7\frac{1}{2}}$ by 15.

(13) .... $\frac{3x+4}{5\frac{1}{2}}$ by 11.

(14) .... $\frac{x-1\frac{2}{3}}{2\frac{1}{3}}$ by 7.

(15) .... $\frac{2\frac{1}{2}-\frac{1}{4}x}{2\frac{1}{2}}$ by 10.

(16) .... $\frac{3x}{2}$ by $\frac{1}{2}$.

(17) .... $\frac{3x}{2}$ by $\frac{2x}{3}$.

(18) .... $\frac{2-3x}{4}$ by $\frac{4}{5}$.

(19) .... $\frac{1}{2x}$ by $\frac{2}{x}$.

(20) .... $\frac{x}{y}+\frac{y}{x}$ by $xy-\frac{1}{xy}$.

(21) Divide $\frac{5x}{2}$ by 5.

(22) …… $\frac{3x}{4}$ by 5.

(23) …… $\frac{3x}{4}$ by 6.

(24) …… $\frac{21ax}{4y}$ by $7a$.

(25) …… $\frac{2mn}{p}$ by $2n$.

(26) Divide $\frac{2x-4xy}{y}$ by $2x$.

(27) …… $\frac{3a+6ab}{4}$ by $3a$.

(28) …… $\frac{5xy}{z}$ by $\frac{2x}{y}$.

(29) …… $\frac{2abc}{3d}$ by $-\frac{ac}{bd}$.

(30) …… $-\frac{a^2xy}{2bc}$ by $-\frac{ay}{4x}$.

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(31) Multiply $x+\frac{1}{x}$ by $x+\frac{1}{x}$.

(32) ………… $\frac{x}{y}+x$ by $\frac{y}{x}+\frac{1}{x}$.

(33) ………… $\frac{1}{1+x}+\frac{1}{1-x}$ by $\frac{1}{2}$.

(34) ………… $1-\frac{2a}{1+a}$ by $1+\frac{2a}{1-a}$.

(35) Multiply $\frac{1}{2}+\frac{m-3}{2}$ by $\frac{1}{3}+\frac{m-2}{3}$.

(36) ………… $\frac{a}{b}+\frac{1}{2}.\frac{b}{a}$ by $\frac{b}{a}-\frac{1}{2}.\frac{a}{b}$.

(37) ………… $\frac{a^2-ax}{b}$ by $\frac{b^2}{a-x}$.

(38) ………… $\frac{a^2+ax+x^2}{a^2-ax+x^2}$ by $\frac{a-x}{a+x}$.

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(39) Divide $2+\frac{1}{x}$ by $1-\frac{2}{x}$.

(40) ……. $\frac{2-x}{y}$ by $\frac{x}{1-x}$.

(41) ……. $\frac{b-3a}{2ab}$ by $\frac{2a-b}{4a}$.

(42) ……. 1 by $1+\frac{x}{4-x}$.

(43) Divide $\frac{1}{2}$ by $\frac{1}{2}-\frac{x}{2}$.

(44) ……. $\frac{1}{1+x}$ by $1-\frac{1}{1+x}$.

(45) ……. ab by $\frac{b^2}{a-x}$.

(46) ……. $\frac{a^3-x^3}{a^3+x^3}$ by $\frac{a-x}{a+x}$.

EXERCISES. K.

Square each of the following quantities:

(1) $5ax$.

(2) $5axy$.

(3) $-7ab$.

(4) a^2bc.

(5) $-7a^2bc^3$.

(6) $\frac{ab}{c}$.

(7) $\frac{3ax}{2by}$.

(8) $\frac{a^2b}{2c}$.

(9) $\frac{4a^2b}{7x^3y^4}$.

(10) $\frac{-3xy^2}{2x^2}$.

(11) $\frac{4}{5a^2bc^3}$.

(12) $a+1$.

(13) $ab+1$.

(14) $x+3$.

(15) $2-y$.

(16) $2m-n$.

(17) $2x-3y$.
(18) $x-\frac{p}{2}$.
(19) $x+\frac{3}{2}$.
(20) $mx+n$.
(21) $2mx-n$.
(22) $abx+c$.
(23) $3xy-a$.
(24) $\frac{1}{2}ab+c$.
(25) x^2+2x-2.
(26) $a^2-2ab-2b^2$.
(27) $\frac{3}{2}x-\frac{2}{3}y$.
(28) $x-\frac{1}{2}y+1$.

Cube each of the following quantities:

(29) $3x+2$.
(30) $2x-3$.
(31) $\frac{1}{2}x-\frac{2}{3}y$.
(32) $2ab+3c$.
(33) $\frac{a}{b}-\frac{b}{c}$.
(34) $2.\frac{a}{b}+3.\frac{b}{a}$.
(35) $bx+cx^2$.
(36) $1-2a+\frac{1}{a}$.
(37) a^2+a-1.

Find the 4th power of each of the following quantities:

(38) $2-x$.
(39) $\frac{1}{2}+2x$.
(40) $ax-b$.

Extract the square root of each of the following quantities:

(41) $4a^2b^2$.
(42) $9x^2y^4$.
(43) $100a^2b^4c^6$.
(44) $\frac{9a^2x^2}{4b^2}$.
(45) $\frac{4a^2b^2}{9x^2y^4}$.
(46) $\frac{1}{4}.\frac{m^2x^4}{n^2y^2}$.
(47) $1+x^2-2x$.
(48) $4x^2+4x+1$.
(49) $4a^2+b^2-4ab$.
(50) $9x^2+6x+1$.
(51) $x^2+x+\frac{1}{4}$.
(52) $x^2+\frac{1}{x^2}-2$.

Complete the squares in each of the following cases:

(53) x^2-12x.
(54) x^2-14x.
(55) x^2+11x.
(56) x^2+2x.
(57) x^2-x.
(58) $x^2+\frac{4x}{5}$.
(59) $x^2-\frac{2x}{7}$.
(60) $x^2+\frac{1}{2}x$.
(61) $x^2-\frac{1}{3}x$.
(62) $x^2-\frac{5}{6}x$.
(63) $x^2-\frac{3x}{4}$.
(64) $x^2-\frac{7x}{10}$.

Extract the square root of each of the following quantities:

(65) $a^6-4a^5-2a^4+12a^3+9a^2$.
(66) $x^4+2x^3-x+\frac{1}{4}$.
(67) $4x^6-12x^5y+29x^4y^2-30x^3y^3+25x^2y^4$.
(68) $9a^2b^4-12a^3b^3+34a^4b^2-20a^5b+25a^6$.

(69) $\frac{9}{4}+6x-17x^2-28x^3+49x^4$.

(70) $\frac{9}{16}-\frac{7}{6}a+\frac{49}{81}a^2$.

(71) $\frac{9}{4}x^8-2x^4y^3+\frac{4}{9}y^6$.

(72) $\frac{4x^2-4x+1}{9x^2+6x+1}$.

(73) $\frac{9-12x+4x^2}{25+30x+9x^2}$.

(74) $3+\frac{a^4+b^4+2ab^3+2a^3b}{a^2b^2}$.

Extract the cube root of

(75) $8-36a+54a^2-27a^3$.

(76) $a^6+3a^5-5a^3+3a-1$.

(77) $125x^3-300x^2y+240xy^2-64y^3$.

EXERCISES. K*.

Simplify the following expressions :

(1) $2a\sqrt{b}+3a\sqrt{b}+a\sqrt{b}$.

(2) $7\sqrt[3]{a}+2\sqrt[3]{a}-8\sqrt[3]{a}$.

(3) $3\sqrt{2}+10\sqrt{2}-5\sqrt{2}$.

(4) $\frac{1}{2}\sqrt{5}+2\sqrt{5}-\frac{3}{2}\sqrt{5}$.

(5) $\sqrt{24}+\sqrt{54}-\sqrt{96}$.

(6) $2\sqrt{18}-3\sqrt{8}+2\sqrt{50}$.

(7) $\frac{3}{2}\sqrt{\frac{5}{9}}+\sqrt{80}-\frac{1}{4}\sqrt{20}$.

(8) $8\sqrt{\frac{5}{16}}+10\sqrt{\frac{20}{25}}-2\sqrt{1\frac{1}{4}}$.

(9) $\sqrt[3]{81}+\sqrt[3]{24}-\sqrt[3]{192}$.

(10) $3\sqrt[3]{40}+2\sqrt[3]{625}-4\sqrt[3]{320}$.

(11) $\sqrt{a^2c}-\sqrt{b^2c}+\sqrt{c^3}$.

(12) $2b\sqrt{27b}-\sqrt{3b^3}-\sqrt{12b^3}$.

(13) $\sqrt{4a^2b}+\sqrt{25a^2b}-\sqrt{81a^2b}$.

(14) $\sqrt{45x^3}+\sqrt{80x^3}+\sqrt{5xy^2}$.

(15) $\sqrt[3]{24a^3b}+\sqrt[3]{81b^4}$.

(16) $\sqrt{8x^2-16xy+8y^2}+\sqrt{8y^2}$.

(17) $\sqrt{a^2b-6ab+9b}+\sqrt{9b}$.

(18) $\sqrt{(x^2-y^2)(x+y)}-\sqrt{xy^2-y^3}$.

(19) $\sqrt{\frac{4a^3-8a^2+4a}{3x^3}}+\sqrt{\frac{4a}{3x^3}}$.

(20) $\sqrt[4]{32a}+\sqrt[4]{162a}-\sqrt[4]{512a}$.

EXERCISES. L.

Simplify the following expressions :

(1) $\sqrt{2}\times\sqrt{3}\times\sqrt{4}\times\sqrt{5}$.

(2) $2\sqrt{2}\times3\sqrt{3}\times5\sqrt{5}$.

(3) $a\sqrt{a}\times b\sqrt{b}\times c\sqrt{c}$.

(4) $\sqrt[3]{4}\times\sqrt[3]{2}\times2\sqrt[3]{8}$.

(5) $3\sqrt[3]{2a}\times5\sqrt[3]{3ab}$.

(6) $\frac{1}{2}\sqrt{xy}\times\frac{3}{4}x\sqrt{\frac{y}{x}}\times\frac{2}{3}x\sqrt{xy^2}$.

(7) $\sqrt[3]{2a}\times\sqrt[3]{a^2b}\times\sqrt[3]{4b^2}$.

(8) $3\sqrt{a}\times2\sqrt[3]{a}\times4\sqrt[4]{a}$.

(9) $5\sqrt[3]{x^2}\times2\sqrt{x^3}\times\sqrt[6]{x}$.

(10) $2\sqrt{a^2b^3}\times\sqrt{a^4c}$.

(11) $\sqrt[3]{2x-1}\times\sqrt[3]{2x+1}$.

(12) $\sqrt{a^2+ab}\times\sqrt{a^2-ab}$.

(13) $(2\sqrt{3}+3\sqrt{2})\times(\sqrt{3}-\sqrt{2})$.

(14) $(3\sqrt{12}-\frac{1}{2}\sqrt{24})\times(3\sqrt{12}+\sqrt{6})$.

(15) $(\sqrt{a+b}+\sqrt{b})\times(\sqrt{a+b}-\sqrt{b})$.

(16) $(\sqrt{a+x}+\sqrt{a-x})\times(\sqrt{a+x}-\sqrt{a-x})$.

(17) $\sqrt{8+\sqrt{39}}\times\sqrt{8-\sqrt{39}}$.

(18) $\sqrt{a^3-\sqrt{a^6-x^6}}\times\sqrt{a^3+\sqrt{a^6-x^6}}$.

(19) $(a+\sqrt{x})\times(b+\sqrt{y})$.

(20) $(\sqrt{a+x}-\sqrt{a-x})\times\sqrt{a+x}$.

EXERCISES. M.

Simplify each of the following expressions :

(1) $\sqrt{2axy}\div\sqrt{bxy}$.

(2) $6\sqrt{a^3b}\div2\sqrt{ab^3}$.

(3) $12ab\sqrt{12b^3c^3}\div3a\sqrt{bc}$.

(4) $14\sqrt[3]{9a}\div2\sqrt[3]{4a}$.

(5) $\sqrt[3]{4x^2}\div\sqrt{2x}$.

(6) $\sqrt[4]{a^3}\div\sqrt{a}$.

(7) $a\div\sqrt[5]{a^3}$.

(8) $\sqrt[6]{\frac{a}{2x}}\div\sqrt[4]{\frac{3a}{x}}$.

(9) $\sqrt[8]{\frac{ax}{2by^2}}\div\sqrt[6]{\frac{2a^2x^2}{3by^3}}$.

(10) $3x\sqrt[3]{a}\div\frac{3}{4}\sqrt[3]{ax}$.

(11) $\sqrt{b\sqrt{a^3}+a\sqrt{b^3}}\div\sqrt{ab}$.

(12) $a\sqrt{\frac{mn+mx}{n}}\div m\sqrt{\frac{ap-ax}{p}}$.

EXERCISES. N.

Simplify the following expressions :

(1) $(\sqrt[3]{x^2y})^2$.

(2) $(\sqrt{2a^5bc^2})^3$.

(3) $(\sqrt{x}\sqrt[3]{ab^2})^4$.

(4) $(\sqrt[8]{b^2c^4x})^4$.

(5) $\left(\sqrt[6]{\frac{1}{a^2}}\right)^5$.

(6) $(\sqrt[7]{-128x^8y})^2$.

(7) $\sqrt[3]{\sqrt[2]{a^6}}$.

(8) $\sqrt{4\sqrt[3]{81a^2}}$.

(9) $\sqrt[3]{2\sqrt{32}}$.

(10) $\sqrt[6]{(ax+b)^{12}}$.

(11) $\sqrt[10]{(4a^2-12a+9)^5}$.

(12) $\sqrt[n]{a^{3n}b^{2n}c^n}$.

(13) $(\sqrt[3]{2}\times\sqrt{5})^6$.

(14) $\left(\sqrt[4]{\frac{a}{b}}\cdot\sqrt{\frac{b}{a}}\right)^8$.

(15) $\left(\sqrt[m]{\frac{a}{b}}\cdot\sqrt[n]{\frac{b}{c}}\right)^{mn}$.

EXERCISES. O.

Find expressions equivalent to each of the following *with rationa denominators :*

(1) $\frac{a}{2\sqrt{b}}$.

(2) $\frac{3x}{4\sqrt{2y}}$.

(3) $\frac{2a}{\sqrt[3]{7}}$.

(4) $\frac{2ab}{\sqrt[4]{2}}$.

(5) $\frac{a}{\sqrt{2}-1}$.

(6) $\frac{2}{\sqrt{3}+\sqrt{2}}$.

(7) $\dfrac{9}{\sqrt{15}+\sqrt{6}}$.

(8) $\dfrac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$.

(9) $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.

(10) $\dfrac{1}{\sqrt{2}-\sqrt{3}}$.

(11) $\dfrac{1}{2\sqrt{a}+3\sqrt{x}}$.

(12) $\dfrac{1}{\sqrt{mx}+\sqrt{ny}}$.

(13) $\dfrac{3-\sqrt{2}}{\sqrt{3}-\sqrt[4]{2}}$.

(14) $\dfrac{1}{\sqrt[3]{3}+\sqrt[3]{4}}$.

(15) $\dfrac{1}{3\sqrt[3]{5}-2}$.

(16) $\dfrac{1}{\sqrt{x+1}+\sqrt{x-1}}$.

(17) $\dfrac{1}{\sqrt[3]{a}+\sqrt[4]{x}}$.

(18) $\dfrac{1}{a-\sqrt{a^2-x^2}}$.

EXERCISES. P.

Extract the square root of each of the following quantities :

(1) $7+2\sqrt{10}$.

(2) $11+6\sqrt{2}$.

(3) $30-10\sqrt{5}$.

(4) $37+20\sqrt{3}$.

(5) $4\frac{1}{3}-\frac{4}{3}\sqrt{3}$.

(6) $28\pm5\sqrt{12}$.

(7) $\frac{3}{4}-\sqrt{\frac{1}{2}}$.

(8) $31-\sqrt{600}$.

(9) $\frac{5}{6}-\sqrt{\frac{2}{3}}$.

(10) $1\frac{5}{12}+\sqrt{2}$.

(11) $x-2\sqrt{x-1}$.

(12) $4x+2\sqrt{4x^2-1}$.

(13) $2a-2\sqrt{2ax-x^2}$.

(14) $a+b-c-2\sqrt{ab-ac}$.

(15) $4x-y-2\sqrt{4x^2-2xy}$.

(16) $a+\sqrt{a^2-4x^2}$.

(17) $ax-2a\sqrt{ax-a^2}$.

(18) $1+\sqrt{1-x^2}$.

(19) $\frac{1}{2}(a+b)+\sqrt{(a+2)(b-2)}$.

(20) $2+2(1-x)\sqrt{1+2x-x^2}$.

EXERCISES. Q.

Find the value of x in each of the following equations :

(1) $6x-10 = 5x-4$.

(2) $13x+1 = 9x+5$.

(3) $3x+30 = 2x+36$.

(4) $4x-2x = 24-x$.

(5) $7x-11+5 = 8x-9$.

(6) $15-2x+6 = 3x+1$.

(7) $3x-6 = 12-4x-4$.

(8) $12-8x = 15-3x-8$.

(9) $121 = 14x+1-3x+10$.

(10) $500 = 30x+12+32x-8$.

(11) $7x-2x+5 = 13x-4x-15$.

(12) $12x-6x+4x = 3x+84$.

(13) $2x+\frac{1}{2} = 3x-\frac{1}{2}$.

(14) $15x-3\frac{1}{2} = 3\frac{1}{2}+x$.

(15) $x+\frac{x}{2}=6.$

(16) $2x-\frac{x}{2}=18.$

(17) $3x+\frac{x}{3}=4x-6.$

(18) $\frac{4x}{3}+\frac{2}{3}=x+3.$

(19) $\frac{3x}{5}-\frac{x}{5}=x-6.$

(20) $\frac{x}{3}+\frac{x}{6}=15.$

(21) $\frac{x}{5}-\frac{x}{10}=\frac{1}{2}.$

(22) $x-\frac{x}{2}+\frac{x}{3}-\frac{2}{3}=3\tfrac{1}{2}.$

(23) $\frac{2x}{7}+\frac{x}{6}-\frac{1}{6}=x-4.$

(24) $\frac{3x}{7}-1=\frac{x}{5}+\frac{3}{5}.$

(25) $\frac{x}{2}-\frac{x}{3}-\frac{x}{4}+\frac{4}{3}=\frac{3}{4}.$

(26) $\frac{3x}{2}-\frac{2x}{3}+\frac{1}{2}=\frac{x}{6}+9\tfrac{5}{6}.$

(27) $\frac{x}{5}+\frac{x}{4}+\frac{x}{3}-\frac{x}{2}=17.$

(28) $x-\frac{x}{6}-\frac{x}{12}-\frac{x}{7}=\frac{x}{2}+9.$

(29) $\frac{3x}{14}-\frac{2x}{21}+\frac{1}{3}=\frac{x}{4}-4\tfrac{1}{4}.$

(30) $\frac{3x}{7}-\frac{x}{4}-\frac{x}{6}=\frac{5}{21}-\frac{x}{28}.$

(31) $2x-\frac{2x}{5}-2\tfrac{1}{5}-\frac{4x}{11}=\frac{8x}{7}-1\tfrac{6}{11}.$

(32) $\frac{x}{8}+\frac{2x}{5}=\frac{7x}{15}-\frac{x}{60}+\frac{3}{20}.$

(33) $\frac{7x}{8}-\frac{3x}{7}+1\tfrac{1}{8}=\frac{9x}{4}+\frac{9x}{14}-20\tfrac{25}{28}.$

(34) $\frac{3x}{16}+\frac{7x}{15}-\frac{7x}{20}=2\tfrac{19}{60}-\frac{3}{16}.$

(35) $\frac{14x}{3}-\frac{8x}{5}=10\tfrac{1}{3}+\frac{2x}{1\frac{1}{2}}-3\tfrac{2}{5}.$

(36) $\frac{x}{4}-4\tfrac{1}{2}+\frac{x}{5\frac{1}{2}}+\frac{x}{2}=\frac{16\frac{1}{4}}{5\frac{1}{2}}.$

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(37) $6x+2(11-x)=3(19-x).$

(38) $3(x+1)+2(x+2)=32.$

(39) $3x-2(5x+4)=2(4x-9).$

(40) $5(2x-2)-3(2x+1)=27.$

(41) $6(3-2x)=24-4(4x-5).$

(42) $45-4(x-2)=5(x+2).$

(43) $7x=8-\frac{1-9x}{2}.$

(44) $\frac{2x}{7}+4=x-\frac{x-1}{6}.$

(45) $\frac{3x+1}{2}-\frac{x-1}{6}=\frac{2x}{3}+10.$

(46) $\frac{1}{4}(x+6)-\frac{1}{12}(16-3x)=4\tfrac{1}{6}.$

(47) $\frac{1}{16}(3x+3)+\frac{1}{15}(7x-4)-\frac{1}{20}(7x+1)=2.$

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(48) $\frac{6}{x}-\frac{4}{x}+1=\frac{5}{x}+\frac{1}{4}.$

(49) $\frac{2}{3x}+\frac{3}{2x}=13.$

(50) $\frac{4}{5x}+\frac{5}{4x}=41.$

(51) $\frac{a}{bx}+\frac{b}{ax}=a^2+b^2.$

(52) $\frac{6x-4}{21}+\frac{x-2}{5x-6}=\frac{2x}{7}.$

(53) $\frac{9x-16}{36}=\frac{12-4x}{4-5x}+\frac{x-4}{4}.$

(54) $$\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}.$$

(55) $$\frac{x-7}{x+7}+\frac{1}{2(x+7)}=\frac{2x-15}{2x-6}.$$

(56) $$\frac{3}{x}-\frac{2}{x+1}=\frac{5}{4(x+1)}.$$

(57) $$\frac{17}{6x+17}-\frac{10}{3x-10}=\frac{1}{1-2x}.$$

EXERCISES. R.

Find the value of x in each of the following equations:

(1) $\sqrt{12+x}=2+\sqrt{x}.$

(2) $\sqrt{x}+\sqrt{4+x}=\dfrac{2}{\sqrt{x}}.$

(3) $2+\sqrt{3x}=\sqrt{5x+4}.$

(4) $\sqrt{x}+\sqrt{9+x}=\dfrac{45}{\sqrt{9+x}}.$

(5) $\sqrt{8+x}-\sqrt{x}=2\sqrt{1+x}.$

(6) $\sqrt{2x-45}=3\sqrt{15}-\sqrt{2x}.$

(7) $\sqrt{2x-27a}=9\sqrt{a}-\sqrt{2x}.$

(8) $\sqrt{x+4a+4b}=2\sqrt{b+x}-\sqrt{x}.$

(9) $\dfrac{28+\sqrt{x}}{28-\sqrt{x}}=\dfrac{9+3\sqrt{x}}{9+2\sqrt{x}}.$

(10) $\dfrac{\sqrt{nx+1}+\sqrt{nx}}{\sqrt{nx+1}-\sqrt{nx}}=\dfrac{\sqrt{n}+1}{\sqrt{n}-1}.$

(11) $\dfrac{\sqrt{2x}+\sqrt{3-2x}}{\sqrt{2x}-\sqrt{3-2x}}=\dfrac{3}{2}.$

(12) $\dfrac{13-2\sqrt{x-5}}{13+2\sqrt{x-5}}=\dfrac{3}{23}.$

(13) $\sqrt{4+x}-\sqrt{1+x}=2\sqrt{x-2}.$

(14) $\sqrt{x^2+2x}+\sqrt{x^2-2x}=2\sqrt{5}.$

(15) $\sqrt{4+x}=\sqrt[4]{x^2+20x+9}.$

(16) $\sqrt[3]{2+\frac{1}{2}x}=\frac{1}{2}\sqrt[6]{16x^2+8x+320}.$

EXERCISES. S.

Find the values of x and y in the following equations:

(1) $$\left.\begin{aligned} x+y&=17,\\ 2x-y&=19.\end{aligned}\right\}$$

(2) $$\left.\begin{aligned} 4x-7y&=26,\\ 4x+5y&=50.\end{aligned}\right\}$$

(3) $$\left.\begin{aligned} 5x+y&=32,\\ 3x-2y&=14.\end{aligned}\right\}$$

(4) $$\left.\begin{aligned} 3x-7y&=2,\\ 11y-3x&=2.\end{aligned}\right\}$$

(5) $$\left.\begin{aligned} 3x+4y&=11,\\ 15x-2y&=11.\end{aligned}\right\}$$

(6) $$\left.\begin{aligned} 13x-6y&=31,\\ 11x-3y&=47.\end{aligned}\right\}$$

(7) $$\left.\begin{aligned} 7x-6y&=10,\\ 6x-7y&=3.\end{aligned}\right\}$$

(8) $$\left.\begin{aligned} 35x+2y&=76,\\ 12y-x&=34.\end{aligned}\right\}$$

(9) $$\left.\begin{aligned} 5x+2y&=16,\\ 9y+2x&=31.\end{aligned}\right\}$$

(10) $$\left.\begin{aligned} 11x-7y&=72,\\ 7x-11y&=0.\end{aligned}\right\}$$

(11) $$\left.\begin{aligned} 36x-45y&=0,\\ 2x+5y&=1\tfrac{1}{2}.\end{aligned}\right\}$$

(12) $$\left.\begin{aligned} 9x+5y&=65,\\ 7x-2\tfrac{1}{2}\cdot y&=25.\end{aligned}\right\}$$

(13) $$\left.\begin{aligned} 15x-y&=143,\\ 35y+x&=255.\end{aligned}\right\}$$

(14) $$\left.\begin{aligned} 11x-13y&=16,\\ 20x-19y&=43.\end{aligned}\right\}$$

(15) $$\left.\begin{aligned}45x+8y&=350,\\21y-13x&=132.\end{aligned}\right\}$$

(16) $$\left.\begin{aligned}101x-24y&=63,\\103x-28y&=29.\end{aligned}\right\}$$

(17) $$\left.\begin{aligned}64x+90y&=237,\\63x-218y&=80.\end{aligned}\right\}$$

(18) $$\left.\begin{aligned}3\tfrac{1}{2}.x-4\tfrac{1}{2}.y&=12,\\7x+9y&=60.\end{aligned}\right\}$$

(19) $$\left.\begin{aligned}2\tfrac{1}{4}.x+3y&=45,\\4\tfrac{1}{2}.x+10y&=216.\end{aligned}\right\}$$

(20) $$\left.\begin{aligned}4\tfrac{1}{4}.x-3\tfrac{1}{2}.y&=6,\\\tfrac{1}{2}x-\tfrac{1}{4}y&=2.\end{aligned}\right\}$$

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(21) $$\left.\begin{aligned}3(4x-5y)&=2(x+y)+3,\\4(3x-2y)&=5(x-y)+11.\end{aligned}\right\}$$

(22) $$\left.\begin{aligned}3x+\frac{y}{3}&=36,\\\frac{6y-2x}{4}&=8.\end{aligned}\right\}$$

(23) $$\left.\begin{aligned}\frac{3x-2y}{2}-3&=\frac{2x-y}{4},\\\frac{5x-4y}{2}-3&=\frac{4x-3y}{3}.\end{aligned}\right\}$$

(24) $$\left.\begin{aligned}\frac{2x-3}{2}+y&=7,\\5x-13y&=33\tfrac{1}{2}.\end{aligned}\right\}$$

(25) $$\left.\begin{aligned}\frac{x+3}{y}&=\frac{1}{3},\\\frac{x}{y-1}&=\frac{1}{5}.\end{aligned}\right\}$$

(26) $$\left.\begin{aligned}\frac{x}{18}+\frac{y}{9}&=32,\\\frac{x}{9}+\frac{y}{18}&=28.\end{aligned}\right\}$$

(27) $$\left.\begin{aligned}\frac{x}{6}+\frac{y}{11}&=26,\\\frac{x}{2}-\frac{y}{7}&=46.\end{aligned}\right\}$$

(28) $$\left.\begin{aligned}\tfrac{1}{2}(x+y)&=\tfrac{1}{3}(2x+4),\\\tfrac{1}{3}(x-y)&=\tfrac{1}{2}(x-24).\end{aligned}\right\}$$

(29) $$\left.\begin{aligned}\frac{1}{2}(3x-5y)+3&=\frac{1}{5}(2x+y),\\8-\frac{1}{4}(x-2y)&=\frac{1}{2}x+\frac{1}{3}y.\end{aligned}\right\}$$

(30) $$\left.\begin{aligned}\tfrac{1}{3}(3x-7y)&=\tfrac{1}{6}(2x+y+1),\\8-\tfrac{1}{6}(x-y)&=6.\end{aligned}\right\}$$

(31) $$\left.\begin{aligned}\frac{x-2}{5}-\frac{10-x}{3}&=\frac{y-10}{4},\\\frac{2y+4}{3}-\frac{2x+y}{8}&=\frac{x+13}{4}.\end{aligned}\right\}$$

(32) $$\left.\begin{aligned}\frac{2x-y}{3}+6&=\frac{2y-x}{2}+4\tfrac{1}{2},\\\frac{3x+y}{5}+1&=\frac{x+3y+13}{10}.\end{aligned}\right\}$$

(33) $$\left.\begin{aligned}\frac{x}{3}+\frac{y}{4}&=4,\\\frac{7y}{2}-11&=\frac{3x}{2}+y.\end{aligned}\right\}$$

(34) $$\left.\begin{aligned}\frac{1}{4}x+\frac{1}{7}y&=20,\\7y+4x&=584.\end{aligned}\right\}$$

(35) $$\left.\begin{aligned}\frac{1}{2}x-\frac{1}{3}y&=1,\\6(x+y)-3(x-y)&=39.\end{aligned}\right\}$$

(36) $$\left.\begin{aligned}\frac{2x}{3}-3y&=1\tfrac{2}{3},\\\frac{4x}{15}-y&=1\tfrac{2}{3}.\end{aligned}\right\}$$

(37) $$\left.\begin{aligned}\frac{3}{x}+\frac{4}{y}&=2,\\\frac{4}{x}+\frac{3}{y}&=2\tfrac{1}{12}.\end{aligned}\right\}$$

(38) $$\left.\begin{aligned}\frac{4y-6}{x+y}&=2,\\\frac{8x-5}{y-x}&=9.\end{aligned}\right\}$$

(39) $$\left.\begin{aligned} \frac{5}{3}x-y &= \frac{1}{2}, \\ 8x+\frac{3}{4}y &= 4\tfrac{1}{4}. \end{aligned}\right\}$$

(40) $$\left.\begin{aligned} \frac{3}{4}y+\frac{1}{2}x &= 17, \\ \frac{1}{2}y-\frac{1}{3}x &= 4\tfrac{2}{3}. \end{aligned}\right\}$$

EXERCISES. T.

Find the values of x, y, z, in the following equations:

(1) $$\left.\begin{aligned} x+y &= 6, \\ x+z &= 8, \\ y+z &= 10. \end{aligned}\right\}$$

(2) $$\left.\begin{aligned} 2x+y &= 9, \\ 2y+x &= 15, \\ 2z+y &= 27. \end{aligned}\right\}$$

(3) $$\left.\begin{aligned} 3x+5y &= 34, \\ 4x+2z &= 26, \\ 5y+z &= 32. \end{aligned}\right\}$$

(4) $$\left.\begin{aligned} 9x+10y &= 180, \\ 10y+11z &= 178, \\ 11z+12y &= 196. \end{aligned}\right\}$$

(5) $$\left.\begin{aligned} y+\frac{x}{2} &= 41, \\ x+\frac{z}{4} &= 20\tfrac{1}{2}, \\ y+\frac{z}{5} &= 34. \end{aligned}\right\}$$

(6) $$\left.\begin{aligned} x+y+z &= 15, \\ x+z-y &= 5, \\ x+y-z &= -3. \end{aligned}\right\}$$

(7) $$\left.\begin{aligned} x+y+z &= 36, \\ 5x-2y &= 10, \\ 9y-7x &= 110. \end{aligned}\right\}$$

(8) $$\left.\begin{aligned} \frac{1}{2}x+y+\frac{1}{2}z &= 162, \\ \frac{1}{4}x+\frac{1}{8}y &= 26, \\ 5y &= 4z. \end{aligned}\right\}$$

(9) $$\left.\begin{aligned} x+\frac{1}{2}y+\frac{1}{3}z &= 10, \\ \frac{1}{2}(x+z)+y &= 9, \\ \frac{1}{4}(x-z)+7 &= 2y. \end{aligned}\right\}$$

(10) $$\left.\begin{aligned} \frac{1}{3}(x+y)+2z &= 21, \\ 3x-\frac{1}{2}(y+z) &= 65, \\ x+\frac{1}{2}(x+y-z) &= 38. \end{aligned}\right\}$$

(11) $$\left.\begin{aligned} \frac{2x}{3} &= \frac{y}{5}+5, \\ \frac{1}{3}y+\frac{1}{8}z &= 8, \\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z &= 17. \end{aligned}\right\}$$

(12) $$\left.\begin{aligned} 3x+2y+\frac{1}{z} &= 14, \\ 3(x+y)+\frac{3}{z} &= 18, \\ 2x &= 3y. \end{aligned}\right\}$$

EXERCISES. U.

(1) **What number is that which added to its half makes 24?**

(2) **What number is that which increased by two-thirds of itself becomes 20?**

(3) What number is that of which the half exceeds the third part by 3?

(4) What number is that of which the fourth part exceeds the fifth part by 3?

(5) There is a certain number which, upon being diminished by 6, and the remainder multiplied by 6, produces the same result as if it were diminished by 4, and the remainder multiplied by 4. What is the number?

(6) Divide 40 into two such parts, that one-tenth of the smaller part taken from one-fifth of the greater will leave 5 for a remainder.

(7) Divide 25 into two such parts, that one shall be three-fourths of the other.

(8) Find two numbers which produce the same result, 7, whether the one be subtracted from the other, or the latter be divided by the former.

(9) Divide £1 among 4 children so that the oldest shall have 1*s.* more than the second, the second 1*s.* more than the third, and the third 1*s.* more than the youngest.

(10) Divide a line 33 feet long into 4 parts, the second of which is $1\frac{1}{2}$ feet greater than the first, the third $2\frac{1}{2}$ feet greater than the second, and the fourth $3\frac{1}{2}$ feet greater than the third.

(11) A banker was asked to pay £10 in sovereigns, and half-crowns, and so that the number of the latter should be exactly twice that of the former. How must he do it?

(12) Thirteen shillings is the sum of exactly *the same number* of shillings, sixpences, pence, and half-pence. What is the number?

(13) I went to the bank with a cheque for 6 guineas, and asked to have for it exactly *the same number* of sovereigns, half-sovereigns, shillings, and sixpences. The banker was puzzled: what is the number?

(14) I have exactly 5 times as many shillings as half-crowns; and altogether my money amounts to £3. How many have I of each coin?

(15) A father is 4 times as old as his son; but 3 years ago he was 7 times as old as his son. What is the age of each?

(16) The ages of two brothers, who differ only by a single year, when added together amount to the age of their father; and if the father's age be increased by one-fourth of that of the elder brother, it will amount to fourscore years. What is the age of each?

(17) The ages of a man and his wife together amount to 80 years, and 20 years ago the woman was exactly two-thirds the age of the man. What is the age of each?

(18) There is a certain fraction whose denominator is greater than its numerator by 1; and if 1 be taken from the numerator and added to the denominator, the fraction becomes equal to $\frac{1}{2}$. What is the fraction?

(19) A certain fraction has its numerator less than its denominator by 2, and if 1 be taken from the numerator, and the numerator be added to the denominator to form a new denominator, the resulting fraction is equal to $\frac{1}{4}$. What is the fraction?

(20) A boy being asked to divide one half of a certain number by 4, and the other half by 6, and to add together the quotients, attempted to obtain the required result *at one step* by dividing the whole number by 5; but his answer was too small by 2. What was the number?

(21) Find the time between 12 and 1 o'clock when the hour and minute hands of a clock point exactly in opposite directions.

(22) A person, being asked what o'clock it was, answered that it was between 1 and 2, and that the hour and minute hands were together. Required the time of day.

(23) A servant is dispatched on an errand to a town 8 miles off, and walks at the rate of 4 miles an hour: ten minutes afterwards another is sent to fetch him back, walking $4\frac{1}{2}$ miles per hour. How far from the town will the latter overtake the former?

(24) A student has just an hour and a half for exercise. He starts off on a coach which travels 10 miles an hour, and after a time he dismounts, and walks home at the rate of 4 miles an hour. What is the greatest distance he can travel by the coach, so as to keep within his time?

(25) A cistern which holds 820 gallons is filled in twenty minutes by 3 pipes, one of which conveys 10 gallons more, and another 5 gallons less, per minute, than the third. How much flows through each pipe per minute?

(26) A man and a boy engaged to draw a field of turnips for 21*s*., but when two-fifths of the work was done, the boy ran away, and the man then finished it alone. The consequence was that the work occupied $1\frac{1}{4}$ days more than it should have done. Now the boy could do only half a man's work, and is paid in proportion. What did each receive per day?

(27) The date of the accession of GEO. III. is represented by $1800-2x$, that of GEO. IV. by $1800+\frac{1}{2}.2x$, that of WILL. IV. by $1800+\frac{1}{2}.3x$; and if GEO. III.rd's reign be increased by $2x$, it will amount to 100 years. What are the actual dates?

(28) Her Majesty QUEEN VICTORIA was born May 24, A.D. x, and Prince Albert was born Aug. 26, A.D. $\overline{x+1}$. Now their united ages on the 26th of Aug. 1848 amounted to *three* times the age of Prince Albert on the birth-day immediately preceding his marriage, which took place Feb. 10, 1840. What is the year of our Lord in which each was born?

(29) The interest of the National Debt being reckoned at 30 millions sterling per annum, and 3 per cent. the average rate of interest paid, what reduction in the rate of interest would give the same relief to taxation as the paying off 200 millions of debt, and allowing the interest to be paid on the remainder to continue the same?

(30) Says Charles to William, If you give me 10 of your marbles, I shall then have just *twice* as many as you: but says William to Charles, If you give me 10 of yours, I shall then have *three times* as many as you. How many had each?

(31) A man, who has two purses containing money, receives £10 to add to them, and finds that if he puts £5 into each, one will then contain exactly twice as much as the other, but if he puts the whole £10 into that which already contains the most, its contents will be just *three times* the value of the other. How much was there in each purse to begin with?

(32) A party consists of men and women, and there are 6 men to every 5 women; but if there had been 2 men less and 2 women more, the number of each would have been the same. How many are there?

(33) A clergyman, who had a dole of £5. 10*s.* to distribute amongst a certain number of old men and widows, found that, if he gave them 3*s.* each, he would be 1*s.* out of pocket; but, if he gave each of the men 2*s.* 2*d.* and each of the widows 3*s.* 6*d.*, he would have 6*d.* to spare. How many were there of each?

(34) There is a certain fraction which becomes equal to $\frac{1}{2}$, when both numerator and denominator are diminished by 1; but, if 2 be taken from the numerator and added to the denominator, it becomes equal to $\frac{1}{3}$. What is the fraction?

(35) What is the fraction in which twice the sum of the numerator and denominator is equal to three times their difference?

(36) Find two numbers such that one shall be as much above 10, as the other is below it, and one-tenth of their sum equal to one-fourth of their difference.

(37) Find two numbers such that the half of one added to a third of the other is 12, but a third of the former added to half the other is 13.

(38) A person has two casks with a certain quantity of wine in each. He draws out of the first into the second as much as there was in the second to begin with: then he draws out of the second into the first as much as was left in the first: and then again out of the first into the second as much as was left in the second. There are then exactly 8 gallons in each cask. How much was there in each at first?

(39) In the course of last century the change took place, called '*the change of Style*,' which consisted in beginning the year with Jan. 1, instead of March 25, as heretofore, and for that year only, calling the day after Sept. 2, the 14th, instead of the 3rd. Now the year of our Lord in which this happened, possesses the following properties:—The first digit being 1 for thousands, the second is the sum of the third and fourth, the third is the *third* part of the sum of all four, and the fourth is the *fourth* part of the sum of the first two. Determine the year.

(40) Iron, worth £10 in its raw state, is manufactured half into knife-blades, and half into razors, and is then worth £444. But if *one-third* of it had been made into razors and the rest into knife-blades, the produce would have been worth £30 more than in the former case. How much is the value of the original material increased by these respective manufactures?

EXERCISES. V.

Find the values of x in each of the following equations:

(1) $3x^2-5 = \frac{8x^2}{3}+7.$

(2) $(x+1)^2 = 2x+17.$

(3) $(x+2)^2 = 4x+5.$

(4) $(2x-5)^2 = x^2-20x+73.$

(5) $x^2-\frac{3x^2-2}{5} = 3-\frac{2x^2-5}{3}.$

(6) $\frac{2x^2+10}{15} = 7-\frac{50+x^2}{25}.$

(7) $\frac{x^2}{5}-\frac{x^2}{15}+\frac{x^2}{25} = 4\frac{1}{3}.$

(8) $13\frac{3}{4}-\frac{x^2}{2} = 2x^2-8\frac{3}{4}.$

(9) $\frac{3}{1+x}+\frac{3}{1-x} = 8.$

(10) $\frac{1}{x^2}-\frac{2}{3x^2+1} = \frac{5}{4(3x^2+1)}.$

(11) $\frac{14x^2+16}{21}-\frac{2x^2+8}{8x^2-11} = \frac{2x^2}{3}.$

(12) $\left(x-\frac{3}{4}\right)^2 = \frac{1}{4}.$

(13) $x^2 = 3x+10.$

(14) $x^2 = 5x-4.$

(15) $x^2-9x = x-16.$

(16) $x^2-14x = 120.$

(17) $12x-20 = x^2.$

(18) $4x-x^2 = 4.$

(19) $7x-x^2 = 6.$

(20) $x = x^2-30.$

(21) $x^2+\frac{x}{2} = 3.$

(22) $x^2-\frac{3x}{2} = 27.$

(23) $x^2+\frac{9x}{2} = 63.$

(24) $9x-5x^2 = 2\frac{1}{4}.$

(25) $7x+3x^2 = 6.$

(26) $\frac{x^2}{3}+\frac{3x}{2} = 21.$

(27) $x^2-\frac{x}{3} = 34.$

(28) $11x^2-9x = 11\frac{1}{4}.$

(29) $3x^2-5x+2 = 0.$

(30) $\frac{1}{2}x^2-\frac{1}{3}x-2\frac{2}{3} = 0.$

(31) $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8} = 8.$

(32) $\frac{3}{4}x^2-\frac{2}{3}x = 1\frac{2}{3}.$

(33) $5(x^2+1)-3(x-1) = 22.$

(34) $x^2-4 = 16-(x-2)^2.$

(35) $3(x-2)^2-3 = 8(x+2).$

(36) $\frac{3}{4}(x^2-3) = \frac{1}{8}(x-3).$

(37) $3(2-x)+2(3-x) = 2(4+3x^2).$

(38) $x^2+(x+1)^2 = \frac{13}{6}x(x+1).$

(39) $\frac{x}{x+60} = \frac{7}{3x-5}.$

(40) $3\left(x-\frac{1}{4}\right)-\frac{x-1}{x+2} = 5.$

(41) $\frac{48}{x+3}+5=\frac{165}{x+10}$.

(42) $4x+4\sqrt{x+2}=7$.

(43) $\sqrt{x+3}+\sqrt{x+6}=3\sqrt{x}$.

(44) $\frac{4x}{5-x}-\frac{4(5-x)}{x}=15$.

(45) $\frac{3x-7}{x}=3\frac{1}{2}-\frac{4(x-2\frac{1}{2})}{x+5}$.

(46) $\frac{4x-3}{3x-7}-\frac{2x-3}{x-1}=3$.

(47) $\frac{7x+1}{6\frac{1}{2}-3x}=\frac{80}{3}\cdot\frac{x-\frac{1}{2}}{x-\frac{2}{3}}$.

(48) $\frac{3x-5}{3x+5}+\frac{135}{176}=\frac{3x+5}{3x-5}$.

(49) $\frac{3x+2}{3x-2}+\frac{3x-2}{3x+2}=\frac{15x+11}{3x+2}$.

(50) $\frac{1}{2}(x-1)(x-2)=2\frac{1}{4}(x-2\frac{2}{3})$.

(51) $\frac{2x+3}{10-x}=\frac{2x}{25-3x}-6\frac{1}{2}$.

(52) $\frac{x+8}{x+12}+\frac{5}{x+4}=\frac{3x+14}{3x+8}$.

EXERCISES. W.

Find the values of x in each of the following equations :

(1)* $3x^2+2x=85$.

(2) $5x^2-9x+2\frac{1}{4}=0$.

(3) $7x^2-11x=6$.

(4) $(a-b)x^2-(a+b)x+2b=0$.

(5) $x+6\sqrt{x}=27$.

(6) $x^4-6x^2=27$.

(7) $2\sqrt{x}+\frac{2}{\sqrt{x}}=5$.

(8) $x+4+\sqrt{x+4}=12$. 5.

(9) $x^2=21+\sqrt{x^2-9}$. 5.

(10) $x-\sqrt{x+5}=1$. 4

(11) $2x+\sqrt{2x+3}=3$.

(12) $x^2+6\sqrt{x^2-2x+5}=11+2x$.

(13) $2x^2+3x-5\sqrt{2x^2+3x+9}+3=0$.

(14) $9x-4x^2+\sqrt{4x^2-9x+11}=5$.

(15) $3x(3-x)=11-4\sqrt{x^2-3x+5}$.

$\frac{3\pm\sqrt{5}}{2}$

EXERCISES. X.

12, 13. (1) Find the two consecutive numbers whose product is 156.

4, 5 (2) Find the three consecutive numbers whose sum is equal to the product of the first two.

16, 4. (3) Divide 20 into two such parts, that one is the square of the other.

196, 14 (4) Divide 210 into two such parts, that one is the square of the other.

12, 13. (5) Divide 25 into two such parts, that the *sum* of their squares shall be 313.

20, 10. (6) Divide 30 into two such parts, that the *difference* of their squares shall be 300.

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* This and the three following by the *Hindoo* method.

(7) The product of two numbers is 144, and if each number be increased by 2, their product will then be 200. What are the numbers?

(8) Find the number whose square exceeds the number itself by 156.

(9) Find the fraction which is greater than its square by $\frac{1}{4}$. $\frac{1}{2}$

(10) The sum of £4. 10*s*. is equally divided among a certain number of persons, and each receives as many half-crowns as there are persons altogether. What is the number? 6.

(11) A person bought a lot of pigs for £4. 16*s*. which he sold again at 13*s*. 6*d*. per head, and gained by the whole as much as one pig cost him. What number did he buy? 8 pigs.

(12) A gardener, who had no knowledge of Arithmetic, undertook to plant a certain number of trees at equal distances apart, and in the form of a square. In the first attempt, when he had finished his square, he had 11 trees to spare. He then added one of these to each row, as far as they would go, and found that he wanted 24 trees more to complete his square. How many trees were there? 300.

(13) A printer, reckoning the cost of printing a book at so much per page, made the whole book come to £16. It turned out however that the book contained 5 pages more than he reckoned, and an abatement also was made of 2 shillings per page. He received £13. 10*s*. How many pages did the book contain? $\frac{27}{2x} + \frac{1}{10} = \frac{16}{x-5}$. $x = 45$.

(14) There are 4 consecutive numbers, of which if the first two be taken for the digits of a number, that number is the product of the other two. What are the 4 numbers? 1, 2, 3, 4. Or, 5, 6, 7, 8.

(15) Two trains start at the same time to perform a journey of 156 miles, but one travels a mile an hour faster than the other, and reaches the end of its journey just one hour before the other; at what rate did each train travel? 12, 13, an hour.

(16) A student travelled on a coach 6 miles into the country, and walked back at a rate 5 miles less per hour than that of the coach. He found that he was 50 minutes more in returning than going. What was the speed of the coach? 9 miles an hr.

(17) A person distributed £5 in equal portions among a certain number of poor men; and another person did the same, but by giving each man a shilling less, relieved 5 more. What was the number of recipients in each case? $\frac{5}{x} - \frac{1}{20} = \frac{5}{x+5}$; $x = 20$

(18) A person distributed £36 in equal portions among the poor of a certain place. The next year the same amount was distributed, but the number of recipients was diminished by 6, and consequently each received 1*s*. 8*d*. more than in the year before. What was the number of recipients in each year? 54 at 13/4, 48 at 15/–

(19) Two travellers *A* and *B* start at the same time from two places distant 180 miles to meet each other. *A* travelled 6 miles per day more

than *B*, and *B* travelled as many miles per day as was equal to twice the number of days before they met. How many miles did each travel per day? 18, 12.

(20) Twenty persons contribute to send a donation of £2. 8*s*. to the Society for Promoting Christian Knowledge, one half of the whole being furnished in equal portions by the women, and the other half by the men; but each man gave a shilling more than each woman. How many were there of each sex, and what did each person contribute? 8 men at 3/- = 12 women at 2/-

(21) A person, who can walk forwards four times as fast as he can walk backwards, undertakes to walk a certain distance (one-fourth of it backwards) in a stated time. He finds that, if his speed per hour backwards were one-fifth of a mile less, he must walk forwards 2 miles an hour faster, to gain his object. What is his speed? 1, 4.

EXERCISES. Y.

Find the *integral* values of x in the following '*Inequalities*':

✓(1) $2x-5>31$, and $3x-7<2x+13$.

✓(2) $x+7<15$, and $2x+10>20$.

✓(3) $7x-15>4x+30$, and $\frac{1}{2}x-\frac{1}{3}x<3$.

✓(4) $2\frac{1}{2}.x+\frac{1}{3}x<8$, and $3\frac{1}{3}.x-\frac{1}{2}x>5$.

Solve the following 'Inequalities':

✓(5) $4x-3>\frac{5}{3}x-\frac{5}{6}$. ✓(6) $\frac{8}{7}-\frac{6}{5}x<9-3x$. ✓(7) $\frac{3}{2}x+\frac{2}{3}>\frac{2}{3}x+\frac{1}{2}$.

(8) If $\frac{a}{b}$ be any fraction whatever, shew that $\frac{a}{b}+\frac{b}{a}>2$.

(9) If a and x be both positive, shew that $a^3+x^3>ax^2+a^2x$, unless $a=x$.

✓(10) Which is greater $\frac{\sqrt{2}}{\sqrt[3]{3}}$, or $\frac{\sqrt{3}}{\sqrt[3]{5}}$?

(11) Shew that $2x^3>$ or $<x+1$, according as $x>$ or <1.

(12) If $a>b$, and both of the same sign, shew that

$$a^3-b^3<3a^2(a-b), \text{ and } >3b^2(a-b).$$

✓(13) Which is greater $\sqrt{2}+\sqrt{7}$ or $\sqrt{3}+\sqrt{5}$?

(14) Shew that $\frac{mx+ny}{my+nx}>$ the least and $<$ the greatest of the fractions $\frac{x}{y}$, $\frac{y}{x}$.

EXERCISES. Z.

Find the value of each of the following *Ratios :*

✓(1) $3a : 15a$.
✓(2) $2x : 10x^2$.
✓(3) $ax : bx$.
✓(4) $abc : bc$.
✓(5) $axy : 2x$.
✓(6) $3abx : 2a^2x$.
✓(7) $a^2pc : 3acx$.
✓(8) $3x^3y^2 : 12x^2y^3$.
✓(9) $ac+bc : c^2$.
✓(10) $2ax+x^2 : mx$.
✓(11) $1-x^2 : 1-x$.
✓(12) $a^2-b^2 : a+b$.

Simplify each of the following *Ratios :*

✓(13) $5ax : 4x$.
✓(14) $16xy : 20x^2$.
✓(15) $\frac{1}{2}ax : \frac{3}{4}bx$.
✓(16) $2x^2y : \frac{1}{4}x^3$.
✓(17) $\frac{7axy}{1\times2\times3} : \frac{5ay^2}{2\times3\times4}$.
✓(18) $\frac{n(n-1)}{1\times2}ax^3 : na^2x^2$.

✓(19) Which is the greater 15 : 16, or 16 : 17? $16 : 17 > 15 : 16$

✓(20) Which is the greater $2ax : 3by$, or $3a : 2b$, when $x : y :: 2 : 1$?

✓(21) What is the ratio compounded of $2a : b$, and $bx : a$?

✓(22) Compound the ratios $a+x : x$, and $x^2 : a^2-x^2$.

✓(23) Compound the ratios $a : 1$, $2a : 1$, and $3a : 1$.

✓(24) Divide 27 into two parts in the ratio of 7 : 2.

✓(25) Divide 20 into 3 parts such that the ratio of the first two shall be 2 : 5, and that of the last two 5 : 3.

✓(26) Find two numbers in the ratio of $1\frac{1}{2} : 2\frac{2}{3}$, and such that, when each number is increased by 15, they shall be in the ratio of $1\frac{2}{3} : 2\frac{1}{2}$.

✓(27) The numbers of boys in the three classes of a school were as the numbers 5, 7, 8. At the next inspection the first class was found increased by 4 boys, the 2nd had gained two-sevenths of its former number, the 3rd was doubled—and the whole number of additional scholars was 34. What were the numbers in the classes at the 1st inspection?

EXERCISES. *Za*.

✓(1) Find a 4th proportional to a, $2a$, and $3a$. ✓

✓(2) Find a 4th proportional to $1\frac{1}{2}$, $2\frac{1}{2}$, and $3\frac{1}{2}$. ✓

✓(3) Find a 4th proportional to $a-x$, a^2-x^2, and $a+x$. $(a+x)^2$

✓(4) If $a : b :: c : d$, shew that $5a : 6b :: 5c : 6d$.

✓(5) If $a : b :: c : d$, shew that $\frac{2}{3}a : \frac{4}{5}b :: \frac{2}{3}c : \frac{4}{5}d$.

(6) If $2a : b :: b : 2c$, shew that $a : c :: 4a^2 : b^2$. NB.

(7) Convert the proportion $a : a+x :: a-x : b$ into an equation.

(8) Convert $x : y :: y : 2a-x$ into an equation.

(9) If $a+x : a-x :: 11 : 7$, find the value of $a : x$.

(10) Find two numbers in the ratio $2 : 3$, and the sum of which : their product :: $5 : 12$.

(11) The 1st, 3rd, and 4th, terms of a proportion are ax, $3cx$, and $\frac{6bcy}{a}$, what is the second term? $2by$

(12) There are two numbers in the ratio $3 : 4$, and if each of them b increased by 5, the resulting numbers are in the ratio $4 : 5$. What are th numbers? 15, 20.

(13) If $a : b :: b : c$, and $b : c :: c : d$, shew that $a+b : b+c :: b+ : c+d$. $\frac{a}{b}+1=\frac{b}{c}+1$; $\frac{a+b}{b}=\frac{b+c}{c}$. Multiply both sides by $\frac{b}{b+c}$, Then $\frac{a+b}{b+c}=$ -

(14) If $6x-a : 4x-b :: 3x+b : 2x+a$, find x. $\frac{a^2-b^2}{4a-b}$

W. 4 (15) Shew that $a : b$ is double of the ratio $a+c : b+c$, if c be a mean proportional between a and b.

(16) Find the ratio of the value of a gold coin to a silver one, when 13 gold coins together with 12 silver ones are worth 3 times as much as 3 gold and 40 silver.

Try be [illegible]

NB. (17) If $a : b :: c : d$, shew that $(a+b)^2 : ab :: (c+d)^2 : cd$.

, they are so when [illegible]

(18) Find two numbers, the greater of which : the less :: their sum : 42, and :: their difference : 6. $\frac{x}{y}=\frac{x+y}{42}=\frac{x-y}{6}$ ∴ $\frac{x}{y}=\frac{2x}{y+48}$ · $y=24$

(19) Divide the number n into two parts so that one shall be to the other in the ratio $n : 1$.

(20) There are two vessels, A, B, each containing a mixture of wate and wine, the wine : water in A :: $2 : 3$, and in B :: $3 : 7$. What proportion of each must be taken to form a third mixture in which the wine : water :: $5 : 11$?

EXERCISES. *Zb.*

(1) Given that $y \propto x$, and when $x = 2$, $y = 20$, state the resulting proportion.

(2) If $y \propto x$, and when $x = 2$, $y = 4a$, find the *equation* between x and y.

(3) If $y \propto$ inversely as x, and when $x = \frac{1}{2}$, $y = 8$, find the *equation* between x and y.

(4) If $1+x \propto 1-x$, shew that $1+x^2 \propto x$.

(5) If $2x+3y \propto 4x+5y$, shew that $x \propto y$.

(6) If $x^2 \propto y^3$, and $x = 2$, when $y = 3$, find the equation between x and y.

(7) If $y =$ the sum of 3 quantities, of which the 1st $\propto x^2$, the 2nd $\propto x$, and the 3rd is constant; and when $x = 1, 2, 3$, $y = 6, 11, 18$, respectively, find the equation between x and y.

(8) If $y =$ the sum of 3 quantities, of which the 1st is constant, the 2nd $\propto x$, and the 3rd $\propto x^2$; also when $x = 3, 5, 7$, $y = 0, -12, -32$, respectively, find the equation between x and y.

(9) Given that the solid content of a globe *varies* as the cube of its diameter, what is the diameter of a globe formed by melting down two other globes whose diameters are 6 in. and 7 in.?

(10) What ratio does the solid content of a globe whose diameter is 4 in. bear to that of a globe whose diameter is 8 in.?

(11) Given that the illumination from a source of light *varies inversely* as the *square* of the distance, how much farther from a candle must a book, which is now 3 inches off, be removed, so as to receive just *half* as much light?

(12) Given that the content of a cylinder *varies* as its height and the square of its diameter *jointly*, compare the contents of two cylinders, one of which is *twice* as high as the other, but with only half its diameter.

EXERCISES. Zc.

Find the 15th, and the 20th, terms in each of the following series:

(1) 1, 6, 11, &c.
(2) 16, 15, 14, &c.
(3) $\frac{1}{3}, \frac{2}{3}, 1$, &c.
(4) $mx, 2mx, 3mx$, &c.
(5) $1+x, 1+3x, 1+5x$, &c.
(6) $\frac{a}{2}, \frac{3a}{2}, \frac{5a}{2}$, &c.

Find the sum of 20 terms of each of the following series:

(7) 1, 6, 11, &c.
(8) 5, 8, 11, 14, &c.
(9) 100, 110, 120, &c.
(10) 100, 97, 94, &c.
(11) 15, 11, 7, &c.
(12) $\frac{1}{2}, \frac{3}{4}, 1$, &c.
(13) $\frac{1}{3}, \frac{2}{3}, 1$, &c.
(14) $mx, 2mx, 3mx$, &c.
(15) $3x, 5x, 7x$, &c.
(16) $\frac{1}{a}, \frac{2}{a}, \frac{3}{a}$, &c.
(17) $25a, 24a, 23a$, &c.
(18) $\frac{a-1}{10}, \frac{a-2}{10}, \frac{a-3}{10}$, &c.

(19) Find the *Arith. Mean* between $\frac{1}{4}$ and $\frac{1}{9}$.

(20) Find the *Arith. Mean* between $1+x$, and $1-x$.

(21) Find the *Arith. Mean* between $\frac{a}{2}$ and $\frac{b}{2}$.

(22) Insert 2 *Arith. Means* between 5 and 14.

(23) Insert 3 *Arith. Means* between 1 and 3.

(24) Insert 4 *Arith. Means* between 100 and 80.

(25) There is a series of terms in *Arith. Prog.* of which the sum of the first two terms is $2\frac{1}{2}$, and the 4th term is $2\frac{1}{2}$. What is the series?

(26) The first and last of 40 numbers in *Arith. Prog.* are $1\frac{1}{3}$, and $1\frac{2}{3}$; what are the intervening terms? And what is the sum of the whole series?

(27) An insolvent tradesman agreed to pay a certain debt by weekly instalments, beginning with *5s.*, and increasing by *3s.* every week. His last payment was £15. *2s.* For how many weeks did he pay, and what was the whole amount of his debt?

(28) Find the series in A.P. in which the sum of the first 5 terms is one-fourth of the sum of the next 5 terms, the first term being 1.

(29) How many terms of the series 1, 3, 5, 7, &c. amount to 1234321?

(30) The sum of n terms of a series in A.P. is $3n+5n^2$, find the 6th term.

EXERCISES. *Zd.*

Find the '*Common Ratio*' in each of the following series in *Geom. Prog.*:

(1) 100, 200, 400, &c.

(2) $2\frac{1}{2}$, 5, 10, &c.

(3) $\frac{1}{3}$, 1, 3, &c.

(4) $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c.

(5) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, &c.

(6) 0·1, 0·01, 0·001, &c.

(7) 1·25, 2·5, 5, &c.

(8) ax, $2a^2x$, $4a^3x$, &c.

(9) $\frac{x}{r}$, $\frac{nx}{r^2}$, $\frac{n^2x}{r^3}$, &c.

(10) The first two terms of a series in *Geom. Prog.* are $\frac{1}{3}$, and 1, what are the next two terms?

(11) The first two terms of a series in *Geom. Prog.*, are 125, and 25, what are the 6th and 7th terms?

(12) Find the sum of 5 terms of a series in *Geom. Prog.*, of which the 1st term is $\frac{1}{9}$, and the 5th is 9.

(13) Find the sum of 4 terms of a series in *Geom. Prog.*, of which the 1st term is $\frac{16}{27}$, and the 4th is 2.

(14) Find the *Geom. Mean* between 30, and $7\frac{1}{2}$.

(15) Find the *Geom. Mean* between $\frac{1}{3}$, and $\frac{3}{4}$.

(16) Insert two *Geom. Means* between 5, and 320.

(17) Insert two *Geom. Means* between 1, and $\frac{1}{8}$.

(18) Insert three *Geom. Means* between 6, and 486.

(19) Insert three *Geom. Means* between 100, and $2\frac{14}{25}$.

(20) Which is greater the *Arith. Mean*, or the *Geom. Mean*, between 1 and $\frac{1}{9}$? and how much greater?

(21) Are $\frac{x}{y}$, x, xy in *Geom. Prog.?* and if so, what is the 'Common Ratio'?

(22) A series of terms are in *Geom. Prog.;* the sum of the first two is $1\frac{1}{3}$, and the sum of the next two is 12. Find the series.

(23) A farmer sowed a peck of wheat, and used the whole produce for seed the following year, the produce of this 2nd year again for seed the 3rd year, and the produce of this again for the 4th year. He then sells his stock after harvest, and finds that he has $12656\frac{1}{4}$ bushels to dispose of. Supposing the increase to have been always in the same proportion to the seed sown, what was the annual increase?

(24) If a servant agrees with his master to receive for his wages, a farthing for the 1st month, a penny for the 2nd, fourpence for the 3rd, and so on; what will twelve months' wages amount to?

EXERCISES. *Ze.*

(1) The first three terms of a series in H.P. are 3, 4, 6; what are the next two terms?

(2) The first two terms of a series in H.P. are a, and b, find the next two terms.

(3) Two terms in a series, which is in H.P., are $\frac{1}{2}$, and $\frac{7}{4}$, what are the two terms immediately preceding?

(4) Continue as far as three terms more the Harmonic series 2, $\frac{4}{3}$, 1.

(5) If a, b, c, be in H.P., shew that $a : c :: 2a-b : b$.

(6) Shew that a, b, c, are in A.P., G.P., or H.P., according as

$$\frac{a-b}{b-c}=\frac{a}{a},\ \frac{a}{b},\ \text{or}\ \frac{a}{c}.$$

(7) Find the *Harm^c. Mean* between 2 and 6.

(8) Find the *Harm^c. Mean* between $\frac{3}{7}$ and $\frac{3}{10}$.

(9) Insert *two Harm^c. Means* between 1 and 2.

(10) Insert *two Harm^c. Means* between $2\frac{2}{3}$ and 6.

(11) Insert *three Harm^c. Means* between 1 and $\frac{1}{9}$.

(12) Insert *three Harm^c. Means* between $\frac{3}{4}$ and $\frac{3}{10}$.

(13) The *Arith^c.* and *Harm^c. Means* between two numbers are 2, and $1\frac{1}{2}$, respectively. What are the numbers?

EXERCISES. *Zf*.

(1) In how many different ways may a class of 9 boys stand up to read?

(2) With a peal of 6 bells what difference will be made, in the number of changes which can be rung, by the absence of one ringer?

(3) If the number of permutations of n things 4 together = 12 times the number taken 2 together, find n.

(4) If the number of permutations of n things 3 together = 6 times the number of *combinations* of n things 4 together, find n.

(5) The number of permutations of n things r together : number taken $(r-1)$ together :: 10 : 1; and number of *combinations* r together : number $(r-1)$ together :: 5 : 3; find n and r.

(6) Find the number of combinations of 100 things taken 98 together.

(7) In how many ways can 4 men be taken out of 20?

(8) Out of a dozen friends how many different parties may be made consisting of *no more* than 8?

(9) The number of combinations of $\overline{(n+1)}$ things 4 together = 9 times the number of combinations of n things 2 together; find n.

(10) Find the number of permutations of the letters in the word '*Susannah*' taken all together.

EXERCISES. *Zg*.

Expand the following powers of a binomial:

(1) $(1+x)^5$.
(2) $(1+x)^7$.
(3) $(a+b)^9$.
(4) $(a-b)^6$.
(5) $(x+2y)^4$.
(6) $(a-3x)^5$.
(7) $(1+2x)^6$.
(8) $(m+nx)^4$.
(9) $(a^2+b^2)^5$.
(10) $(a+bx)^6$.
(11) $(a^3-x^3)^4$.
(12) $(x^4+x^2y^2$

(13) Find the 6th term (independently of the rest) in the expansion of $(a+2x)^7$.

(14) Find the 10th term in $(2a+b)^{11}$.

(15) Find the 8th term in $(a-x)^9$.

(16) Find the term containing x^6 in $(a-bx)^8$.

(17) Find the term containing x^{48} in $(a-x)^{50}$.

(18) Find the 4th term, and the 98th, in $(a-b)^{100}$.

(19) Find the middle term of $(a-b)^{16}$.

(20) Find the middle term of $(2a^{\frac{1}{2}}-b^{\frac{3}{2}})^{10}$.

Expand the following *fractional* powers of a binomial:

(21) $(a+x)^{\frac{5}{2}}$.
(22) $(a-x)^{\frac{1}{2}}$.
(23) $(1+2x)^{\frac{1}{2}}$.
(24) $(1-3x)^{\frac{2}{3}}$.
(25) $(ax+by)^{\frac{4}{5}}$.
(26) $(2a^2-3b^2)^{\frac{3}{2}}$.

Expand the following *negative* powers of a binomial:

(27) $(a+x)^{-4}$.
(28) $(a-x)^{-5}$.
(29) $(1+2x)^{-6}$.
(30) $(1-3x)^{-7}$.
(31) $(a^2-x^2)^{-\frac{3}{4}}$.
(32) $(a^5-x^5)^{-\frac{1}{5}}$.

(33) Find the 5th term of $(ax-by)^{-10}$.

(34) Find the 5th term of $(a^2-x^2)^{\frac{2}{3}}$.

(35) Find the 4th term of $(ax-x^2)^{-\frac{1}{3}}$.

(36) Expand $(1+2x+3x^2)^4$.
(37) Expand $(a-b-c)^3$.
(38) Expand $(1-\sqrt{x}+x)^3$.
(39) Expand $(1+2x-3x^2)^5$.
(40) Expand $(1-x-x^2)^{-1}$.
(41) Expand $(1-x-x^2)^{-2}$.

EXERCISES. *Zh.*

Extract the square root of each of the following surds:

(1) $19+8\sqrt{3}$.
(2) $12-\sqrt{140}$.
(3) $2\sqrt{21}+22$.
(4) $10\frac{1}{4}-6\sqrt{2}$.
(5) $3\frac{1}{2}\pm\sqrt{10}$.
(6) $11\pm4\sqrt{7}$.
(7) $3\sqrt{2}-4$.
(8) $5\sqrt{2}-4\sqrt{3}$.
(9) $3\sqrt{6}+4\sqrt{3}$.
(10) $8\sqrt{3}-6\sqrt{5}$.
(11) $1\frac{1}{7}.\sqrt{21}-2\sqrt{3}$.
(12) $4\sqrt{3}-\sqrt{21}$.

Extract the 4th root of each of the following:

(13) $49+20\sqrt{6}$.
(14) $51-36\sqrt{2}$.
(15) $\frac{3}{2}\sqrt{5}+3\frac{1}{2}$.

Extract the cube root of each of the following:

(16) $38+17\sqrt{5}$.
(17) $10\sqrt{7}+22$.
(18) $10-6\sqrt{3}$.
(19) $3-1\frac{2}{3}\sqrt{6}$.
([illegible]0) $26+15\sqrt{3}$.
(21) $16+8\sqrt{5}$.

EXERCISES. *Zi*.

Expand by the method of *Indeterminate Coefficients:*

(1) $\dfrac{1}{1-x}$.

(2) $\dfrac{1+2x}{1-x-x^2}$.

(3) $\dfrac{1}{(1-x)^2}$.

(4) $\dfrac{1+2x}{1-3x}$.

(5) $\left(\dfrac{1-x}{1+x}\right)^2$.

(6) $\left(\dfrac{1-x}{1+x}\right)^3$.

(7) $\dfrac{a+x}{1-bx}$.

(8) $\sqrt{1+x^2}$.

(9) $\sqrt[3]{1+x}$.

Resolve the following into their *partial* fractions:

(10) $\dfrac{2a-x}{a^2-x^2}$.

(11) $\dfrac{2x+3}{(x-1)(x+2)}$.

(12) $\dfrac{3x+2}{x^2-1}$.

(13) $\dfrac{x^2}{(x-2)(x^2-1)}$.

(14) $\dfrac{1-x+x^2}{(1+x)x^2}$.

(15) $\dfrac{1+5x+3x^2}{(1+x)^2(1+2x)^2}$.

(16) Resolve $8x^2+2x-3$ into two factors of the first degree.

(17) Resolve $10x^2-6x-28$ into two factors of the first degree.

(18) Resolve $5x^2+9x+3$ into two factors of the first degree.

(19) Revert the series $y = x+\dfrac{x^2}{1.2}+\dfrac{x^3}{1.2.3}+\ldots\ldots$

(20) Revert the series $y = x-\dfrac{x^3}{1.2.3}+\dfrac{x^5}{1.2.3.4.5}-\ldots\ldots$

EXERCISES. *Zk*.

Express in the form of *Continued Fractions* the following:

(1) $\dfrac{27}{19}$.

(2) $\dfrac{47}{257}$.

(3) $\dfrac{67}{88}$.

(4) $\dfrac{365}{224}$.

(5) $\dfrac{251}{764}$.

(6) $\dfrac{907}{18564}$.

Find the *convergents* to the following fractions:

(7) $\dfrac{1051}{329}$. (8) $\dfrac{251}{764}$. (9) $\dfrac{182}{25}$. (10) $\dfrac{743}{611}$.

Express in the form of continued fractions the following:

(11) $\sqrt{10}$. (12) $\sqrt{26}$. (13) $\sqrt{50}$. (14) $\sqrt{101}$.

(15) $\sqrt{7}$. (16) $\sqrt{40}$. (17) $\sqrt{52}$. (18) $\sqrt{67}$.

Find fractions the nearest to the value of the following fractions having only 2 digits in their denominators:

(19) $\dfrac{251}{764}$. (20) $\dfrac{743}{611}$. (21) $\dfrac{5065}{13891}$. (22) $\dfrac{13957}{59476}$.

EXERCISES. *Zl.*

Solve in *positive integers* the following equations:

(1) $9x+7y=57$.
(2) $5x+21y=240$.
(3) $3x+17y=121$.
(4) $11x+5y=254$.
(5) $7x+15y=225$.
(6) $3x+4y=39$.
(7) $8x-23y=19$.
(8) $39x-56y=11$.
(9) $20x-31y=7$.

(10) Find the positive integral values of x for which at the same time $\frac{x-1}{2}$, $\frac{x-2}{3}$ and $\frac{x-3}{5}$, are positive integers.

(11) Find the positive integral values of x for which at the same time $\frac{x-3}{5}$, $\frac{x-5}{7}$ and $\frac{x-8}{11}$, are positive integers.

(12) Find the least positive integral value of x, which will make $\frac{x-1}{14}$, $\frac{x-3}{5}$, $\frac{2x-5}{9}$, and $\frac{\frac{1}{2}(x-1)}{21}$, positive integers.

Solve in *positive integers* the following equations:

(13) $$\left.\begin{aligned}3x+5y+7z&=560,\\ 9x+25y+49z&=2920.\end{aligned}\right\}$$

(14) $$\left.\begin{aligned}x+y+z&=100,\\ \frac{7x}{2}+\frac{4y}{3}+\frac{z}{2}&=100.\end{aligned}\right\}$$

(15) Find *one* solution in positive integers of $4x-18y+27z=100$.

(16) Divide 142 into two parts so that one shall be exactly divisible by 9, and the other by 14.

(17) A boy has a certain quantity of nuts, which he knows to be between 200 and 300; he makes them into parcels of 13 each, and finds that he has 9 over: he then makes them into parcels of 17, and finds he has 14 over: how many nuts had he?

(18) A person having a basket of oranges, between 50 and 70, takes them out in parcels of 4, and finds he has 1 over: he then takes them out in parcels of 3, and finds he has none over: how many had he?

(19) A farmer bought nearly the same number of cows and calves for £135; each cow cost £8, and each calf £3. How many were there of each?

(20) Thirty persons in an excursion spend 30 shillings altogether; each man spends 5*s*., each woman 1*s*., and each child 3*d*.; how many men, women, and children, were there?

(21) An officer of police finds that if he sends his men out 2, 4, 8, or 10, together, he has always 1 left; but if he sends them 6, or 12, together, he has 5 left. How many men had he?

EXERCISES. *Zm*.

Find the *general* solutions of the following equations:

(1) $11x+5y=254$.
(2) $3x+7y=39$.
(3) $17x-23y=19$.
(4) $7x-13y=152$.
(5) $13x-9y=17$.
(6) $27x-19y=43$.

(7) $\left.\begin{aligned}2x+3y+z&=15,\\ 10x-4y+3z&=10.\end{aligned}\right\}$

(8) $\left.\begin{aligned}x+y+z&=30,\\ 7x+5\tfrac{1}{2}.y+4\tfrac{1}{2}.z&=180.\end{aligned}\right\}$

Find the number of positive integral solutions of

(9) $11x+5y=254$.
(10) $3x+4y=39$.
(11) $7x+15y=225$.
(12) $5x+8y=42$.
(13) $13x-9y=17$.
(14) $2x+7y=125$.

(15) In how many ways may £80 be paid with sovereigns and guineas?

(16) In how many ways may £500 be paid in guineas and £5 notes?

(17) Find the number of ways in which I can mix together 40 gallons of wine, some at 15*s*., some at 19*s*., and some at 12*s*., per gallon, so as to produce a mixture worth 16*s*. per gallon, an *integral* number of gallons of each sort being always taken.

(18) How many fractions are there with denominators 3 and 4, whose sum is $3\frac{1}{4}$?

(19) How many fractions are there with denominators 12 and 18, whose sum is $\frac{25}{36}$?

Solve *in positive integers* the following equations:

(20) $3xy-7x=7y+5$.
(21) $5xy=2x+3y+18$.
(22) $xy-(x+y)=34$.
(23) $5xy-3x=24$.
(24) $y(3x-2)=3x^2+1$.
(25) $3xy+2x^2=3x+2y+5$.
(26) $x^2+xy=2x+3y+29$.
(27) $xy+2x+3y=42$.

EXERCISES. *Zn*.

Transform the following numbers from the *Denary* to the *Senary Scale:*

(1) 182061.
(2) 5002001.
(3) 211115600.

Transform the following from scale 5 to scale 7:

(4) 4321.
(5) 110423.
(6) 100311.

(7) Transform 37704 from scale 9 to scale 8.

(8) Transform 13256 from scale 7 to scale *twelve*.

(9) Transform 1341120 from the senary to the duodenary scale.

(10) Transform 654321 from the septenary to the duodenary scale.

(11) Subtract 20404020 from 103050301 in the octenary scale.

(12) Extract the square root of the result in the last Ex.

(13) Divide 51117344 by 675 in the octenary scale.

(14) Find the radix of the scale in which 40501 is equivalent to 5365 in the denary scale.

(15) In what scale is the denary number 2704 written 20304?

(16) Extract the square root of 1010001 in the binary scale, and reduce the result to the denary scale.

(17) Apply the duodenary notation to find the square of 4ft. 2in. $0'$. $2''$. $10'''$.

(18) Apply the duodenary notation to find the cube of 16ft. 10in.

EXERCISES. *Zo*.

Transform the following quantities from scale 10 to scale 5:

(1) 221·342. (2) 357·234. (3) 101·265.

(4) Transform 179·25 from the denary to the senary scale.

(5) Transform 23·32 from the denary to the duodenary scale.

(6) Transform $\frac{25}{36}$ from the denary to the duodenary scale.

(7) Transform $21\frac{1}{8}$ from the denary to the octenary scale.

(8) Transform 7304·513 from scale 8 to scale 4.

(9) Transform $3t{\cdot}97e$ from the duodenary to the octenary scale.

(10) Transform 345·6273 from the octenary to the ternary scale.

(11) Transform $\frac{11}{17}$ from the denary to the duodenary scale.

(12) A certain number is 125 in the scale whose radix is x, 78 in the scale whose radix is y, and 49 in the scale whose radix is $x+y$; find the number in the scale whose radix is 10.

EXERCISES. *Zp*.

(1) If p and q are any positive whole numbers, and $p+q$ is even, shew that $p-q$ is also even.

(2) Shew that the difference between any number and its square is always an even number.

(3) Shew that the difference between any number and its cube is always divisible by 6 without remainder.

(4) Shew that the product of two odd numbers will always be odd.

(5) Shew that the *sum* of any two *consecutive* odd numbers will always be divisible by 4.

(6) Shew that the *product* of any two *consecutive* even numbers is divisible by 8.

(7) Shew that every odd square number, greater than 1, leaves a remainder 1, when divided by 8.

(8) Shew that every perfect cube is of one of the forms $7m$, or $7m \pm 1$.

(9) Shew that upon any number, greater than 12, which is a perfect square, being divided by 12, the remainder, if there be any, is a square.

(10) Shew that the difference of the squares of any two odd numbers is exactly divisible by 8.

(11) Shew that the square of any number prime to 4 is of the form $4p+1$.

(12) Shew that the difference of the squares of any two prime numbers, each of which is greater than 3, is divisible by 24.

EXERCISES. *Zq.*

Find the value of each of the following fractions:

(1) $\dfrac{x^2-1}{x^2+2x-3}$, when $x=1$.

(2) $\dfrac{x^2+5x+6}{x^2+7x+12}$, when $x=-3$.

(3) $\dfrac{x^2+2x-35}{x^2-6x+5}$, when $x=5$.

(4) $\dfrac{2x^3-7x^2+12}{x^3-7x+6}$, when $x=2$.

(5) $\dfrac{x^3-2x^2-x+2}{x^3-7x+6}$, when $x=2$.

(6) $\dfrac{x^3+2ax^2-a^2x-2a^3}{x^3-13a^2x+12a^3}$, when $x=a$.

(7) $\dfrac{x^3-5x^2+3x+9}{x^3-x^2-21x+45}$, when $x=3$.

(8) $\dfrac{81-a^4}{3-a}$, when $a=3$.

(9) $\dfrac{\sqrt{x+1}-\sqrt{2}}{x-1}$, when $x=1$.

(10) $\dfrac{\sqrt{3x+1}-2}{x-1}$, when $x=1$.

(11) $\dfrac{\sqrt[4]{5x-1}-\sqrt{2}}{x-1}$, when $x=1$.

(12) $\dfrac{x^5-a^5}{x^4-a^4}$, when $x=a$.

(13) Find the value of $\dfrac{x^2+y^2+2xy-4}{x+y-2}$, when $x=1$, and $y=1$.

(14) Find the value of $\dfrac{xy-ay-bx+ab}{xy-ay-cx+ac}$, when $x=a$.

EXERCISES. *Zr*.

(1) Given $\log a$, find the log. of $\sqrt[n]{a^m}$.

(2) Given $\log 2 = 0\cdot30103$, and $\log 3 = 0\cdot47712$, find $\log\left(\frac{3}{2}\right)^{20}$.

(3) Given $\log 2$, and $\log 3$, as in last example, find $\log\left(\frac{5}{6}\right)^{10}$.

(4) Given $\log n$, find the log. of $\frac{\sqrt{n}\,.\sqrt[3]{n^2}}{\sqrt[4]{n^3}}$.

(5) Given $\log 2$, and $\log 3$, as in Ex. (2), find $\log 12$.

(6) Given $\log 2$, find the log. of $\frac{5}{4}$.

(7) Given $\log 2$, find the log. of $6\cdot4$.

(8) If $a^{mx}b^{nx} = c$, find x.

(9) If $a^{\sqrt{x}} = b$, find x.

(10) If $\sqrt[x]{a} = b$, find x.

(11) If $\log x = \frac{1}{2}\log a - \frac{1}{4}\log b$, find x.

(12) If $\frac{1}{2}\log x = n\log a + m\log b - p\log c$, find x.

(13) Given $\log 2 = 0\cdot30103$, find the log. of 16^{20}.

(14) Given $\log x + \log y = \frac{5}{2}$, and $\log x - \log y = \frac{1}{2}$, find x and y.

(15) Given $\log x - \log y = \log n$, and $ax + by = c$, find x and y.

(16) If $\log_{10} x = 3\log_{10} a - 2$, find x.

(17) If $\log x + \log y = \log a$, and $2\log x - 2\log y = \log b$, find x and y.

EXERCISES. *Zs*.

[N.B. $\log 1\cdot05 = 0\cdot02119$, and $\log 1\cdot04 = 0\cdot01703$.]

(1) What would £200 amount to in 7 years at 4 per cent., compound interest?

(2) How much money must be invested at compound interest to amount to £500 in 12 years at 5 per cent.?

(3) In how many years will a sum of money *double* itself placed out at 4 per cent. compound interest?

(4) A freehold estate which produces a clear rental of £100 a year is sold for £2500; at what rate is interest reckoned?

(5) Find the amount of £100 in 10 years at £100 per cent., compound interest.

(6) If a person returns 100 guineas for the loan of £100 for 3 months, what is the rate of interest allowed?

(7) A person returns £287. 10*s.* for the loan of £250 at the rate of 5 per cent. per annum, simple interest. For what time was the money lent?

(8) Supposing interest paid *half-yearly*, what will £500 amount to in 8 years at 5 per cent., compound interest? (given log 1·025 = 0·0107239.)

(9) How many years' purchase should be given for a freehold estate when money is worth 3½ per cent.?

[N.B. The ANSWERS to all the *Exercises* are printed on a separate sheet, and may be had *gratis*, by any School-Master, Lecturer, or Tutor, on application to Messrs. LONGMAN & Co., 39, Paternoster-Row, London, or to the Rev. THOMAS LUND, Morton Rectory, near Alfreton.]

NOTES.

NOTE 1.

DIVISIBILITY OF AN EXPRESSION BY $x-a$.

In Ex. 7, p. 49, it will not fail to be observed, that the remainder is the same expression as the dividend, with a written in the place of x; this we shall shew to be true in all such cases.

Let it be required to divide the expression

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n$$

by $x-a$, and let Q be the quotient, and R the remainder; then, writing for the sake of convenience $f(x)$ for the dividend, we have,

$$f(x) = Q\times(x-a)+R.$$

Now R cannot contain x, for otherwise the division would not have been completely performed, and therefore R remains unaltered whatever value x assumes; also Q, being evidently of the form

$$q_0x^{n-1}+q_1x^{n-2}+q_2x^{n-3}+\ldots+q_{n-1},$$

cannot become infinite for any finite value of x. Let then $x = a$;

$$\therefore\ f(a) = Q\times 0+R,$$

i.e. $R=f(a)$, the same expression as the dividend, with a written in the place of x.

In order to find the *quotient*, we have

$$f(x) = Q(x-a)+R,$$

$$\textit{i.e.}\ \ p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n = (q_0x^{n-1}+q_1x^{n-2}+\ldots+q_{n-1})(x-a)+R,$$

identically; $\therefore$ performing the multiplication, and equating the coefficients of the several powers of x in each member of this equality, we get

$$\left.\begin{array}{r} p_0 = q_0, \\ p_1 = q_1-aq_0, \\ p_2 = q_2-aq_1, \\ \&\text{c.}, \\ p_{n-1} = q_{n-1}-aq_{n-2}, \\ p_n = R-aq_{n-1}; \end{array}\right\}\ \text{and}\ \therefore\ \left\{\begin{array}{l} q_0 = p_0, \\ q_1 = aq_0+p_1, \\ q_2 = aq_1+p_2, \\ \&\text{c.}, \\ q_{n-1} = aq_{n-2}+p_{n-1}, \\ R = aq_{n-1}+p_n. \end{array}\right.$$

From the second of these sets of equations we discover an easy method of forming the quotient, viz. to obtain any one coefficient, multiply the preceding one by a, and add in the corresponding coefficient in the dividend: it also appears that the *remainder* can be found by this method.

Ex. 1. Required the *remainder*, when $x^4-2x^3+3x^2-x-1$ is divided by $x+2$.

Here $a=-2$, and $\therefore$ the remainder is

$$(-2)^4-2\times(-2)^3+3\times(-2)^2-(-2)-1=45.$$

Ex. 2. Required the *quotient*, and the *remainder*, in the division of $3x^5-2x^4-11x^3-55x+7$ by $x-3$.

Here $a=3$, and, observing that one term in the dividend is wanting, and writing the coefficients in order, we obtain

$$\begin{array}{cccccc}
3 & -2 & -11 & 0 & -55 & +7 \\
 & \underline{9} & \underline{21} & \underline{30} & \underline{90} & \underline{105} \\
 & 7 & 10 & 30 & 35 & 112
\end{array}$$

The quotient, therefore, is $3x^4+7x^3+10x^2+30x+35$, and the remainder 112.

In the equality $f(x)=Q(x-a)+R$, if $f(x)=0$, when $x=a$, we have $R=f(a)=0$; $\therefore$ if a be a root of the equation $f(x)=0$, then $x-a$ will divide the expression $f(x)$ without remainder. This has already been perceived to be the case in quadratic equations (Art. 205), and ought to be borne in mind.

By applying the above method we can easily see, that x^m-a^m is always divisible by $x-a$, and also by $x+a$ when m is even; and that x^m+a^m is only divisible by $x+a$, and that when m is odd.

But the simplest way of shewing, that x^m-a^m is divisible by $x-a$, is as follows:

$$\frac{x^m-a^m}{x-a}=x^{m-1}+\frac{x^{m-1}-a^{m-1}}{x-a}.a,$$

from which it appears, that if $x^{m-1}-a^{m-1}$ be divisible by $x-a$, so also is x^m-a^m. But we know, that x^2-a^2 is divisible by $x-a$; *therefore* also is x^3-a^3; and if x^3-a^3, *therefore* also x^4-a^4; and so on generally.

NOTE 2.

EXTRACTION OF SQUARE OR CUBE ROOT OF NUMBERS.

The rules for *pointing* in the operation of extracting the square and cube roots of arithmetical quantities (see pp. 77, 79) will perhaps be more clearly perceived by considering what part of the square or cube arises from the several terms of the root.

(i) *For square root.* Let a be the part already obtained, and b the next digit, so that if there be n places of figures after a, the root is $a.10^n+b.10^{n-1}+\&c.$ The part of the square which depends upon this is $a^2.10^{2n}+2ab.10^{2n-1}+b^2.10^{2n-2}$; whence we see, that a is to be determined from the first term, which, as it involves an even power of 10, has an even number of figures following it; and that the introduction of b brings in two inferior powers of 10, *i.e.* every fresh figure in the root involves two fresh places of figures in the square; we have therefore to bring down at every trial division two figures. If then we point according to the rule, we shall satisfy both the above conditions, and consequently extract the root correctly.

(ii) *For cube root.* If $a.10^n+b.10^{n-1}+\&c.$ be the root, the part of the cube which depends on this is

$$a^3.10^{3n}+3a^2b.10^{3n-1}+3ab^2.10^{3n-2}+b^3.10^{3n-3}.$$

The first term has therefore a number of figures following it which is a multiple of 3, and the introduction of a fresh figure in the root brings in 3 inferior powers of 10; and therefore at every trial division we have to bring down 3 figures. These conditions are satisfied by the ordinary rule for pointing, and therefore the operation will be correct.

NOTE 3.

THEORY OF INDICES.

The theory of *Indices* is often established in the following manner: With our original definition of an *index* there is no meaning to be attached to a fractional or negative one. We are therefore at liberty to assign any meaning we please to them; this may, or may not, make fractional and negative indices follow the same laws as positive integral ones; but it is evidently most convenient that they *should* follow the same laws: if then we *assume* these laws to hold good in all cases, we shall be guided to an interpretation of our indices.

The fundamental laws that have been proved for positive integral indices are these:—$a^m.a^n=a^{m+n}$, and $a^m \div a^n = a^{m-n}$, when n is less than m, and $=\dfrac{1}{a^{n-m}}$, when n is greater than m. Let these hold universally; then we shall have

$$a^{\frac{p}{q}}.a^{\frac{p}{q}}.a^{\frac{p}{q}}\dots\dots \text{to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\dots\text{to } q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p;$$

i. e. $a^{\frac{p}{q}}$ is a quantity such that, when multiplied into itself q times, it produces a^p, and therefore $=\sqrt[q]{a^p}$; which gives us the meaning of a *positive fractional* index.

Again, for a *negative* index we have

$$a^m.a^{-n}=a^{m-n}=a^m\div a^n, \text{ or } a^m.\frac{1}{a^n};\ \therefore\ a^{-n}=\frac{1}{a^n},$$

which gives a meaning to a *negative* index whether integral or fractional.

Lastly, we have $a^n . a^{-n} = a^0 = a^n . \frac{1}{a^n} = 1$; $\therefore$ $a^0 = 1$.

It will be observed, that this manner of establishing the theory of indices is the *inverse* of that pursued in the text. (See Art. 162.)

NOTE 4.

ALGEBRAICAL AND GEOMETRICAL RATIO.

In Arts. 257, 260*, it is shewn, that ratios, which are *equal* according to the algebraical definition, are equal also according to the geometrical definition, and *vice versâ*. It may also be *strictly* proved, that if ratios are *unequal* algebraically, they are *unequal* geometrically.

Let $\frac{a}{b}$, $\frac{c}{d}$, be two positive* algebraical, and *unequal*, ratios, of which $\frac{a}{b}$ is the greater; and let $\frac{1}{f}$ be a fraction, which is readily found, less than the difference between $\frac{a}{b}$ and $\frac{c}{d}$. Multiply $\frac{1}{f}$ successively by 2, 3, 4, &c. till its multiple exceeds $\frac{c}{d}$; and let $\frac{e}{f}$ be the *first* multiple which exceeds $\frac{c}{d}$. Then $\frac{e-1}{f}$ is not greater than $\frac{c}{d}$; and $\frac{1}{f} < \frac{a}{b} - \frac{c}{d}$;

$$\therefore \frac{e-1}{f} + \frac{1}{f} < \left(\frac{a}{b} - \frac{c}{d}\right) + \frac{c}{d}, \text{ or } \frac{e}{f} < \frac{a}{b}; \ \therefore fa < eb.$$

$$\text{Also} \qquad \frac{c}{d} < \frac{e}{f}, \ \therefore fc < ed.$$

Hence it appears, that equimultiples fa, fc, have been taken of a and c, and equimultiples eb, ed, of b and d, such that the multiple of a is greater than the multiple of b, but the multiple of c is less than that of d; that is, the ratios $a : b$, and $c : d$, are proved unequal according to the test stated by Euclid. Q. E. D.

NOTE 5.

NEW PROOF OF THE BINOMIAL THEOREM†.

By actual Multiplication the successive powers of the binomial $a+b$ are found to be as follow:

$$(a+b)^1 = a+b,$$
$$(a+b)^2 = a^2+2ab+b^2,$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3,$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4, \text{ and so on;}$$

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* Euclid is concerned with *positive* magnitudes only, and treats of none other.

† This proof is taken with some slight alterations from *Principes D'Algèbre*, par E. E. Bobillier, 1857.

which may be written thus,

$$\frac{(a+b)^1}{1}=\frac{a^1}{\underline{|1}}+\frac{b^1}{\underline{|1}},$$

$$\frac{(a+b)^2}{\underline{|2}}=\frac{a^2}{\underline{|2}}+\frac{a^1}{\underline{|1}}\cdot\frac{b^1}{\underline{|1}}+\frac{b^2}{\underline{|2}},$$

$$\frac{(a+b)^3}{\underline{|3}}=\frac{a^3}{\underline{|3}}+\frac{a^2}{\underline{|2}}\cdot\frac{b^1}{\underline{|1}}+\frac{a^1}{\underline{|1}}\cdot\frac{b^2}{\underline{|2}}+\frac{b^3}{\underline{|3}},$$

$$\frac{(a+b)^4}{\underline{|4}}=\frac{a^4}{\underline{|4}}+\frac{a^3}{\underline{|3}}\cdot\frac{b^1}{\underline{|1}}+\frac{a^2}{\underline{|2}}\cdot\frac{b^2}{\underline{|2}}+\frac{a^1}{\underline{|1}}\cdot\frac{b^3}{\underline{|3}}+\frac{b^4}{\underline{|4}};\ \text{and so on,}$$

the same *law* of formation of the terms of the expansion being found to hold for $(a+b)^5$, $(a+b)^6$, &c.

Suppose, then, the same law to hold for $(a+b)^n$, so that

$$\frac{(a+b)^n}{\underline{|n}}=\frac{a^n}{\underline{|n}}+\frac{a^{n-1}}{\underline{|n-1}}\cdot\frac{b^1}{\underline{|1}}+\frac{a^{n-2}}{\underline{|n-2}}\cdot\frac{b^2}{\underline{|2}}+\ldots\ldots+\frac{a^1}{\underline{|1}}\cdot\frac{b^{n-1}}{\underline{|n-1}}+\frac{b^n}{\underline{|n}},\quad(1),$$

where the 1st and last terms are $\frac{a^n}{\underline{|n}}$, and $\frac{b^n}{\underline{|n}}$, n being a positive integer, and the general term, expressed by the $\overline{r+1}^{\text{th}}$, is $\frac{a^{n-r}}{\underline{|n-r}}\cdot\frac{b^r}{\underline{|r}}$.

Now multiply (1) by $\frac{a+b}{n+1}$, or $\frac{a}{n+1}+\frac{b}{n+1}$; then we have

$$\frac{(a+b)^{n+1}}{\underline{|n+1}}=\frac{a^{n+1}}{\underline{|n+1}}+\ldots\ldots\ldots\ldots+\frac{b^n}{\underline{|n+1}},$$

a series of which the general, or $\overline{r+1}^{\text{th}}$, term, *i.e.* the term involving b^r, is

$$\frac{a}{n+1}\cdot\frac{a^{n-r}}{\underline{|n-r}}\cdot\frac{b^r}{\underline{|r}}+\frac{b}{n+1}\cdot\frac{a^{n-r+1}}{\underline{|n-r+1}}\cdot\frac{b^{r-1}}{\underline{|r-1}},$$

$$\text{or}\quad\frac{a^{n-r+1}}{\underline{|n-r+1}}\cdot\frac{b^r}{\underline{|r}}\cdot\left\{\frac{n-r+1}{n+1}+\frac{r}{n+1}\right\},$$

$$\text{or}\quad\frac{a^{(n+1)-r}}{\underline{|(n+1)-r}}\cdot\frac{b^r}{\underline{|r}}.$$

Hence it is *proved*, that the expansion of $\frac{(a+b)^{n+1}}{\underline{|n+1}}$ follows the same law as that of $\frac{(a+b)^n}{\underline{|n}}$. But the law above stated has been shewn to hold, when n is 1, 2, 3, or 4; therefore it holds for 5; and if for 5, for 6; and so on; and therefore *generally* for any positive integer whatever.

Thus it follows, that for all positive integral values of n,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}.b^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1\ .\ 2\ \ldots\ r}a^{n-r}b^r + \ldots + b^n,$$

which is the *Binomial Theorem.*

COR. The general term of $\dfrac{(a+b+c)^n}{\lfloor\underline{n}}$ is $\dfrac{a^{n-m}}{\lfloor\underline{n-m}}.\dfrac{(b+c)^m}{\lfloor\underline{m}}$, being the $\overline{m+1}^{\text{th}}$ term, and the general term of $\dfrac{(b+c)^m}{\lfloor\underline{m}}$ is $\dfrac{b^{m-r}}{\lfloor\underline{m-r}}.\dfrac{c^r}{\lfloor\underline{r}}$, being the $\overline{r+1}^{\text{th}}$ term; $\therefore$ the general term of $\dfrac{(a+b+c)^n}{\lfloor\underline{n}}$ is $\dfrac{a^{n-m}}{\lfloor\underline{n-m}}.\dfrac{b^{m-r}}{\lfloor\underline{m-r}}.\dfrac{c^r}{\lfloor\underline{r}}$; or if $n-m=p$, $m-r=q$, observing that $n-m+m-r=p+q$, so that $p+q+r=n$, the general term is $\dfrac{a^p}{\lfloor\underline{p}}.\dfrac{b^q}{\lfloor\underline{q}}.\dfrac{c^r}{\lfloor\underline{r}}$.

Again, writing $c+d$ for c, the general term of

$$\frac{(a+b+c+d)^n}{\lfloor\underline{n}} \text{ is } \frac{a^p}{\lfloor\underline{p}}.\frac{b^q}{\lfloor\underline{q}}.\frac{c^r}{\lfloor\underline{r}}.\frac{d^s}{\lfloor\underline{s}}, \text{ where } p+q+r+s=n.$$

And so on, giving us the *Multinomial Theorem* for a positive integral index.

NOTE 6*.

LEMMA. *If* f(m, n) *represent a positive integral function of* m *and* n, *of not more than* μ *dimensions in* m, *nor more than* ν *dimensions in* n, *which vanishes for each of* $\mu+1$ *given different values of* m, *combined with any one of* $\nu+1$ *given different values of* n, *in every way in which these can be combined, then will* f(m, n) *be identically* zero.

For arrange $f(m, n)$ according to ascending powers of m, so that $f(m, n)$ assumes the form

$$f_0(n) + f_1(n).m + f_2(n).m^2 + \ldots + f_r(n).m^r + \ldots + f_\mu(n).m^\mu, \ldots\ldots(\text{A}),$$

where each coefficient of m and of its powers,

$$f_0(n),\quad f_1(n),\quad f_2(n),\ldots\ldots f_r(n),\ldots\ldots f_\mu(n),$$

will, by hypothesis, be a positive integral function of n, of not more than ν dimensions.

Let the given values of m and n be

$$m_0,\ m_1,\ m_2,\ldots\ldots m_r,\ldots\ldots m_\mu,$$

and $\quad n_0,\ n_1,\ n_2,\ldots\ldots n_r,\ldots\ldots n_\nu,$ respectively;

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* **For this Note I am indebted to the Rev. Henry Geo. Day, M.A. Fellow of St John's College, Cambridge.**

substitute in (A) one of the values of n, viz. n_0; then

$$f_0(n_0)+f_1(n_0).m+f_2(n_0).m^2+\ldots+f_r(n_0).m^r+\ldots+f_\mu(n_0).m^\mu$$

will vanish for more than μ different values of m. Hence

$$f_0(n_0)=0,\quad f_1(n_0)=0,\quad f_2(n_0)=0,\ldots\ldots f_r(n_0)=0,\ldots\ldots f_\mu(n_0)=0.$$

Similarly, $f_0(n_1)=0,\quad f_2(n_1)=0,\ldots\ldots f_r(n_1)=0,\ldots\ldots f_\mu(n_1)=0$;

and so on, for the other given values of n.

Hence any one of the coefficients in (A), as $f_r(n)$, which is a positive integral function of n, of not more than ν dimensions, will vanish for more than ν different values of n, and $\therefore f_r(n)=0$, *identically*.

Similarly, $f_0(n)=0,\quad f_1(n)=0,\ldots\ldots f_\mu(n)=0$; or, substituting these values in (A),

$$f(m,\ n)=0,\ \textit{identically}.\quad \text{Q.E.D.}$$

PROOF OF THE BINOMIAL THEOREM FOR NEGATIVE AND FRACTIONAL INDICES.

As long as n is a *positive integer*, the series

$$1+\frac{n}{1}.x+\frac{n(n-1)}{1.2}.x^2+\frac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}.x^3+\&c.$$

whose $\overline{r+1}^{\text{th}}$, or general, term is $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1\ .\ 2\ .\ 3\ \ \ldots\ \ r}\,x^r$, will terminate in a finite number of terms, and will have for its value $(1+x)^n$.

But if n be *negative*, or *fractional*, the series will not terminate, and will consequently have no *arithmetical* significance, unless it be *convergent*.

Now, as long as x lies between -1 and 1, the series *is* convergent, and we may safely assume, under those circumstances, that the *limit* of

$$1+\frac{n}{1}.x+\frac{n(n-1)}{1.2}.x^2+\frac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}.x^3+\ldots\ldots in\ inf.$$

has a definite numerical value dependent only on the values of n and x.

Assume, therefore, that this series, which we shall write

$$1+N_1x+N_2x^2+N_3x^3+\ldots+N_rx^r+\ldots in\ inf.$$

is equal to $f(n)$; and, similarly, that

$$1+M_1x+M_2x^2+M_3x^3+\ldots+M_rx^r+\ldots in\ inf.$$

is equal to $f(m)$.

Now, these series being *convergent*, and therefore having definite *arithmetical* values, are subject to the arithmetical laws of multiplication, &c. Hence

$$f(m)\times f(n)=(1+M_1x+M_2x^2+\ldots).(1+N_1x+N_2x^2+\ldots),$$

or the series $1+(M_1+N_1)x+(M_2+N_2+M_1N_1)x^2+\ldots$ obtained by the multiplication of the two former series will itself be convergent, and have an arithmetical value represented by $f(m)\times f(n)$.

But the coefficient of x^r in the resulting series will be of the form $M_r+M_{r-1}N_1+M_{r-2}N_2+\ldots+M_1N_{r-1}+N_r=A_r$, suppose, the law of its formation being obvious.

Now as long as m and n are both positive integers, and each is not less than r,

$\left.\begin{matrix} M_1,\ M_2,\ M_3,\ldots M_r, \\ \text{and}\ N_1,\ N_2,\ N_3,\ldots N_r, \end{matrix}\right\}$ will be coefficients of the successive powers of x in the expansions of $(1+x)^m$, and $(1+x)^n$, respectively; and A_r will be the coefficient of x^r in their product, or in the expansion of $(1+x)^{m+n}$.

Hence, whenever m and n are *positive integers*, and each not less than r,

$$A_r-(M+N)_r=0,\ \textit{identically}.$$

But A_r, and $(M+N)_r$, and therefore $A_r-(M+N)_r$, are integral functions of m and n, of not more than r dimensions in either m or n. And $A_r-(M+N)_r$ vanishes for each integral value of m from r to $2r$, combined with each integral value of n from r to $2r$, that is, for each of more than r different values of m, combined with each of more than r different values of n, in whatever manner these can be combined. Hence, by the Lemma,

$A_r=(M+N)_r$, *identically*, for every integral value of r however large.

Now $f(m)\times f(n)$ has been shewn to be equal to

$$1+A_1x+A_2x^2+\ldots+A_rx^r+\ldots in\ inf.$$

$$\therefore\ f(m)\times f(n)=1+(M+N)_1x+(M+N)_2x^2+\ldots+(M+N)_rx^r+\ldots in\ inf.$$
$$=f(m+n).$$

Multiply each side of this *identity* by $f(p)$; then

$f(m)\times f(n)\times f(p)=f(m+n)\times f(p)=f(m+n+p)$, by what has been proved; and this may be continued to any number of factors; therefore,

$$\textit{generally},\ f(m).f(n).f(p).f(q).\&c.=f(m+n+p+q+\&c.)$$

I. Let $n=\dfrac{a}{b}$, a positive fraction, where a and b are both positive integers; then

$$f\left(\frac{a}{b}\right).f\left(\frac{a}{b}\right).\&c.\ \text{to}\ b\ \text{factors}=f\left(\frac{a}{b}+\frac{a}{b}+\&c.\ \text{to}\ b\ \text{terms}\right),$$
$$=f\left(\frac{ba}{b}\right)=f(a)=(1+x)^a,\ (\because a\ \text{is a positive integer}),$$

that is, $\left\{f\left(\dfrac{a}{b}\right)\right\}^b=(1+x)^a$, or $f\left(\dfrac{a}{b}\right)=(1+x)^{\frac{a}{b}}$, which proves the Theorem for a *positive fractional* index.

II. Next, let n be a negative index, integral or fractional, $=-m$, suppose. Take s any *positive* integer or fraction greater than m; then

$$f(s).f(n)=f(s+n)=f(s-m),$$
$$=(1+x)^{s-m},\ (\because\ s-m \text{ is positive}),$$
$$=\frac{(1+x)^s}{(1+x)^m};$$

$\therefore\ f(n)=\dfrac{(1+x)^s}{(1+x)^m}\div(1+x)^s=(1+x)^{-m}=(1+x)^n$, which proves the Theorem for a *negative* index, integral or fractional.

Collecting these results, it is proved, that, whenever x lies between -1 and 1, *for all commensurable values of* n, the arithmetical value of $(1+x)^n$ is equal to the limit of the series

$$1+\frac{n}{1}.x+\frac{n(n-1)}{1\,.\,2}.x^2+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}.x^3+\ldots in\ inf.$$

III. Lastly, suppose n *incommensurable;* and take a series of fractions c_1, c_2, c_3,......converging to n. Then, ultimately, the series $f(c)$ will tend to equality with $f(n)$, and in the *limit* will be equal to it.

Also, by the law of indices, $(1+x)^c$ tends to equality with $(1+x)^n$, and the *limits* of these are equal. But $(1+x)^c=f(c)$; therefore $(1+x)^n$ and $f(n)$ will have the same *limit*, that is, for *incommensurable* indices

$$(1+x)^n=f(n).$$

NOTE 7.

CONTINUED FRACTIONS.

The law of the formation of the convergents to a continued fraction investigated in Art. 346, may also be proved in the following way:

We have, as in the text, $\dfrac{N_4}{D_4}=\dfrac{q_4N_3+N_2}{q_4D_3+D_2}$, and $\dfrac{N_4}{D_4}$ is in its lowest terms therefore, by Art. 258, $q_4N_3+N_2$ and $q_4D_3+D_2$ are either equal to, or are equimultiples of, N_4 and D_4; *i.e.* $q_4N_3+N_2=k.N_4$, $q_4D_3+D_2=k.D_4$, where $k=1$, or some other integer. Now it is observed for the first two or three fractions that $N_2D_1-N_1D_2=\pm1$. Let this be true as far as $\dfrac{N_3}{D_3}$, so that

$$N_3D_2-N_2D_3=\pm1.$$

Then $$N_4D_3-N_3D_4=\frac{1}{k}\{(q_4N_3+N_2)D_3-N_3(q_4D_3+D_2)\},$$

$$=\frac{1}{k}\cdot(N_2D_3-N_3D_2)=\mp\frac{1}{k}.$$

But as the N's and D's are all integers, the left-hand member of this equation is an integer, and therefore also the right-hand member. This cannot be unless $k = 1$, which consequently is the case, and then

$$N_4 = q_4 N_3 + N_2, \quad D_4 = q_4 D_3 + D_2,$$

which proves the rule. It will be observed that this proves each convergent to be in its lowest terms, and also establishes independently the truth of Art. 349.

NOTE 8.

CONTINUED FRACTIONS.

It has been assumed in the formation of the continued fraction which represents $\sqrt{n}$ in Art. 348, that the quantity corresponding to r'' will at some stage of the operation become equal to 1, when the quotients will recur. It is evident, that the continued fraction will never terminate; the recurrence of the quotients will be seen by investigating the process of formation.

We have $a'+a'' = r'b'$; $r'r'' = n-a''^2$; and similar equations for determining successively the values of the symbols. Now taking the quotient b'', the complete quotient is $\frac{\sqrt{n}+a''}{r''}$: and if $\frac{N}{D}$, $\frac{N_1}{D_1}$, be the two preceding convergents,

$$\sqrt{n} = \frac{\frac{\sqrt{n}+a''}{r''}N_1 + N}{\frac{\sqrt{n}+a''}{r''}D_1 + D} = \frac{\sqrt{n}.N_1 + a''N_1 + r''N}{\sqrt{n}.D_1 + a''D_1 + r''D};$$

∴ multiplying up, and equating the rational and irrational parts, by virtue of Art. 179, we have

$$\left.\begin{aligned} nD_1 &= a''N_1 + r''N, \\ N_1 &= a''D_1 + r''D; \end{aligned}\right\}$$

$$\therefore \left.\begin{aligned} nDD_1 - NN_1 &= a''(N_1D - ND_1), \\ N_1^2 - nD_1^2 &= r''(N_1D - ND_1). \end{aligned}\right\}$$

Now $\frac{N}{D}$, $\sqrt{n}$, $\frac{N_1}{D_1}$, are in order of magnitude, and $\sqrt{n}$ is nearer to the latter than to the former; therefore, according as this order is ascending or descending,

$$n \gtrless \frac{NN_1}{DD_1}, \quad n \lessgtr \frac{N_1^2}{D_1^2}, \quad N_1D - ND_1 = \pm 1;$$

therefore, in both cases, a'' and r'' are positive integers. And as $\frac{\sqrt{n}+a}{r}$, &c., are all >1, b, b', b'', &c. are positive integers, therefore all the symbols in this investigation are positive integers. The equation $r'r'' = n - a''^2$, and those similar to it, shew that a'', &c. are all less than $\sqrt{n}$, therefore a', a'', &c. are not $> a$, which is the greatest whole number $< \sqrt{n}$; and then from the equation $a' + a'' = r'b'$ and those similar to it, we find that none of the quantities corresponding to r' and b' can exceed $2a$. As then these are restricted to lie within the above limits, since the operation never terminates, it is clear that they must recur. If then we make the comparison of these recurring quotients in a retrograde order, we shall see that there must be one corresponding to the first that we obtain; here then the quantity r'' must be equal to the corresponding quantity at the beginning, *i. e.* 1. Now if, instead of $\sqrt{n}$, it had been proposed to converge to $\sqrt{n} + a$, the whole operation would have been the same as that we have been performing, excepting that the first step would have been exactly similar to all the following ones, and the first quotient would have been $2a$ instead of a. It is evident, then, from these considerations, that when the recurrence begins, the quotient is $2a$. Consequently in performing the operation, we need only carry it on until we obtain a quotient $2a$, and afterwards write the quotients over and over again as far as we please.

In the equation $N_1^2 - nD_1^2 = r''(N_1 D - N D_1)$, if we put $r'' = 1$, (which value it does take periodically), we have $N_1^2 - nD_1^2 = \pm 1$, according as

$$\frac{N_1}{D_1} \gtrless \frac{N}{D}, \text{ and } \therefore \gtrless \sqrt{n},$$

i.e. according as $\frac{N_1}{D_1}$ occupies an even or an odd place among the series of converging fractions. But as $r'' = 1$, the next quotient $= 2a$, and therefore $\frac{N_1}{D_1}$ is the fraction preceding any one formed by stopping at any quotient $2a$. Now, if the number of recurring quotients be even, $\frac{N_1}{D_1}$ is always in an even place, and therefore $N_1^2 - nD_1^2 = +1$: but if the number of recurring quotients be odd, $\frac{N_1}{D_1}$ is alternately in an odd and an even place, and therefore $N_1^2 - nD_1^2$ alternately equals -1 and $+1$. Therefore by forming the convergents to $\sqrt{n}$, and taking those corresponding to quotients immediately preceding the quotients $2a$, we have $x = N_1$, $y = D_1$, for a solution of the equation $x^2 - ny^2 = 1$, when the number of recurring quotients is even; and $x = N_1$, $y = D_1$, a solution of the equations $x^2 - ny^2 = -1$, $x^2 - ny^2 = 1$, alternately. The equation $x^2 - ny^2 = 1$ can, therefore, always be solved in positive integers; but $x^2 - ny^2 = -1$, only when the number of recurring quotients in the convergence to $\sqrt{n}$ is odd.

Ex. 1. $x^2 - 3y^2 = 1$.

In the convergence to $\sqrt{3}$, the quotients are

$$1,\ 1,\ 2,\ 1,\ 2,\ \&c.;$$

and the converging fractions are

$$\frac{1}{1},\ \frac{2}{1},\ \frac{5}{3},\ \frac{7}{4},\ \frac{19}{11},\ \frac{26}{15},\ \&c.;$$

therefore, as the number of recurring quotients is even, the solutions are

$$x = 2,\ 7,\ 26,\ \&c.,$$

$$y = 1,\ 4,\ 15,\ \&c.$$

And $x^2 - 3y^2 = -1$ is impossible.

Ex. 2. $x^2 - 13y^2 = \pm 1$.

In the convergence to $\sqrt{13}$, the quotients are

$$3,\ 1,\ 1,\ 1,\ 1,\ 6;\ 1,\ 1,\ 1,\ 1,\ 6;\ \&c.;$$

and the fractions are

$$\frac{3}{1},\ \frac{4}{1},\ \frac{7}{2},\ \frac{11}{3},\ \frac{18}{5},\ \frac{119}{33};\ \frac{137}{38},\ \frac{256}{71},\ \frac{393}{109},\ \frac{649}{180},\ \&c.;$$

also the number of recurring quotients is odd;

$$\therefore\ x = 18,\ y = 5,$$

is a solution of $x^2 - 13y^2 = -1$;

and $x = 649$, $y = 180$,

is a solution of $x^2 - 13y^2 = +1$.

NOTE 9.

DISTINCTION AND CONNECTION BETWEEN "ARITHMETICAL" AND "SYMBOLICAL" ALGEBRA.

[DR WOOD, in the foregoing work, has nowhere distinctly recognised the division of *Algebra* into two parts, *Arithmetical*, and *Symbolical*. But of late years the tendency has been towards increased generalization; and, as a consequence, *Symbolical* (or, as some will call it, *Formal*) Algebra ha received greater attention. Whether indeed much *practical* benefit will arise from keeping up a marked line of distinction between *Arithmetical* and *Symbolical* Algebra may perhaps be doubted; nevertheless it is due to an exact science, that its foundations be laid sure and strong, and the discussion of first principles is at all times to be encouraged.

The only book in our language, in which this distinctive teaching i fully carried out, is the admirable "*Treatise on Algebra*," in 2 vols., by th late Professor Peacock; and to that work the higher class of Students wil doubtless have recourse. But for the sake of others, with the kind permission of the deceased Author's literary representative (the Rev. Jas. Raine) a few extracts are here reprinted, which may serve at least to give a sor of notion of what is meant by *Symbolical* Algebra, and of its relativ importance.]

I. *Definitions, and Explanation, of Terms.*

"In *Arithmetical* Algebra, *zero* and infinity are the extreme limits of the values of the symbols which we employ, though the circumstances of their usage will very generally confine them within a much less extensive range."

"Thus, in the expression $a-b$, a may have every value between infinity and b, whilst b may have every value between a and 0. The expression $a-b$ may have every value between 0 and infinity."

"But in *Symbolical* Algebra, a, b, and $a-b$, may severally have every value between *positive* and *negative* infinity, *zero* being included in their number."

II. *Statement of "the Principle of the Permanence of Equivalent Forms."*

"This principle, which is made the foundation of the operations and results of *Symbolical* Algebra, may be stated as follows:—

"*Whatever algebraical forms are equivalent, when the symbols are general in* form *but specific in* value, *will be equivalent likewise, when the symbols are general in* value *as well as in form.*"

III. *Connection between* Arithmetical, *and* Symbolical *Alegbra.*

"*The Principle of the Permanence of Equivalent Forms* is that which expresses, in the most general terms, the nature of the connection between *Arithmetical* and *Symbolical* Algebra."

"All the conclusions of *Arithmetical* Algebra are considered to be the necessary results of the *defined* operations of addition, subtraction, multiplication and division, involution and evolution, when applied to numbers or quantities whose relations are fully understood; and such conclusions when represented through the medium of symbols, which are general in *form*, but specific in *value*, or by rules, which are general in their *form*, though applied to quantities, which are specific in their *value*, are assumed to be true likewise, when the symbols, which represent such magnitudes, are equally general in their *form* and *representation.*"

"In *Arithmetical* Algebra it is the *definition* of the operation (whether expressed or understood) which determines the result, and also the *rule* for obtaining it. In *Symbolical* Algebra it is the *rule* which determines the meaning of the operation, such *rule* being determined by the principles of *Arithmetical* Algebra, when the symbols, though general in their *form*, are yet so specific in *their value*, as to come under the operation of its *definitions.* The *rules* of operation are the same in *Arithmetical* and *Symbolical* Algebra, and therefore the results are the same as far as they proceed in common; but it is at the point of transition from *Arithmetical* to *Symbolical* Algebra, when the symbols, or the conditions of their usage, cease to be *arithmetical*, that the meaning of the operations must be determined not by *definition*, but *interpretation;* and such interpretations must vary with every change in the circumstances of their application."

"The results of *Arithmetical* Algebra may be said to exist by *necessity*, as consequences of the *definitions;* and those definitions, whether expressed or understood (for they are never formally enunciated), may be considered as derived immediately from the relations of *numbers*, and as consequently involving nothing which is arbitrary or variable in our conceptions of their nature or essence. They may be said, therefore, to possess, in an eminent degree, the character of mathematical *necessity*."

"The case is very different, however, with the results of *Symbolical*, as far as they are not common likewise to *Arithmetical*, Algebra; inasmuch as they may be said to exist by *convention* only, for the *rules* for forming them are not proved as consequences of *definitions*, but are borrowed, or adapted, from a kindred science. And it is only when specific values are assigned to the symbols, that their relations or properties can become the subject of our reasonings, with a view to their *interpretation* in those cases, and in those cases only, where the requisite correspondence, between the symbols and the quantities which they are assumed to represent, can be shewn to exist."

"Again, such interpretations must not be confined to the meaning of the operations performed merely, but may extend likewise to the nature of the connection which exists between the operation and its result. The sign =, which is universally used for this purpose, means arithmetical *equality* in *Arithmetical* Algebra, when placed between the primitive expression and the result of the operation which it involves, whether the result, to which it leads, presents itself under a finite, or indefinite, form. But in *Symbolical* Algebra, in cases which are not likewise common to *Arithmetical* Algebra, and in which the operation which produces the result requires *interpretation*, the sign =, in common with the expressions which it connects, must necessarily be included in it. Its most comprehensive meaning will be, that the expression, which exists on one side of it, is the result of an *operation* (using this term in its largest sense) which is indicated on the other side o it and not performed. This view of its general meaning will include, as consequence, *Arithmetical equality*, or *Algebraical equivalence*, according as either one or the other of them may be shewn to exist."

"The phrase '*algebraical equivalence*,' as distinct from *algebraica identity*, or *arithmetical equality*, would be applied in the case of expression which, though not possessing either of these characters, are capable of reproducing or representing the symbolical properties of the expressions from which they are derived."

IV. Diverging Series, *neither false, nor insecure.*

"It has become a common practice with many distinguished analysts to denounce all *diverging* series as either *false*, or *insecure;* or, in other words as not capable of replacing the expression in which they originate in any algebraical operation, without leading to erroneous results. It would, however, be more in accordance with large and comprehensive views o *Symbolical Algebra* and its operations, if it should be said, that *diverging series* are not *arithmetical*, and therefore incapable of *arithmetical* computation by the aggregation of their terms, that inasmuch as they *rarely, if ever* originate in expressions which are *arithmetical* both in their arrangement and value, it would be more correct to term them *false* under such circumstances, *if* they gave *arithmetical* values of expressions which were not themselves *arithmetical;* and further, that an *algebraical equivalence* may exist between an expression and its developement, when they are not *arithmetically equal.*"

NOTE 10.

ON THE PRINCIPLE OF MATHEMATICAL GENERALISATION.

(By the Rev. J. R. LUNN, M.A.)

In any mathematical investigation it is evidently convenient, that we should have our symbols as free from any restrictions as they can possibly be; for otherwise it might happen at the end of the investigation that a certain symbol employed was not subject to the laws we assumed throughout the operation, and then the whole work would be useless. For instance, at the outset of Algebra we might have been acquainted with the arithmetical operations on *whole numbers* only: if then, in the solution of a problem which required us to perform these, such as multiplication and division, we obtained for any symbol a *fractional* value, we could not rely at all on the correctness of our reasoning, unless we had previously discovered, that the results of multiplication and division in *fractions* take the same symbolical form as they do when those fractions become *integers.*

In order to get rid of the restrictions in such cases, *when we know nothing as yet about the meaning of the symbols excepting under the restrictions,* we have the choice of either of the two following methods: (1) from the original *definition* of our symbols or terms to form a new one which shall merely express in other words what the original one did for the restricted case, and shall not by its phraseology exclude the case to which we wish to generalize: or (2) to take any property which the symbols possess, and assume it to be true *universally,* always provided that it be one which is deduced immediately from the original *definition.* This can very often be seen by assuming the property in question, and deducing the original *definition* from it.

An example of the first method will be found in Art. 129.

It is, however, in most cases not so easy to generalize the *definition,* and consequently the second method is usually adopted; and the principle, or convention, upon which we assume the symbolical proposition to be universally true is called "*the Permanence of Equivalent Forms,*" and is usually stated thus:

When any Algebraical Proposition is proved true for the symbols subject to certain restrictions, but there is nothing in its form which should lead us of necessity *to assume these restrictions, it shall be true without any restriction whatever.* This, of course, is to be adopted under the conditions previously stated (i.e. that we know nothing about the unrestricted symbols, and the proposition in question is obtained directly from the *definitions*): but in cases where we know all about the properties of the symbols when they are unrestricted, it is still of great use in indicating to us the manner upon which we must proceed to prove the proposition we are invest gating.

For instance, in positive integral indices we have proved that $a^m \times a = a^{m+n}$; we wish to know whether indices are *the only things* which posses this property (positive integers being assumed throughout).

Let then $a^m \times a^n = a^{m+n}$ for all positive integral values of m and n, an suppose $a^1 = a$; then $a^m = a^{m-1+1} = a^{m-1} \times a = a^{m-2} \times a \times a =$ &c. $= a \times a \times a \ldots$ t m factors; therefore indices, in the usual acceptation of the term, are th only symbols which satisfy the above law: and therefore we assume thi law to be true *universally*, so that it stands in place of a *definition* c *fractional*, or *negative*, indices.

If we had used another notation such as $a_m \times a_n = a_{m+n}$, where $a_1 = a$ we should have $a_m = a \times a \times \ldots$ to m factors $= a^m$, and $a_{\frac{1}{2}} = \sqrt{a}$, $a_{-3} = \frac{1}{a^3}$ &c. thus we see, that if in any expression, involving a symbol m suppose, w write $m+n$ for m, and the result is equivalent to that obtained by *multiplyin* two such expressions together, then the assumed expression is equal to th m^{th} power of the value which it takes, when 1 is written for m.

In the *Binomial Theorem* we have proved, that $(1+x)^m = 1 + mx + \frac{m(m-1)}{\underline{|2}}x^2 +$ &c. as long as m is a positive integer: there is nothing in the *form* of this equality to lead us to suppose that m must be s restricted; we should therefore expect, that it will be *universally* true but this cannot be *assumed*, since we have already discovered the propertie of *fractional*, and *negative*, *indices*. If, however, we wish to prove thi proposition to be true in these cases, we must first investigate whether th above series is subject to the *index law* or not, i.e. whether writing in it $m+n$ in place of m gives the same result as that obtained by the multiplication of two such series together.

Now at the very beginning of *Algebra*, definitions, conventions, and assumptions are made, so that the symbolical forms of addition, subtraction, multiplication, and division, should be always the same, whether the symbols are positive or negative, integral, or fractional; and if the series $1 + mx + \frac{m(m-1)}{\underline{|2}}x^2 +$ &c. be denoted by $f(m)$, or a_m, in the expression $f(m) \times f(n)$, or $a_m \times a_n$, no other operation than these four is involved, therefore the *form* of the result is unalterable, and therefore either *always*, or *never*, $f(m) \times f(n) = f(m+n)$; or, with the other notation, either *always* or *never* $a_m \times a_n = a_{m+n}$.

All then that is now necessary is, that $f(1)$, or a_1, should be equal to $1+x$, and then $f(x)$, or a_m, will be equal to $(1+x)^m$ in all cases.

It will easily be seen, that in the latter part of Euler's proof of the *Binomial Theorem*, every step is analogous to a corresponding step in the method adopted to discover the meaning of a *fractional* or a *negative index*.

THE END.

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